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Some operations on complex fuzzy graphs

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Abstract

In this paper we discussed about some types of complex fuzzy graphs which is the extension of fuzzy graph. As the membership value of elements of fuzzy graph is in between 0 and 1. In complex fuzzy graph it will be extended to unit circle of complex plane. Therefore we have to consider the amplitude value as well as phase term value. We are considering complex fuzzy graphs in polar forms. Some operations on complex fuzzy graphs such as union, intersection, composition, and Cartesian products are introduced.Some basic theorems with respect to the above mentioned operations are proved with examples.
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1 Introduction

L.A.Zadeh [1965] gave the idea of fuzzy set which is similar to probability function. The idea of complex fuzzy set (CFS) was first given by Ramot et al. [2002]. Accordingly a CFS is an extension of the fuzzy set whose range is extended from [0,1] to a disc of unit radius in complex plane. Xueling et al. [2019] introduced some basic operations on complex fuzzy set. They have developed a new algorithm in signals and system by using complex fuzzy sets. After that some results have been given by P.Bhattacharya [1987] about fuzzy graphs. Thirunavukarasu et al. [2016] extended the fuzzy graph to complex fuzzy graph. Fuzzy graph were narrated by Mordeson and Peng [1994]. Nagoorgani and Latha [2015] gave some operations such as cartesian product, conjuction, disjunction in fuzzy graph. Shannon and Atanassov [1994] defined intuitionistic fuzzy graphs. After that many authors added their ideas to intuitionistic fuzzy graph. Naveed et al. [2019] introduced the complex intuitionistic fuzzy graphs with certain notions of union, join and composition. For Crisp graph we can refer Graph theory by Harary [1969].

In this paper, we introduced some types of complex fuzzy graphs with examples. Also some operations on complex fuzzy graphs such as union, intersection, composition and cartesian product with suitable examples.

2 Premilinaries

Definition 2.1. *A complex fuzzy graph* $G_c = (\sigma_c, \mu_c)$ *is defined on a graph* $G = (V, E)$ *is a pair of complex functions* $\sigma_c : V \to r(z)e^{i\theta(z)}, \mu_c : E \subseteq V \times V \to R(e)e^{i\phi(e)}$ *such that* $\mu_c(z_1, z_2) = R(e)e^{i\phi(e)}$, *where* $R(e) \leq min\{r(z_1), r(z_2)\}$ *and* $\phi(e) \leq min\{\theta(z_1), \theta(z_2)\}$ *for all* $z_1, z_2 \in V$ *and* $0 \leq r(z_1), r(z_2) \leq 1, 0 \leq \theta(z_1), \theta(z_2) \leq 2\pi$.

Example 2.1. *Consider the complex fuzzy graph* $G_c = (\sigma_c, \mu_c)$
where $\sigma_c = \{z_1/0.2e^{i\pi}, z_2/0.5e^{i0.5\pi}, z_3/0.7\}$,
$\mu_c = \{(z_1, z_2)/0.1e^{i0.5\pi}, (z_1, z_3)/0.1, (z_2, z_3)/0.5\}$

Definition 2.2. *A complex fuzzy graph* $\overline{G_c} = (\overline{\sigma_c}, \overline{\mu_c})$ *is said to be a complement of CFG* G_c *if*

i) $\overline{\sigma_c(z)} = \sigma_c(z)$ *and*

ii) $\overline{\mu_c(z_1, z_2)} = \overline{R(e)}e^{i\overline{\phi(e)}}$, *where* $\overline{R(e)} = \min\{r(z_1), r(z_2)\} - R(e)$ *and* $\overline{\phi(e)} = \min\{\theta(z_1), \theta(z_2)\} - \phi(e), \forall z_1, z_2 \in V$.

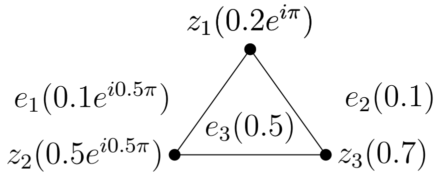

Figure 1: Complex fuzzy graph

Example 2.2. *Consider the example 2.1, The complement of a CFG given in the figure 1 is given by*
$\overline{\sigma_c} = \{z_1/0.2e^{i\pi}, z_2/0.5e^{i0.5\pi}, z_3/0.7\}$, $\overline{\mu_c} = \{(z_1, z_2)/0.1, (z_1, z_3)/0.1\}$

Definition 2.3. *The order p and size q of a CFG $G_c = (\sigma_c, \mu_c)$ defined on $G = (V, E)$ are defined by*
$p = \sum_{z\in V} r(z).e^{i\sum_{z\in V}\theta(z)}$ *;* $q = \sum_{e=(z_i,z_j)\in E} R(e).e^{i\sum_{e=(z_i,z_j)\in E}\phi(e)}$,
where $R(e) \leq min\{r(z_i), r(z_j)\}$ and $\phi(e) \leq min\{\theta(z_i), \theta(z_j)\}$ for all $z_i, z_j \in V$.

Example 2.3. *Consider the example 2.1, the order of G_c is $p = 1.4e^{i1.5\pi}$,*
the size of G_c is $q = 0.7e^{i0.5\pi}$

Definition 2.4. *The degree of a vertex z_i in a CFG $G_c = (\sigma_c, \mu_c)$ defined on $G = (V, E)$ is defined by $d(z_i) = \sum_{e=(z_i,z_j)\in\mu_c} R(e)\ .\ e^{i\sum_{e=(z_i,z_j)\in\mu_c}\phi(e)}$ such that $\mu_c(z_i, z_j) = R(e)\ .\ e^{i\phi(e)}$, for all $z_j \in \sigma_c$.*

Example 2.4. *Consider the example 2.1, $d(z_1) = 0.2e^{i0.5\pi}$; $d(z_2) = 0.6e^{i0.5\pi}$;*
$d(z_3) = 0.6$

Definition 2.5. *A CFG is said to be complete for every pair of vertices $\mu_c(z_1, z_2) = R(e)e^{i\phi(e)}$ where $R(e) = \min\{r(z_1), r(z_2)\}$ and $\phi(e) = \min\{\theta(z_1), \theta(z_2)\}$ for all $z_1, z_2 \in V$*

Example 2.5. *Let $G_c = (\sigma_c, \mu_c)$ be a CFG,where*
$\sigma_c = \{z_1/0.5e^{i0.7\pi}, z_2/0.8e^{i\pi}, z_3/0.6e^{i\pi}\}$
$\mu_c = \{(z_1, z_2)/0.5e^{i0.7\pi}, (z_1, z_3)/0.5e^{i0.7\pi}, (z_2, z_3)/0.6e^{i\pi}\}$

Definition 2.6. *A CFG is regular if $d(z_i) = d(z_j)$ for all $z_i, z_j \in \sigma_c$.*

Definition 2.7. *In a CFG $G_c = (\sigma_c, \mu_c)$ for all $z_i, z_j \in \sigma_c$, the neighbourhood of z_i is defined by $N(z_i) = \{z_j \in \sigma_c/(z_i, z_j) \in \mu_c\}$*

Definition 2.8. *A path P in a CFG $G_c = (\sigma_c, \mu_c)$ is a sequence of distinct vertices $z_0, z_1, z_2, \cdots z_n \in V$ (except possibly $z_0 and z_n$) such that $\mu_c(z_{i-1}, z_i)$= $R(e_i)e^{i\phi(e_i)}, R(e_i) > 0, \phi(e_i) \geq 0, i = 1, 2, ..., n$. Here n is called the length of the path. The consecutive pairs are called edges of the path. The strength of the path in a CFG is defined by $\mu_c(z_{i-1}, z_i) = \min\{R(e_i)\} e^{i \min \phi(e_i)}$,*
$i = 1, 2, 3, ..., n$. It is denoted by $S(p)$.

Definition 2.9. *The strength of connectedness between two vertices z_i abd z_j which is defined as the maximum amplitude and maximum phase term values of the strength of all paths between z_i and z_j. In symbol we denote it as $\mu_c^{\infty}(z_i, z_j) = CONN_{G_c}(z_i, z_j)$, $\mu_c^{\infty}(z_i, z_j) = T(e)e^{i\psi(e)}, 0 \leq T(e) \leq 1, 0 \leq \psi(e) \leq 2\pi$, where $T(e)$ is maximum amplitude value of all paths between z_i and z_j and $\psi(e)$ is maximum phase term value of all paths between z_i and z_j.*
For any arc (z_i, z_j), if $R(e) \geq T(e)$ and $\phi(e) \geq \psi(e)$ then the arc (z_i, z_j) is said to be strong.

Definition 2.10. *The strong degree of a vertex z in a CFG is defined by sum of membership values of strongarcs incident at z ,and it is denoted by $d_s(z)$.*

Definition 2.11. *The strong neighbourhood of z_i in a CFG is defined by*
$N_s(z_i)$=$\{z_j \in V/(z_i, z_j) is\ a\ strong\ arc\}$.

Definition 2.12. *A CFG $G_c = (\sigma_c, \mu_c)$ is said to be bipartite if the vertex σ_c can be partitioned into two non-empty sets σ_{c1} and σ_{c2} such that $\mu_c(z_i, z_j) = 0$ if $z_i, z_j \in \sigma_{c1}$ and $z_i, z_j \in \sigma_{c2}$.*

Definition 2.13. *A CFG $G_c = (\sigma_c, \mu_c)$ is said to be complete bipartite if the vertex σ_c can be partitioned into two non-empty sets σ_{c1} and σ_{c2} such that $\mu_c(z_i, z_j) = R(e).e^{i\phi(e)}$, where $R(e) = \min\{r(z_i), r(z_j)\}, \phi(e) = \min\{\theta(z_i), \theta(z_j)\}$ for $z_i \in \sigma_{c1}$ and $z_j \in \sigma_{c2}$.*

Definition 2.14. *A vertex z_i of a complex fuzzy graph $G_c = (\sigma_c, \mu_c)$ is said to be an isolated vertex if $\mu_c(z_i, z_j) = 0, \forall z_j \in V - \{z_i\}, (i.e)N(z_i) = \emptyset$.*

3 Operations on complex fuzzy graph

In this section the operations union, intersection, composition and cartesian product of CFGs are defined with examples. Some propositions based on the above operations are stated and proved.

Definition 3.1. *Let the two complex fuzzy graphs $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$, and $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ defined on two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. Let G_{c_1} be a pair of complex functions $\sigma_{c_1} : V_1 \to r_1(z)e^{i\theta_1(z)}, \mu_{c_1} : E_1 \subseteq$*

$V_1 \times V_1 \to R_1(e).e^{i\phi_1(e)}$ such that $\mu_c(z_1, z_2) = R_1(e)e^{i\phi_1(e)}$ where $R_1(e) \leq \min\{r_1(z_1), r_1(z_2)\}, \phi_1(e) \leq \min\{\theta_1(z_1), \theta_1(z_2)\}$. Also G_{c_2} is a pair of complex functions $\sigma_{c_2} : V_2 \to r_2(z)e^{i\theta_2(z)}, \mu_{c_2} : E_2 \subseteq V_2 \times V_2 \to R_2(e).e^{i\phi_2(e)}$ such that $\mu_c(z_1, z_2) = R_2(e)e^{i\phi_2(e)}$ where $R_2(e) \leq \min\{r_2(z_1), r_2(z_2)\}, \phi_2(e) \leq \min\{\theta_2(z_1), \theta_2(z_2)\}$. Then the union of two complex fuzzy graphs $G_c = (\sigma_c = \sigma_{c_1} \cup \sigma_{c_2}, \mu_c = \mu_{c_1} \cup \mu_{c_2})$ on $G = (V = V_1 \cup V_2, E = E_1 \cup E_2)$ is defined as follows

(i) $\sigma_c(z) = (\sigma_{c_1} \cup \sigma_{c_2})(z) = \begin{cases} r_1(z)e^{i\theta_1(z)}, \forall z \in V_1 \textit{and } z \notin V_2 \\ r_2(z)e^{i\theta_2(z)}, \forall z \in V_2 \textit{and } z \notin V_1 \\ \max\{r_1(z), r_2(z)\} e^{i\max\{\theta_1(z),\theta_2(z)\}}, z \in V_1 \cap V_2 \end{cases}$

(ii) $\mu_c(z_1, z_2) = (\mu_{c_1} \cup \mu_{c_2})(z_1, z_2) =$
$\begin{cases} R_1(e)e^{i\phi_1(e)}, \forall\ (z_1, z_2) \in E_1 \textit{ and} (z_1, z_2) \notin E_2 \\ R_2(e)e^{i\phi_2(e)}, \forall\ (z_1, z_2) \in E_2 \textit{ and } (z_1, z_2) \notin E_1 \\ \max\{R_1(e), R_2(e)\} e^{i\max\{\phi_1(e),\phi_2(e)\}}, (z_1, z_2) \in E_1 \cap E_2 \end{cases}$

Proposition 3.1. *Prove that the union of two complex fuzzy graph is also a complex fuzzy graph.*

Proof. Let G_{c_1} and G_{c_2} be two CFGs, then the union $G_c = G_{c_1} \cup G_{c_2}$ is discussed in three different case. For vertices,

i) Suppose that $z \in V_1$ and $z \notin V_2$ then
$$\begin{aligned}(\sigma_{c_1} \cup \sigma_{c_2})(z) &= \max\{r_1(z), r_2(z)\}.e^{i\max\{\theta_1(z),\theta_2(z)\}} \\ &= \max\{r_1(z), 0\}.e^{i\max\{\theta_1(z),0\}} \\ &= r_1(z).e^{i\theta_1(z)}, \forall z \in V_1 and\ z \notin V_2.\end{aligned}$$

ii) Similarly we can prove, for $z \notin V_1$ and $z \in V_2$

iii) Suppose that $z \in V_1 \cap V_2$, then
$$\begin{aligned}(\sigma_{c_1} \cup \sigma_{c_2})(z) &= \max\{\sigma_{c_1}, \sigma_{c_2}\} \\ &= \max\left\{r_1(z)e^{i\theta_1(z)}.r_2(z)e^{i\theta_2(z)}\right\} \\ &= \max\{r_1(z), r_2(z)\}.e^{i\max\{\theta_1(z),\theta_2(z)\}}, \forall z \in V_1 \cap V_2.\end{aligned}$$

For edges,

i) Suppose that $(z_1, z_2) \in E_1, (z_1, z_2) \notin E_2$
$$\begin{aligned}(\mu_{c_1} \cup \mu_{c_2})(z_1, z_2) &= \max\{\mu_{c_1}(z_1, z_2), \mu_{c_2}(z_1, z_2)\} \\ &\leq \max\{\min\{r_1(z_1), r_1(z_2), \min\{r_2(z_1), r_2(z_2)\}\}\}. \\ e^{i\max\{\min\{\theta_1(z_1),\theta_1(z_2)\},\min\{\theta_2(z_1),\theta_2(z_2)\}\}} \\ &= \max\{R_1(e), R_2(e)\}.e^{i\max\{\phi_1(e),\phi_2(e)\}} \\ &= \max\{R_1(e), 0\}.e^{i\max\{\phi_1(e),0\}} \\ &= R_1(e).e^{i\phi_1(e)}, \forall (z_1, z_2) \in E_1, (z_1, z_2) \notin E_2\end{aligned}$$

ii) Similarly, we can prove that, $(z_1, z_2) \notin E_1, (z_1, z_2) \in E_2$

iii) Suppose that $(z_1, z_2) \in E_1 \cap E_2$

$$(\mu_{c_1} \cup \mu_{c_2})(z_1, z_2) = \max\{\mu_{c_1}(z_1, z_2), \mu_{c_2}(z_1, z_2)\}$$
$$\leq \max\{\min\{r_1(z_1), r_1(z_2)\}, \min\{r_2(z_1), r_2(z_2)\}\}.$$
$$e^{i\max\{\min\{\theta_1(z_1),\theta_1(z_2)\},\min\{\theta_2(z_1),\theta_2(z_2)\}\}}$$
$$= \max\{R_1(e), R_2(e)\}.e^{i\max\{\phi_1(e),\phi_e(z)\}} \text{ for } (z_1, z_2) \in E_1 \cap E_2$$

$Therefore$ the union of two CFGs is also a CFG. □

Example 3.1. *Consider the two complex fuzzy graphs* $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$ *and* $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ *given below* $\sigma_{c_1} = \{z_1/0.4e^{i0.8\pi}, z_2/0.7e^{i2\pi}, z_3/0.5e^{i1.2\pi}\}$, $\mu_{c_1} = \{(z_1, z_2)/0.4e^{i0.8\pi}, (z_2, z_3)/0.4e^{i\pi}\}$ *and* $\sigma_{c_2} = \{z_1/0.5e^{i\pi}, z_3/0.5e^{i1.4\pi}, z_4/0.8e^{i2\pi}$, $\mu_{c_2} = \{(z_1, z_4)/0.4e^{i0.6\pi}, (z_3, z_4)/0.5e^{i1.4\pi}\}$ *Then the union* $G_c = (\sigma_{c_1} \cup \sigma_{c_2} = \sigma_c, \mu_{c_1} \cup \mu_{c_2} = \mu_c)$ *on* $G = (V_1 \cup V_2, E_1 \cup E_2)$ *can be written as*

$\sigma_c = \{z_1/0.5e^{i\pi}, z_2/0.7e^{i2\pi}, z_3/0.5e^{i1.4\pi}, z_4/0.8e^{i2\pi}\}$

$\mu_c = \{(z_1, z_2)/0.4e^{i0.8\pi}, (z_1, z_4)/0.4e^{i0.6\pi}, (z_3, z_4)/0.5e^{i1.4\pi}, (z_2, z_4)/0.4e^{i\pi}\}$

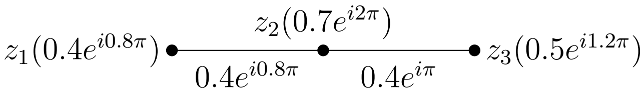

Figure 2: G_{c_1}

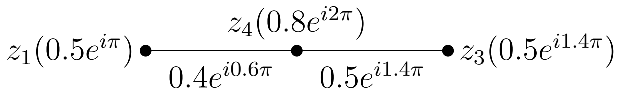

Figure 3: G_{c_2}

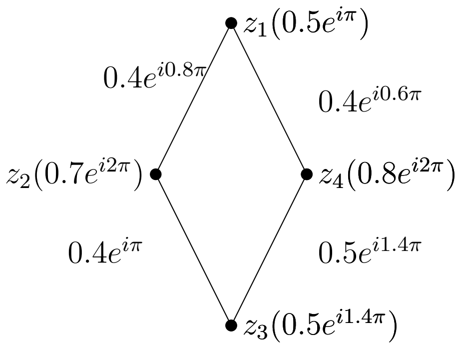

Figure 4: $G_{c_1} \cup G_{c_2}$

Remark 3.1. *Union of two strong CFG need not be a strong CFG.*

Example 3.2. *Consider the Strong CFGs $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$ and $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ where $\sigma_{c_1} = \{z_1/0.4e^{i2\pi}, z_2/0.6e^{i\pi}\}$, $\mu_{c_1} = \{(z_1, z_2)/0.4e^{i\pi}\}$ and $\sigma_{c_2} = \{z_1/0.8e^{i0.2\pi}, z_2/0.2e^{i\pi}\}$, $\mu_{c_2} = \{(z_1, z_2)/0.2e^{i0.2\pi}\}$ then the union is given by $\sigma_c = \{z_1/0.8e^{i2\pi}, z_2/0.6e^{i\pi}\}$, $\mu_c = \{(z_1, z_2)/0.4e^{i0.2\pi}\}$.*
However this is not a strong CFG.

Definition 3.2. *Let the two complex fuzzy graphs $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$, and $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ defined on two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. Let G_{c_1} be a pair of complex functions $\sigma_{c_1} : V_1 \to r_1(z)e^{i\theta_1(z)}, \mu_{c_1} : E_1 \subseteq V_1 \times V_1 \to R_1(e).e^{i\phi_1(e)}$ such that $\mu_c(z_1, z_2) = R_1(e)e^{i\phi_1(e)}$ where $R_1(e) \leq \min\{r_1(z_1), r_1(z_2)\}, \phi_1(e) \leq \min\{\theta_1(z_1), \theta_1(z_2)\}$. Also G_{c_2} is a pair of complex functions $\sigma_{c_2} : V_2 \to r_2(z)e^{i\theta_2(z)}, \mu_{c_2} : E_2 \subseteq V_2 \times V_2 \to R_2(e).e^{i\phi_2(e)}$ such that $\mu_c(z_1, z_2) = R_2(e)e^{i\phi_2(e)}$ where $R_2(e) \leq \min\{r_2(z_1), r_2(z_2)\}, \phi_2(e) \leq \min\{\theta_2(z_1), \theta_2(z_2)\}$. Then the intersection of two complex fuzzy graphs $G_c = (\sigma_c = \sigma_{c_1} \cap \sigma_{c_2}, \mu_c = \mu_{c_1} \cap \mu_{c_2})$ is defined as follows*

(i) $\sigma_c(z) = (\sigma_{c_1} \cap \sigma_{c_2})(z) = \min\{r_1(z), r_2(z)\} e^{i\min\{\theta_1(z),\theta_2(z)\}}, z \in V_1 \cap V_2$

(ii) $\mu_c(z_1, z_2) = (\mu_{c_1} \cap \mu_{c_2})(z_1, z_2) =$
$$\begin{cases} \min\{R_1(e), R_2(e)\} e^{i\min\{\phi_1(e),\phi_2(e)\}}, (z_1, z_2) \in E_1 \cap E_2 \\ 0, otherwise \end{cases}$$

Proposition 3.2. *Prove that the intersection of two complex fuzzy graphs is also a complex fuzzy graph.*

Proof. Let G_{c_1} and G_{c_2} be two complex fuzzy graphs, then

i) $(\sigma_{c_1} \cap \sigma_{c_2})(z) = \min\{r_1(z), r_2(z)\}.e^{i\min\{\theta_1(z),\theta_2(z)\}}$
$Itistrivial.$

ii) For $(z_1, z_2) \in E_1 \cap E_2$
$$\begin{aligned}(\mu_{c_1} \cap \mu_{c_2})(z_1, z_2) &= \min\{\mu_{c_1}(z_1, z_2), \mu_{c_2}(z_1, z_2)\}.e^{i\min\{\phi_1(z),\phi_2(z)\}} \\ &\leq \min\{\min\{r_1(z_1), r_1(z_2)\}, \min\{r_2(z_1), r_2(z_2)\}\}. \\ &e^{i\min\{\min\{\theta_1(z_1),\theta_1(z_2)\},\min\{\theta_2(z_1),\theta_2(z_2)\}\}} \\ &= \min\{\min\{r_1(z_1), r_2(z_1)\}, \min\{r_1(z_2), r_2(z_2)\}\}. \\ &e^{i\min\{\min\{\theta_1(z_1),\theta_1(z_2)\},\min\{\theta_2(z_1),\theta_2(z_2)\}\}} \\ &= \min\{R_1(z), R_2(z)\}.e^{i\min\{\phi_1(z),\phi_2(z)\}}\end{aligned}$$
where $R_1(z) \leq \min\{r_1(z_1), r_2(z_1)\}, R_2(z) \leq \min\{r_1(z_2), r_2(z_2)\},$
$\phi(z_1) \leq \min\{\theta_1(z_1), \theta_2(z_1)\},$
$\phi(z_2) \leq \min\{\theta_2(z_1), \theta_2(z_2)\}$

Hence the $G_{c_1} \cap G_{c_2}$ is a CFG. $\square$

Example 3.3. *Consider the two complex fuzzy graphs* $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$ *where* $\sigma_{c_1} = \{z_1/0.5e^{i0.7\pi}, z_2/0.8, z_3/0.7e^{i1.5\pi}\}$; $\mu_{c_1} = \{(z_1, z_2)/0.5, (z_2, z_3)/0.7\}$ *and*
$G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ *where* $\sigma_{c_2} = \{z_1/0.4e^{i2\pi}, z_2/0.6e^{i0.8\pi}, z_4/0.7e^{i\pi}\}$;
$\mu_{c_2} = \{(z_1, z_4)/0.4e^{i\pi}, (z_1, z_2)/0.4e^{i0.8\pi}\}$
Then the intersection of G_{c_1} *and* G_{c_2} *on a pair* $G = (V_1 \cap V_2, E_1 \cap E_2)$ *is given by* $V = \{z_1/0.4e^{i0.7\pi}, z_2/0.6\}$, $E = \{(z_1, z_2)/0.4\}$ *where* $V = V_1 \cap V_2$; $E = E_1 \cap E_2$.

Remark 3.2. *Intersection of two strong CFGs is also a strong CFG.*

Definition 3.3. *Let the two complex fuzzy graphs* $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$, *and* $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ *defined on two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *respectively. Let* G_{c_1} *be a pair of complex functions* $\sigma_{c_1} : V_1 \to r_1(a)e^{i\theta_1(a)}, \mu_{c_1} : E_1 \subseteq V_1 \times V_1 \to R_1(e).e^{i\phi_1(e)}$ *such that* $\mu_c(a_1, a_2) = R_1(e)e^{i\phi_1(e)}$ *where* $R_1(e) \leq \min\{r_1(a_1), r_1(a_2)\}, \phi_1(e) \leq \min\{\theta_1(a_1), \theta_1(a_2)\}$. *Also* G_{c_2} *is a pair of complex functions* $\sigma_{c_2} : V_2 \to r_2(a)e^{i\theta_2(a)}, \mu_{c_2} : E_2 \subseteq V_2 \times V_2 \to R_2(e).e^{i\phi_2(e)}$ *such that* $\mu_c(a_1, a_2) = R_2(e)e^{i\phi_2(e)}$ *where* $R_2(e) \leq \min\{r_2(a_1), r_2(a_2)\}, \phi_2(e) \leq \min\{\theta_2(a_1), \theta_2(a_2)\}$. *Then the composition of two complex fuzzy graphs is defined as follows*

(i) $(\sigma_{c_1} \circ \sigma_{c_2})(a_1, a_2) = \min\{r_1(a_1), r_2(a_2)\} e^{i\min\{\theta_1(a_1), \theta_2(a_2)\}}, \forall a_1, a_2 \in V = V_1 \circ V_2$

(ii) $(\mu_{c_1} \circ \mu_{c_2})((a, a_2), (a, b_2)) = \min\{r_1(a), R_2(a)\} e^{i\min\{\theta_1(a), \phi_2(a)\}}$
where $R_2(a) \leq \min\{r_2(a_2), r_2(b_2)\}; \phi_2(a) \leq \min\{\theta_2(a_2), \theta_2(b_2)\}, \forall a \in V_1, (a_2, b_2) \in E_2$

(iii) $(\mu_{c_1} \circ \mu_{c_2})((a_1, a), (b_1, a)) = \min\{R_1(a), r_2(a)\} e^{i\min\{\phi_1(a), \theta_2(a)\}}$
where$R_1(a) \leq \min\{r_1(a_1), r_1(b_1)\}; \phi_1(a) \leq \min\{\theta_1(a_1), \theta_1(b_1)\}, \forall a \in V_2, (a_1, b_1) \in E_1$

(iv) $(\mu_{c_1} \circ \mu_{c_2})((a_1, a_2), (b_1, b_2)) = \min\{r_2(a_2), r_2(b_2), R_1(a)\}$.
$e^{i\min\{\theta_2(a_2), \theta_2(b_2), \phi_1(a)\}}$,$\forall a_2, b_2 \in V_2, a_2 \neq b_2, (a_1, b_1) \in E_1$, *where*$R_1(a) \leq \min\{r_1(a_1), r_1(b_1)\}; \phi_2(a) \leq \min\{\theta_1(a_1), \theta_1(b_1)\}$,

Proposition 3.3. *Prove that the composition of two complex fuzzy graphs is also a complex fuzzy graph.*

Proof. Let G_{c_1} and G_{c_2} be two CFGs then we prove that $G_{C1} \circ G_{C2}$ is a CFG.

(i) $(\sigma_{c_1} \circ \sigma_{c_2}) = (a_1, a_2) = \min\{r_1(a_1), r_2(a_2)\} e^{i\min\{\theta_1(a_1), \theta_2(a_2)\}}, \forall a_1, a_2 \in V$. It is trivial.

(ii) $(\mu_{c_1} \circ \mu_{c_2})((a, a_2)(a, b_2)) = \min\{r_1(a), R_2(a)\} e^{i \min\{\theta_1(a), \theta_2(a)\}}$
$\leq \min\{r_1(a), \min\{r_2(a_2), r_2(b_2)\}\} e^{i \min\{\theta_1(a), \min\{\theta_2(a_2), \theta_2(b_2)\}\}}$
$= \min\{\min\{r_1(a), r_2(a_2)\}, \min\{r_1(a), r_2(b_2)\}\}.$
$e^{i \min\{\min\{\theta_1(a), \theta_2(a_2)\}, \min\{\theta_1(a), \theta_2(b_2)\}\}}$
$= \min\{r(a_1), r(b_1)\} e^{i \min\{\theta(a_1), \theta(b_1)\}}$, where $r(a_1) = \min\{r_1(a), r_2(a_2)\}$, $r(b_1) = \min\{r_1(a), r_2(b_2)\}$, $\theta(a_1) = \min\{\theta_1(a), \theta_2(a_2)\}$, $\theta(b_1) = \min\{\theta_1(a), \theta_2(b_2)\}$ for all $a \in V_1 and (a_2, b_2) \in E_2$.

(iii) $(\mu_{c_1} \circ \mu_{c_2})((a_1, b)(b_1, b)) = \min\{R_1(a), r_2(b)\} e^{i \min\{\theta_1(a), \theta_2(b)\}}$
$\leq \min\{\min\{r_1(a_1), r_1(b_1)\}, r_2(b)\} e^{i \min\{\min\{\theta_1(a_1), \theta_1(b_1)\}, \theta_2(b)\}}$
$= \min\{\min\{r_1(a_1), r_2(b)\}, \min\{r_1(b_1), r_2(b)\}\}.$
$e^{i \min\{\min\{\theta_1(a_1), \theta_2(b)\}, \min\{\theta_1(b_1), \theta_2(b)\}\}}$
$= \min\{r(a_2), r(b_2)\} e^{i \min\{\theta(a_2), \theta(b_2)\}}$, where $r(a_2) = \min\{r_1(a_1), r_2(b)\}$, $r(b_2) = \min\{r_1(b_1), r_2(b)\}$, $\theta(a_2) = \min\{\theta_1(a_1), \theta_2(b)\}$, $\theta(b_2) = \min\{\theta_1(b_1), \theta_2(b)\}$ for all $b \in V_2$ and $(a_1, b_1) \in E_1$.

(iv) For all $a_2, b_2 \in V_2, a_2 \neq b_2, (a_1, b_1) \in E_1$
$(\mu_{c_1} \circ \mu_{c_2})((a_1, a_2), (b_1, b_2)) = \min\{r_2(a_2), r_2(b_2), R_1(a)\}.$
$e^{i \min\{\theta_2(a_2), \theta_2(b_2), \phi(a)\}}$
$\leq \min\{r_2(a_2), r_2(b_2), \min\{r_1(a_1), r_1(b_1)\}\}.$
$e^{i \min\{\theta_2(a_2), \theta_2(b_2), \min\{\theta_1(a_1), \theta_1(b_1)\}\}}$
$= \min\{r_2(a_2), r_2(b_2), r_1(a_1), r_1(b_1)\} .e^{i \min\{\theta_2(a_2), \theta_2(b_2), \theta_1(a_1), \theta_1(b_1)\}}$
$= \min\{r(a), r(b)\} .e^{i \min\{\theta(a), \theta(b)\}}.$
where $r(a) = \min\{r_1(a_1), r_2(a_2)\}$, $r(b) = \min\{r_1(b_1), r_2(b_2)\}$
$\theta(a) = \min\{\theta_1(a_1), \theta_2(a_2)\}$, $\theta(b) = \min\{\theta_1(b_1), \theta_2(b_2)\}$.

Hence $G_{C1} \circ G_{C2}$ is a CFG. □

Example 3.4. *Consider the CFGs,* $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$ *where* $\sigma_{c_1} = \{z_1/0.4e^{i0.7\pi}, z_2/0.6e^{i2\pi}\}; \mu_{c_1} = \{(z_1, z_2)/0.4e^{i0.5\pi}\}$ *and* $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ *where* $\sigma_{c_2} = \{z_3/0.5e^{i1.2\pi}, z_4/0.6e^{i0.7\pi}\}; \mu_{c_2} = \{(z_3, z_4)/0.5e^{i0.7\pi}\}$. *Then the composition of* G_{c_1} *and* G_{c_2} *is given by* $G_c = (\sigma_c, \mu_c)$ *where* $\sigma_c = \{z_1z_3/0.4e^{i0.7\pi}, z_1z_4/0.4e^{i0.7\pi}, z_2z_3/0.5e^{i1.2\pi}, z_2z_4/0.6e^{i0.7\pi}\}$,
$\mu_c = \{(z_1z_3, z_1z_4)/0.4e^{i0.7\pi}, (z_1z_4, z_2z_4)/0.4e^{i0.5\pi}, (z_2z_3, z_2z_4)/0.5e^{i0.7\pi},$
$(z_1z_3, z_2z_3)/0.4e^{i0.5\pi}, (z_1z_3, z_2z_4)/0.4e^{i0.5\pi}, (z_1z_4, z_2z_3)/0.4e^{i0.5\pi}\}$

Remark 3.3. *The Composition of two strong CFG is need not be a strong CFG.*

Definition 3.4. *Let the two complex fuzzy graphs* $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$, *and* $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ *defined on two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *respectively. Let* G_{c_1} *be a pair of complex functions* $\sigma_{c_1} : V_1 \to r_1(a)e^{i\theta_1(a)}, \mu_{c_1} : E_1 \subseteq V_1 \times V_1 \to R_1(e).e^{i\phi_1(e)}$ *such that* $\mu_c(a_1, a_2) = R_1(e)e^{i\phi_1(e)}$ *where* $R_1(e) \leq$

$\min\{r_1(a_1), r_1(a_2)\}, \phi_1(e) \leq \min\{\theta_1(a_1), \theta_1(a_2)\}$. *Also* G_{c_2} *is a pair of complex functions* $\sigma_{c_2} : V_2 \to r_2(a)e^{i\theta_2(a)}, \mu_{c_2} : E_2 \subseteq V_2 \times V_2 \to R_2(e).e^{i\phi_2(e)}$ *such that* $\mu_c(a_1, a_2) = R_2(e)e^{i\phi_2(e)}$ *where* $R_2(e) \leq \min\{r_2(a_1), r_2(a_2)\}, \phi_2(e) \leq \min\{\theta_2(a_1), \theta_2(a_2)\}$.
Then the Cartesian product of two complex fuzzy graphs is defined as follows

(i) $(\sigma_{c_1} \times \sigma_{c_2})(a_1, a_2) = \min\{r_1(a_1), r_2(a_2)\} e^{\min\{\theta_1(a_1), \theta_2(a_2)\}}, for a_1, a_2 \in V$

(ii) $(\mu_{c_1} \times \mu_{c_2})((a, a_2), (a, b_2)) = \min\{r_1(a), R_2(a)\}.e^{\min\{\theta_1(a), \phi_2(a)\}}$
where $R_2(a) \leq \min\{r_2(a_2), r_2(b_2)\}; \phi_2(a) \leq \{\theta_2(a_2), \theta_2(b_2)\}$ *for all* $a \in V_1$ *and* $(a_2, b_2) \in E_2$

(iii) $(\mu_{c_1} \times \mu_{c_2})((a_1, a), (b_1, a)) = \min\{R_1(a), r_2(a)\}.e^{\min\{\phi_1(a), \theta_2(a)\}}$
where $R_1(a) \leq \min\{r_1(a_1), r_1(b_1)\}; \phi_2(a) \leq \{\theta_1(a_1), \theta_1(b_1)\}$ *for all* $a \in V_2$ *and* $(a_1, b_1) \in E_1$

Proposition 3.4. *The cartesian product of two complex fuzzy graphs is also a complex fuzzy graph.*

Proof. Let G_{c_1} and G_{c_2} be two complex fuzzy graphs, then we prove that $G_{c1} \times G_{c2}$ is a CFG.

(i) $\sigma_{c_1} \times \sigma_{c_2}(a_1, a_2) = \min\{r_1(a_1), r_2(a_2)\}.e^{i\min\{\theta_1(a_1, \theta_2(a_2))\}}, \forall a_1, a_2 \in V_1 \times V_2$. It is trivial, we have to verify the conditions only for $E_1 \times E_2$

(ii) $(\mu_{c_1} \times \mu_{c_2})((a, a_2), (a, b_2)) = \min\{r_1(a), R_2(a)\}.e^{i\min\{\theta_1(a), \phi_2(a)\}}$
$\leq \min\{r_1(a), \min\{r_2(a_2), r_2(b_2)\}\}.e^{i\min\{\theta_1(a), \min\{\theta_2(a_2), \theta_2(b_2)\}\}}$
$= \min\{\min\{r_1(a), r_2(a_2)\}, \min\{r_1(a), r_2(b_2)\}\}.$
$e^{i\min\{\min\{\theta_1(a), \theta_2(a_2)\}, \min\{\theta_1(a), \theta_2(b_2)\}\}}$
$= \min\{r(a_1), r(b_1)\}.e^{i\min\{\theta(a_1), \theta(b_1)\}}$
where $r(a_1) = \min\{r_1(a), r_2(a_2)\}, r(b_1) = \min\{r_1(a), r_2(b_2)\},$
$\theta(a_1) = \min\{\theta_1(a), \theta_2(a_2)\}, \theta(b_1) = \min\{\theta_1(a), \theta_2(b_2)\}$
forall $a \in V_1, (a_2, b_2) \in E_2$

(iii) $(\mu_{c_1} \times \mu_{c_2})((a_1, a), (b_1, a)) = \min\{R_1(a), r_2(a)\}.e^{i\min\{\phi_1(a), \theta_2(a)\}}$
$\leq \min\{\min\{r_1(a_1), r_1(b_1)\}, r_2(a)\}.e^{i\min\{\min\{\theta_1(a_1), \theta_1(b_1)\}, \theta_2(a)\}}$
$= \min\{\min\{r_1(a_1), r_2(a)\}, \min\{r_1(b_1), r_2)\}\}.$
$e^{i\min\{\min\{\theta_1(a_1), \theta_2(a)\}, \min\{\theta_1(b_1), \theta_2(a)\}\}}$
$= \min\{r(a_2), r(b_2)\}.e^{i\min\{\theta(a_2), \theta(b_2)\}}$
where $r(a_2) = \min\{r_1(a_1), r_2(a)\}, r(b_2) = \min\{r_1(b_1), r_2(a)\},$
$\theta(a_2) = \min\{\theta_1(a_1), \theta_2(a)\}, \theta(b_2) = \min\{\theta_1(b_1), \theta_2(a)\}$
forall $a \in V_2, (a_1, b_1) \in E_1$

Hence $G_{c1} \times G_{c2}$ is a CFG. □

Example 3.5. *Let G_{c_1} and G_{c_2} be two complex fuzzy graphs.*
$\sigma_{c_1} = \{z_1/0.4e^{i2\pi}, z_2/0.5e^{i1.2\pi}\}$, $\mu_{c_1} = \{(z_1, z_2)/0.4e^{i1.2\pi}\}$,
$\sigma_{c_2} = \{z_3/0.5e^{i0.7\pi}, z_4/0.8e^{i\pi},\ z_5/0.4e^{i1.7\pi},$
$\mu_{c_2} = \{(z_3, z_4)/0.4e^{i0.7\pi}, (z_4, z_5)/0.4e^{i\pi}\}$. *Then the cartesian product is given by*
$\sigma_{c_1} \times \sigma_{c_2} = \{z_1z_3/0.4e^{i0.7\pi}, z_1z_4/0.4e^{i\pi}, z_1z_5/0.4e^{i1.7\pi},$
$z_2z_3/0.5e^{i0.7\pi}, z_2z_4/0.5e^{i\pi}, z_2z_5/0.4e^{i1.2\pi}\}$ $\mu_{c_1} \times \mu_{c_2} = \{(z_1z_3, z_1z_4)/0.4e^{i0.7\pi},$
$(z_1z_4, z_1z_5)/0.4e^{i\pi}, (z_1z_3, z_2z_3)/0.4e^{i0.7\pi}, (z_2z_3, z_2z_4)/0.5e^{i0.7\pi},$
$(z_2z_4, z_2z_5)/0.4e^{i1.2\pi}, (z_1z_4, z_2z_4)/0.4e^{i\pi}, (z_1z_5, z_2z_5)/0.4e^{i1.2\pi}\}$

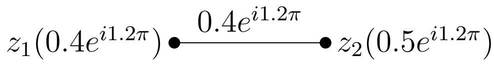

Figure 5: G_{c1}

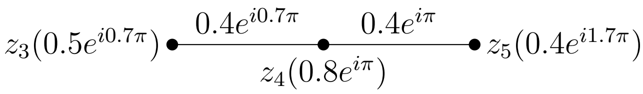

Figure 6: G_{c2}

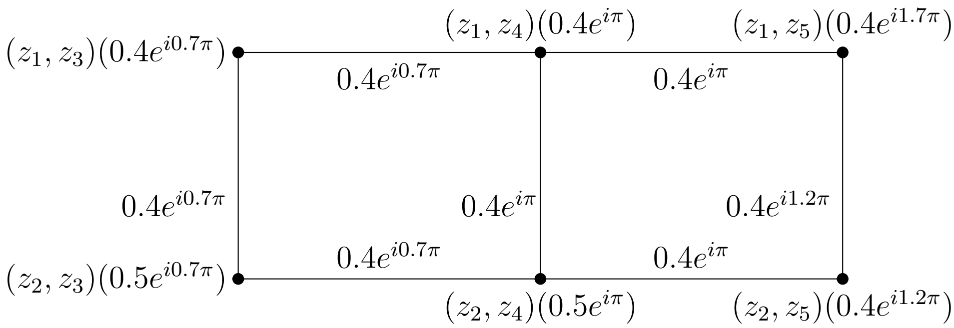

Figure 7: $G_{c1} \times G_{c2}$

Remark 3.4. *Cartesian product of two strong CFG is also a strong CFG.*

4 Conclusions

In this paper, we discussed about some operations on complex fuzzy graphs such as union, intersection, composition and Cartesian product with examples.

Complex fuzzy graph is an extension of a fuzzy graph. We are working to extend the algorithms applied to complex fuzzy graphs are i) Domination on complex fuzzy graphs ii) Connectivity of complex fuzzy graphs and more operations on Complex fuzzy graphs

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