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Anti Q-M-fuzzy normal subgroups

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Abstract

The fuzzy set has been applied in wide area by many researchers. We define the concept of anti-homomorphism in Q-fuzzy subgroups and Q-fuzzy normal subgroups and establish some result in this research article and develop some theory of anti- homomorphism in Q-fuzzy subgroups, normal subgroups and also extend results on Q-fuzzy abelian subgroup and Q- fuzzy normal subgroup. Many research scholars completed their research in field of fuzzy subgroup, anti fuzzy subgroup, Q-fuzzy subgroup, anti Q-fuzzy subgroup, homomorphism, anti homomorphism etc.
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1 Introduction

According to Zadeh.L.A Rosenfeld [1971] was introduce fuzzy sets. It has subsequently been employed in a variety of scientific domains, including engineering, social science, medicine, and pure and applied mathematics. Rosenfeld developed the concept of fuzzy subgroups Asaad [1991]. Biswas.R proposed anti-fuzzy subgroups in Biswas [1990]. Solairaju.A and Nagarajan.R pioneered the structure of Q-fuzzy groups Palaniappan and Muthuraj [2004]. Jacobson.N was the first to use the words M-group and M-subgroup in Jacobson [1951]. In this study, we present and discuss the concepts of group Q- M homomorphism, group anti Q- M homomorphism, and anti Q- M-fuzzy normal subgroup of M-group.

2 Preliminaries

Definition 2.1. *Let X $\neq \phi$. Let X is fuzzy $\theta \subseteq \theta : X \in [0, 1]$.*

Definition 2.2. *Zadeh [1965] A fuzzy $\subseteq \theta \leq N$.*
It is satisfying the axioms,

(i) $\theta(\alpha\beta) \geq \text{lower}\{\theta(\alpha), \theta(\beta)\}$

(ii) $\theta(\alpha^{-1}) = \theta(\alpha), \forall \alpha, \beta \in N$.

Definition 2.3. *Biswas [1990]* $A fuzzy \subseteq \theta$ *of a group G is said to be anti fuzzy $\subseteq$ G if it is satisfying the following conditions,*

(i) $\theta(uv) \leq \text{lower}\{\theta(u), \theta(v)\}$

(ii) $\theta(u^{-1}) = \theta(u), \forall u, v \in G$.

Definition 2.4. *Biswas [1990] Let G: M and $\theta \subseteq G$. Then θ is called M-fuzzy $\subseteq$ G if $\forall u \in G$ and $m \in M$, then $\theta(mx) \leq \theta(u)$*

Definition 2.5. *Jacobson [1951] A Q-fuzzy set θ is $Q - fuzzy \leq G$ if $\forall u, v \in G$, and $\rho \in Q$*

(i) $\theta(uv, \rho) \geq \text{lower}\{\theta(u, \rho), \theta(v, \rho)\}$

(ii) $\theta(u^{-1}, \rho) = \theta(u, \rho)$

Definition 2.6. *Solairaju and Nagarajan [2009] Let fuzzy $\lambda \subseteq X$. For $t\epsilon[0, 1] \subseteq \theta$ is denoted by $[\theta_t = \{u \in U : \theta_\lambda(u) \geq t\}]$*

Definition 2.7. *Sithar Selvam et al. [2014] A Q-fuzzy set θ is called $Q \leq G$ if $\forall u, v \in G$, and $\rho \in Q$ in anti Q fuzzy.*

(i) $\theta(uv,\rho) \leq \text{lower}\{\theta(u,\rho), \theta(v,\rho)\}$

(ii) $\theta(u^{-1},\rho) = \theta(u,\rho)$

Definition 2.8. *Sithar Selvam et al. [2014] An antifuzzy normal* $Q \leq G$. *Then* $G \nearrow \theta$ *of G if* $\forall x, y \in G$ *and* $\rho \in Q$, $\theta(uyx^{-1},\rho) = \theta(v,\rho)$.

3 Anti Q-M- fuzzy normal subgroups and its level subsets

Definition 3.1. *Let* θ *be anti fuzzy* $Q-M- \leq M-groupG$, *then* $\theta \leq M(G)$ *if* $\forall u, v \in G$, $\rho \in Q$, *and* $m \in M$ *such that*
$\theta(m(uvu^{-1}),\rho) = \theta(m(v),\rho)$ *(or)* $\theta(m(uv),\rho) = \theta(m(vu),\rho)$.

Definition 3.2. *Let* θ *be anti fuzzy* $Q-M \leq M-groupG$. *For any* $t \in [0,1]$, *the subset* θ_t *is defined by* $\theta_t = \{u \in G, \rho \in Q, m \in M\theta(m(u),\rho) \leq t\}$ *and it is the subset of* θ.

Theorem 3.1. *If* θ *is a fuzzy* $Q-M- \subseteq$ *of a M-group G, then* θ *is an anti fuzzy* $Q-M- \leq M-groupG$ *iff the level subset* $\theta_t, t \in [0,1]$ *is subgroup of M-group G.*

Proof. Let us assume that θ is an $antifuzzy - Q - M- \leq M - groupG$.
The level subset $\theta_t = \{u \in G, \rho \in Q, m \in M\theta(m(u),\rho) \leq t, t \in [0,1]\}$.
Let $u, v \in \theta_t$, then $\theta(mx,\rho) \leq t$ and $\theta(my,\rho) \leq t$
Now

$$\begin{aligned}\theta(m(uy^{-1}),\rho) &\leq upper\{\theta((mu),\rho), \theta(m(v^{-1}),\rho)\}\\ &= upper\{\theta(mu,\rho), \theta(mv,\rho)\}\\ &\leq upper\{t,t\}\end{aligned}$$

Thus

$$\theta(m(uv^{-1}),\rho) \leq t$$

Hence $xy^{-1} \in \theta_t$. Therefore $\theta_t \leq M(G)$.
Conversely, Let θ_t be a subgroup of a M-group G.
Let $u, v \in \theta_t$. Then $\theta(mu,\rho) \leq t$ and $\theta(mv,\rho) \leq t$.

$$\begin{aligned}\Rightarrow \theta(m(uv^{-1}),\rho) &\leq t \qquad , Because\{uv^{-1} \in \theta_t\}\\ &= upper\{t,t\}\\ &= upper\{\theta(mu,\rho), \theta(mv,\rho)\}\end{aligned}$$

Therefore

$$\theta(m(uv^{-1}), \rho) \leq upper\{\theta(mu, \rho), \theta(mv, \rho)\}$$

Hence θ is an anti fuzzy $Q \leq M - groupG$. □

Definition 3.3. *Let θ be a anti fuzzy $Q - M \leq m - groupG$. The set $N(\theta)$ is defined by $N(\theta) = \{\alpha \in G\theta(m(\alpha ua^{-1}), \rho) = \theta(m(u), \rho)\}$ $\forall u \in G$ and $\rho \in Q$, $m \in M$. and it is called an anti fuzzy Q-M-normalizer of θ.*

Theorem 3.2. *If θ is a fuzzy $Q - M \leq M - groupG$. Then θ is an anti fuzzy $Q - M - fuzzy \leq M - groupG$ iff the level subsets $\theta_t, t \in [0, 1] \leq M(G)$.*

Proof. Let us assume that $\theta \leq Q - M - antifuzzynormalsubgroupofaM(G)$ and the level subsets $\theta_t, t \in [0, 1]$ is a subgroup of a M-group G.
We take $u \in G$ and $\alpha \in \theta_t$, then $\theta(ma, \rho) \leq t$
Now $\theta(m(\alpha xa^{-1}), \rho) = \theta(ma, \rho) \leq t.$
Since θ is an anti fuzzy normal Q-M $\leq M(G)$, $\theta(m(uau{-}1), \rho) \leq t$
Therefore $u\alpha u^{-1} \in \theta_t$, hence $\theta_t \leq M(G)$. □

Theorem 3.3. *If θ is an $\leq Q - M - antifuzzynormalsubgroupofaM(G)$ Then*

(i) $N(\theta) \leq M(G)$.

(ii) *θ is an normal anti fuzzy -Q-M $\leq$ iff $N(\theta) = G$.*

(iii) *θ is an normal fuzzy $anti - Q - M \leq N(\theta)$.*

Proof. Let $\alpha, \beta \in N(\theta)$. (i) Then $\theta(m(\alpha ua^{-1}), \rho) = \theta(mu, \rho) \forall u \in G, \rho \in Q, m \in M$ and $\theta(m(\beta x\beta^{-1}), \rho) = \theta(mu, \rho) \forall u \in G, \rho \in Q, m \in M$. Now

$$\begin{aligned} \theta(m(\alpha\beta u(\alpha\beta)^{-1}), \rho) &= \theta(m(\alpha\beta u\beta^{-1}\alpha^{-1}), \rho) \\ &= \theta(m(\beta u\beta^{-1}), \rho) \\ &= \theta(mu, \rho) \end{aligned}$$

Then we have, $\theta(m(\alpha\beta u(\alpha\beta)^{-1}), \rho) = \theta(mu, \rho)$
$\Rightarrow \alpha\beta \in N(\theta)$
Therefore $N(\theta) \leq M(G)$.
(ii) We know that

$$N\theta \subseteq G, \quad (1)$$

θ is an normal fuzzy anti-$Q - M \leq G$.
Let $\alpha \in G$, then $\theta(m(\alpha u\alpha^{-1}), \rho) = \theta(mu, \rho)$ $\forall$ $u \in G$, $\rho \in Q, m \in M$.
Then

$$\alpha \in N(\theta) \Rightarrow G \subseteq N(\theta) \quad (2)$$

From $(1)\&(2)$, we get
$N(\theta) = G$
Conversely, assume that $N(\theta) = G$
We have, $\theta(m(\alpha x\alpha^{-1}), \rho) = \theta(mx, \rho)\forall\alpha, x \in G, \rho \in Q, m \in M.$
Therefore θ is an fuzzy normal anti $Q - M \leq M(G)$.
(iii) Let θ be an fuzzy normal anti $Q - M \leq M$
We take $\alpha \in G$, then we have
$\theta(m(\alpha ua^{-1}), \rho) = \theta(mu, \rho)\forall u \in G, \rho \in Q, m \in M.$
Therefore $\alpha \in N(\theta) \Rightarrow G \subseteq N(\theta)$.
Hence θ is an fuzzy normal anti $Q - M \leq N(\theta)$ □

Theorem 3.4. *Let θ be an fuzzy normal anti $Q - M \leq M(G)$, then $h\theta h^{-1}$ is also an fuzzy normal anti $Q - M \leq M(G)$ $\forall$ $h \in G$, $\rho \in Q$, $m \in M$.*

Proof. Given $\theta \leq M(G) \leq M(G)$

$$\begin{aligned}
(i)(h\theta h^{-1})(m(uv), \rho) &= \theta(m(h^{-1}(uv)h), \rho) \\
&= \theta(m(h^{-1}(uhh^{-1}v)h), \rho) \\
&= \theta(m((h^{-1}uh)(h^{-1}vh)), \rho) \\
&\leq upper\{\theta(m(h^{-1}uh), \rho), \theta(m(h^{-1}vh), \rho)\} \\
&\leq upper\{h\theta h^{-1}(mu, \rho), h\theta h^{-1}(mv, \rho)\}
\end{aligned}$$

$\forall$ $u, v \in G$, $\rho \in Q$ and $m \in M$.

$$\begin{aligned}
(ii)h\theta h^{-1}(mu, \rho) &= \theta(m(h^{-1}uh), \rho) \\
&= \theta(m(h^{-1}uh)^{-1}, \rho) \\
&= \theta(m(h^{-1}u^{-1}h), \rho) \\
&= h\theta h^{-1}(mu^{-1}, \rho)
\end{aligned}$$

$\forall$ $u, v \in G$, $\rho \in Q$, $m \in M$.
Therefore $h\theta h^{-1}$ is an fuzzy anti $Q - M \leq M(G)$. □

Theorem 3.5. *Let θ fuzzy anti$Q - M \leq M(G)$, then $h\theta h^{-1}$ fuzzy anti$Q - M \leq M(G)$, $\forall h \in G$, $\rho \in Q$, $m \in M$.*

Proof. Given θ is an anti-Q-M-fuzzy normal subgroup of M-group G.
Then $h\theta h^{-1} \leq G$.
Now

$$\begin{aligned}
h\theta h^{-1}(m(uvu^{-1}), \rho) &= \theta(m(h^{-1}(uvu^{-1})h), \rho) \\
&= \theta(m(uvu^{-1}), \rho) \\
&= \theta(mv, \rho) \\
&= \theta(m(hvh^{-1}), \rho) \\
&= h\theta h^{-1}(mv, \rho)
\end{aligned}$$

Therefore $h\theta h^{-1}$ is also an fuzzy normal anti $Q-M \leq M(G)$. □

Theorem 3.6. *The disjoint two fuzzy normal anti $Q-M \leq M(G)$ is also an anti fuzzy anti $Q-M \leq M(G)G$.*

Proof. Let α and β be two anti-Q-M-fuzzy subgroups of a M-group G.
Then

$$\begin{aligned}(\alpha\beta)(m(uv^{-1}),\rho) &= lower\{\alpha(m(uv^{-1}),\rho),\beta(m(uv^{-1}),\rho)\}\\ &\leq lower\{upper\{\alpha(c(u),\rho),\alpha(m(v^{-1}),\rho)\}\\ &\quad ,upper\{\beta(c(u),\rho),\beta(m(v^{-1}),\rho)\}\}\\ &\leq lower\{upper\{\alpha(cu,\rho),\alpha(mv,\rho)\},\\ &\quad upper\{\beta(cu,\rho),\beta(mv,\rho)\}\}\\ &\leq upper\{lower\{\alpha(cu,\rho),\alpha(mv,\rho)\},\\ &\quad lower\{\beta(cu,\rho),\beta(mv,\rho)\}\}\end{aligned}$$

Therefore $\{(\alpha\beta)(c(uv^{-1}),\rho)\} \leq$ upper $\{(\alpha\beta)(cu,\rho),(\alpha\beta)(cv,\rho)\}$
Hence $\alpha\beta$ is an fuzzy normal anti $Q-M \leq M(G)$. □

Theorem 3.7. *If C and D are an anti fuzzy normal Q-M $\leq M(G)$.Then $A\cap B$ is anti fuzzy normal Q-M $\leq M(G)$.*

Proof. For any $x,y \in G$, $q \in Q$, $m \in M$ We have

$$\begin{aligned}(C\cap D)(m(xyx^{-1}),q) &= upper\{C(m(xyx^{-1}),q),D(m(xyx^{-1}),q)\}\\ &= upper\{C(my,q),D(my,q)\}\\ &= (C\cap D)(my,q)\end{aligned}$$

Hence $C\cap$ is an anti fuzzy normal Q-M $\leq M(G)$. □

4 Group Q-M- homomorphism and group anti Q-M- homomorphism

Definition 4.1. *The function $f: G\times Q \rightarrow H\times Q$ is homorphism group Q-M*
(i) $f: G\rightarrow H$ is a homomorphism group and
(ii) $f(m(uv),\rho) = (f(mv).f(mu),\rho)\ \forall\, u,v \in G, \rho\in Q, m\in M$. where G and H are M-groups.

Definition 4.2. *The function $f: G\times Q \rightarrow H\times Q$ is anti homomorphism group of Q-M if* *(i)* $f: G \to H$ *is homomorphism group*

(ii) $f(m(uv), \rho) = (f(mx).f(mv), \rho)\ \forall\, u, v \in G, \rho \in Q, m \in M.$

Theorem 4.1. *If the function* $f : G \times Q \to H \times Q$ *is a group anti Q-M-homomorphism*

(i) If θ *is an anti Q-M-fuzzy normal subgroup of H, then* $f^{-1}(\theta)$ *is an fuzzy normal anti* $Q - M \leq M(G)$.

(ii) If f is an epimorphism and θ *is an fuzzy normal anti* $Q - M \leq M(G)$ *then* $f(\theta)$ *is an anti normal fuzzy Q-M* $\leq H$. *Where G and H are M-groups.*

Proof. (i) Given the function $f : G \times Q \to H \times Q$ is a group anti-Q-M-homomorphism and θ is an anti normal fuzzy Q-M $\leq H$.
For all $u, v \in G, \rho \in Q, m \in M$ we have,

$$\begin{aligned} f^{-1}(\theta)(m(uvu^{-1}), \rho) &= \theta(f(m(uvu^{-1})), \rho) \\ &= \theta(fm(u^{-1}).f(mv).f(mu), \rho) \\ &= \theta(f(mv), \rho) \\ &= f^{-1}(mv, \rho) \end{aligned}$$

Hence $f^{-1}(\theta)$ is an fuzzy normal anti $Q - M \leq M(G)$.

(ii) Given θ is an fuzzy normal anti $Q - M \leq M(G)$. Then $f(\theta)$ is an anti Q-M-fuzzy subgroup of H.
For any $\aleph, \beta \in H$, we have

$$\begin{aligned} f(\theta)(m(\alpha\beta\aleph^{-1}), \rho) &= \inf \theta(mv, \rho) = inf\theta(m(uvu^{-1}), \rho) \\ f(v) &= \alpha\beta\alpha^{-1} = inf\theta(mv, \rho) \\ f(u) = a, f(v) = \beta &= f(\theta)(mb, \rho) \qquad \text{(Since f is an epimorphism)} \end{aligned}$$

Therefore $f(\theta)$ is an fuzzy normal anti $Q - M \leq H$. □

Definition 4.3. *Let A and B be two fuzzy anti* $Q - M \leq M(G)$. *The Product of A and B is defined by* $AB(m(u), \rho) = infupper(A(mv, \rho), vz = u, B(mz, \rho))u \in G, \rho \in Q, m \in M.$

Theorem 4.2. *If A and B are fuzzy normal anti* $Q - M \leq M(G)$*, then AB is an fuzzy normal anti* $Q - M \leq G$.

Proof. Given A and B are two fuzzy normal anti $Q - M \leq M(G)$.

$$\begin{aligned}
(i)\ AB(m(uv),\rho) &= \inf upper\{A(m(u_1y_1),\rho), B(m(u_2y_2),\rho)\} \\
&\quad where\ u = u_1y_1\ and\ v = u_2y_2 \\
&\leq \inf upper\{upper\{A(mu_1,\rho), A(mv_1,\rho)\}, \\
&\quad upper\{B(mu_2,\rho), B(mv_2,\rho)\}\} \\
&\leq upper\{\inf upper\{A(mu_1,\rho), A(mv_1,\rho)\}, \\
&\quad \inf upper\{B(mu_2,\rho), B(mv_2,\rho)\}\} \\
i.e., AB(m(uv),\rho) &\leq upper\{AB(m(u_1y_1),\rho), AB(m(u_2y_2),\rho)\}
\end{aligned}$$

$$\begin{aligned}
(ii)\ AB(m(u^{-1}),\rho) &= inf upper\{B(m(z^{-1}),\rho), A(m(v^{-1}),\rho)\} \\
where (vz)^{-1} = u^{-1} & \\
&= \inf upper\{B(mz,\rho), A(mv,\rho)\} \\
&= \inf upper\{A(mv,\rho), B(mv,\rho)\} \\
&= AB(mv,\rho). \\
AB(mv^{-1},\rho) &= AB(mv,\rho)
\end{aligned}$$

Hence AB is anti fuzzy normal Q-M $\leq M(G)$. □

5 Conclusions

In this research article, we gave some results of anti Q-M-fuzzy normal subgroup, Group Q-M homomorphism and Group anti Q-M homomorphism. This article used to further research in fuzzy algebra.

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