Ratio Mathematica Volume 46, 2023

On fz- domination number of fuzzy graphs

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Abstract

Given a fuzzy graph $\mathcal{G} = (V, \mu, \sigma)$, the fz- domination number, $\gamma_{fz}(\mathcal{G})$, is the least scalar cardinality of an fz- dominating set of $\mathcal{G}$. In this article, we examine several features of fz-domination number of fuzzy graphs as a result of various fuzzy graph operations. We find bounds for the fz-domination number of a few graph products and look at the requirements for the sharpness of these bounds.

Keywords: fuzzy graph; fz-dominating sets; fz-domination number; graph operations

2020 AMS subject classifications:05C69, 05C72. [1]

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[1]Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1078. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Since the initial introduction of fuzzy graphs by Rosenfeld [1975], a large number of researchers have studied the subject. The notion of domination in fuzzy graphs was first proposed by Somasundaram and Somasundaram [1998]. Somasundaram [2005], Gani and Chandrasekaran [2006], Manjusha and Sunitha [2015], Bhutani and Sathikala [2016] also studied domination in fuzzy graphs. Mordeson and Chang-Shyh [1994] developed operations of fuzzy graphs that are comparable to those in crisp graphs.

Different variations of domination in fuzzy graphs found in literature do not consider the situations where we need to take all the non-zero edges incident at a vertex into consideration. These definitions use either the effective or the strong edges of the fuzzy graph.But our model of fuzzy domination in fuzzy graphs [2022] takes into account all the non-zeroedges incident at a vertex, even if they are small in strength. Further most variations of domination in fuzzy graphs found in literature do not considerthe fuzzy subsets of the vertex set, instead considered the crisp subsets of the fuzzy vertex set.But while considering fuzzy graphs and their subset problems it is more apt toconsider fuzzy subsets of the vertex set than their crisp subsets. By taking all these into consideration we defined fz-domination in fuzz graphs[2022].

We, Lekha and Parvathy [2022] developed fz-domination in fuzzy graphs, which coincides with fractional domination in crisp graphs presented by Hedetniemi and Wimer [1987] and explored by Hedetniemi and Mynhardt [1990]. In this article, we examine the effects of several graph operations on fz-domination.

For basic definitions, terminology and notation in fuzzy graphs we refer to Mordeson and Nair [2000].

Definition 1.1. *(Lekha and Parvathy [2022]). Given a fuzzy graph $\mathcal{G} = (V, \mu, \sigma)$, a fuzzy subset μ' of μ is defined as an fz-dominating set of $\mathcal{G}$, if for every $v \in V$,*

$$\mu'(v) + \sum_{x \in V} \Big(\sigma(x, v) \wedge \mu'(x) \Big) \geq \mu(v).$$

A fuzzy subset μ' is a minimal fz-dominating set, if $\mu'' \subset \mu'$ is not an fz-dominating set.

Definition 1.2. *(Lekha and Parvathy [2022]) Fuzzy domination number or fz-domination number of $\mathcal{G}$, denoted by $\gamma_{fz}(\mathcal{G})$, is defined as*

$$\gamma_{fz}(\mathcal{G}) = \textit{min}\ \{|\mu'| : \mu' \textit{ is a minimal fz-dominating set of } \mathcal{G}\}$$

Example 1.1. *For $\mathcal{F} = (\mu, \sigma)$ shown in Fig. 1,*

$$\mu_1 = \{(x, 0.1), (y, 0.5), (z, 0.2)\}$$

$$\mu_2 = \{(x, 0.6), (y, 0), (z, 0.6)\}$$

$$\mu_3 = \{(x, 0.4), (y, 0.2), (z, 0.4)\}$$

$$\mu_4 = \{(x, 0.5), (y, 0.1), (z, 0.5)\}$$

are all minimal fz-dominating sets of $\mathcal{F}$. μ_1 is a minimum fz-dominating set and $\gamma_{fz}(\mathcal{F}) = 0.8$.

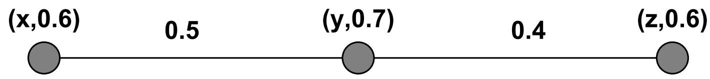

Figure 1: Fuzzy graph, $\mathcal{F}$

2 fz- domination in union of fuzzy graphs

Let $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$. $\mathcal{G} \cup \mathcal{H} = (V, \mu, \sigma)$ where

$$V = V_1 \cup V_2$$

$$\begin{aligned} \mu(u) &= \mu_1(u) \quad \text{if } u \in V_1 \setminus V_2 \\ &= \mu_2(u) \quad \text{if } u \in V_2 \setminus V_1 \\ &= \mu_1(u) \vee \mu_2(u) \quad \text{if } u \in V_1 \cap V_2 \end{aligned}$$

and

$$\begin{aligned} \sigma(u, v) &= \sigma_1(u, v) \quad \text{if } u \in V_1 \setminus V_2, v \in V_1 \\ &= \sigma_2(u, v) \quad \text{if } u \in V_2 \setminus V_1, v \in V_2 \\ &= \sigma_1(u, v) \vee \sigma_2(u, v) \quad \text{if } u, v \in V_1 \cap V_2 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The following theorem gives a general upper bound for the fz-domination number of union of two fuzzy graphs.

Theorem 2.1. *For any two non- trivial fuzzy graphs $\mathcal{G}$ and $\mathcal{H}$,*

$$\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$$

Proof. Consider the fuzzy graphs $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$. Let μ_1' and μ_2' be the minimum fz-dominating sets of $\mathcal{G}$ and $\mathcal{H}$ respectively. Let the fuzzy subset μ' of V be defined by

$$\begin{aligned} \mu'(u) &= \mu_1'(u) \quad \text{if } u \in V_1 \setminus V_2 \\ &= \mu_2'(u) \quad \text{if } u \in V_2 \setminus V_1 \\ &= \mu_1'(u) \vee \mu_2'(u) \quad \text{if } u \in V_1 \cap V_2 \end{aligned}$$

Now let $v \in V$.

Case (i)

If $v \in V_1 \setminus V_2$, then

$$\begin{aligned} \mu(v) &= \mu_1(v) \\ &\leq \Big(\mu_1'(v) + \sum_{x \in V_1} \sigma_1(x, v) \wedge \mu_1'(x)\Big) \\ &= \mu'(v) + \sum_{x \in V} \sigma(x, v) \wedge \mu'(x). \end{aligned}$$

Case (ii)

If $v \in V_2 \setminus V_1$, then

$$\begin{aligned} \mu(v) &= \mu_2(v) \\ &\leq \Big(\mu_2'(v) + \sum_{x \in V_2} \sigma_2(x, v) \wedge \mu_2'(x)\Big) \\ &= \mu'(v) + \sum_{x \in V} \sigma(x, v) \wedge \mu'(x). \end{aligned}$$

Case (iii)

If $v \in V_1 \cap V_2$

$$\begin{aligned} \mu(v) &= \mu_1(v) \vee \mu_2(v) \\ &\leq \Big(\mu_1'(v) + \sum_{x \in V_1} \sigma_1(x, v) \wedge \mu_1'(x)\Big) \vee \Big(\mu_2'(v) + \sum_{x \in V_2} \sigma_2(x, v) \wedge \mu_2'(x)\Big) \\ &\leq (\mu_1'(v) \vee \mu_2'(v)) + \Big(\sum_{x \in V_1 \setminus V_2} (\sigma_1(x, v) \wedge \mu_1'(x)) + \sum_{x \in V_2 \setminus V_1} (\sigma_2(x, v) \wedge \mu_2'(x)) \\ &+ \sum_{x \in V_1 \cap V_2} (\sigma_1(x, v) \vee \sigma_2(x, v)) \wedge (\mu_1'(x) \vee \mu_2'(x))\Big) \\ &\leq \mu'(v) + \sum_{x \in \mathcal{V}} \sigma(x, v) \wedge \mu'(x). \end{aligned}$$

Thus μ' is an fz-dominating set of $\mathcal{G} \cup \mathcal{H}$ and $\mu'(v) \leq \mu_1'(v) + \mu_2'(v)$.
Hence $|\mu'| \leq |\mu_1'| + |\mu_2'|$.
Thus, $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) \leq \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$. □

Remark 2.1. *Obviously equality holds in the above theorem if the vertex sets of $\mathcal{G}$ and $\mathcal{H}$ are disjoint. The following example shows that equality may hold even if they are not disjoint. For the graphs in Fig. 2,* $\gamma_{fz}(\mathcal{G}) = 0.5$, $\gamma_{fz}(\mathcal{H}) = 0.6$ *and* $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) = 1.1$ *so that* $\gamma_{fz}(\mathcal{G} \cup \mathcal{H}) = \gamma_{fz}(\mathcal{G}) + \gamma_{fz}(\mathcal{H})$.

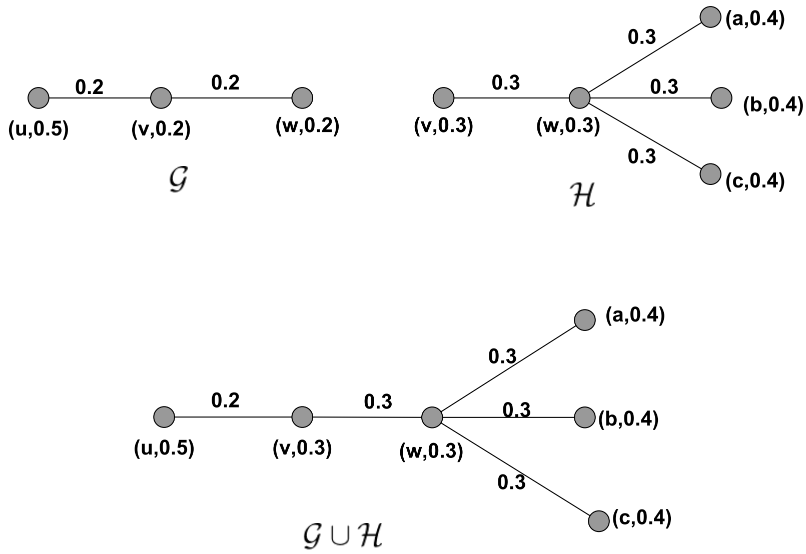

Figure 2: Fuzzy Graphs $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$

3 fz- domination in join of fuzzy graphs

Let $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ whose vertex sets are disjoint. The join $\mathcal{G} + \mathcal{H}$ is defined by $\mathcal{G} + \mathcal{H} = (V, \mu, \sigma)$ where $V = V_1 \cup V_2$,

$$\begin{aligned} \mu(u) &= \mu_1(u) \quad \text{if } u \in V_1 \\ &= \mu_2(u) \quad \text{if } u \in V_2 \end{aligned}$$

and

$$\begin{aligned} \sigma(u,v) &= \sigma_1(u,v) \quad \text{if } u, v \in V_1 \\ &= \sigma_2(u,v) \quad \text{if } u, v \in V_2 \\ &= \mu_1(u) \wedge \mu_2(v) \text{ if } u \in V_1 \text{ and } v \in V_2 \end{aligned}$$

Theorem 3.1. *For any two non- trivial fuzzy graphs $\mathcal{G}$ and $\mathcal{H}$ whose vertex sets are disjoint,*

$$\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq max\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

Proof. Let $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \phi$. Let $\gamma_{fz}(\mathcal{G}) \geq \gamma_{fz}(\mathcal{H})$ and let μ_1' be a minimum fz-dominating set of $\mathcal{G}$.
Define $\mu' \subset \mu$ by

$$\begin{aligned} \mu'(u) &= \mu_1'(u) \quad \text{if } u \in \mathcal{G} \\ &= 0 \quad \text{if } u \in \mathcal{H} \end{aligned}$$

Let m be such that $m = max\{\mu_2(u); u \in \mathcal{H}\}$.
Now $m \leq \gamma_{fz}(\mathcal{H}) \leq \gamma_{fz}(\mathcal{G})$ implies that μ' is an fz-dominating set of $\mathcal{G} + \mathcal{H}$.
Hence,

$$\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq max\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

□

In the following discussion M, m_1 and m_2 denote the maximum membership value of a vertex in $\mathcal{G} + \mathcal{H}$, $\mathcal{G}$ and $\mathcal{H}$ respectively.

Observation 3.1. *It is possible that*

$$\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq min\{\gamma_{fz}(\mathcal{G}), \gamma_{fz}(\mathcal{H})\}$$

For example, if $M \leq \gamma_{fz}(\mathcal{H}) \leq \gamma_{fz}(\mathcal{G})$, then $\mu' \subset \mu$ defined by

$$\begin{aligned} \mu'(u) &= \mu_2'(u) \quad \textit{if } u \in \mathcal{H} \\ &= 0 \quad \textit{if } u \in \mathcal{G} \end{aligned}$$

is an fz-dominating set of $\mathcal{G} + \mathcal{H}$ and hence $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \gamma_{fz}(\mathcal{H})$. Here equality occurs if $M = \gamma_{fz}(\mathcal{H})$. The following example shows that strict inequality can also occur in this relation.

Example 3.1. *Consider the fuzzy graphs $\mathcal{G}_1$, $\mathcal{G}_1$ and $\mathcal{G}_1 + \mathcal{G}_2$ given in Fig.3.*
Here $\gamma_{fz}(\mathcal{G}_1) = 1.6$, $\gamma_{fz}(\mathcal{G}_2) = 1$, $M = 0.8$, $\gamma_{fz}(\mathcal{G}_1 + \mathcal{G}_2) = 0.9$ so that

$$\gamma_{fz}(\mathcal{G}_1 + \mathcal{G}_2) < \gamma_{fz}(\mathcal{G}_2)$$

Observation 3.2. *If $\gamma_{fz}(\mathcal{H}) \leq M \leq |\mu_2|$, then $\gamma_{fz}(\mathcal{G} + \mathcal{H}) = M$.*

Claim: *Define $\mu_2'' \supset \mu_2'$ in $\mathcal{H}$ such that $|\mu_2''| = M$. Then, μ_2'' is an fz-dominating set of $\mathcal{G}+\mathcal{H}$. Hence $\gamma_{fz}(\mathcal{G}+\mathcal{H}) \leq M$. Also, since there is a vertex of membership value M in $\mathcal{G} + \mathcal{H}$, we get $\gamma_{fz}(\mathcal{G} + \mathcal{H}) = M$.*

Observation 3.3. *If $m_1 \leq |\mu_2|$ and $m_2 \leq |\mu_1|$, then $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq m_1 + m_2$.*

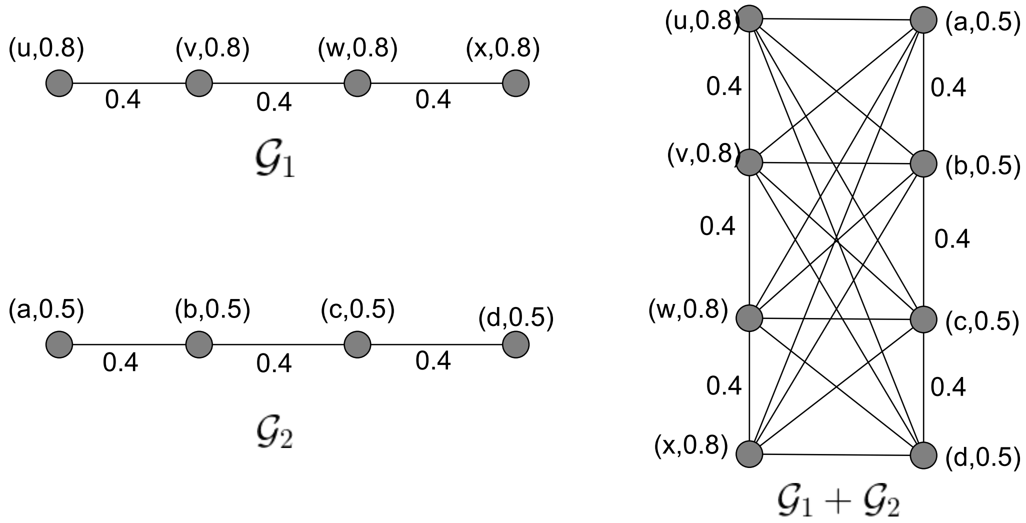

Figure 3: Fuzzy Graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_1 + \mathcal{G}_2$

Claim: *Define $\mu_1' \subset \mu_1$ in $\mathcal{G}$ such that $|\mu_1'| = m_2$ and $\mu_2' \subset \mu_2$ in $\mathcal{H}$ such that $|\mu_2'| = m_1$. Now μ' defined by*

$$\begin{aligned} \mu'(u) &= \mu_1'(u) \quad \textit{if } u \in \mathcal{G} \\ &= \mu_2'(u) \quad \textit{if } u \in \mathcal{H} \end{aligned}$$

is an fz- dominating set in $\mathcal{G} + \mathcal{H}$. Hence $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq m_1 + m_2$.

Observation 3.4. *If $|\mu_2| \leq M$, then $\mu' \subset \mu$ in $\mathcal{G} + \mathcal{H}$ defined by*

$$\begin{aligned} \mu'(u) &= \mu_2(u) \quad \textit{if } u \in \mathcal{H} \\ &= max\{0, \mu_1'(u) - |\mu_2|\} \quad \textit{if } u \in \mathcal{G} \end{aligned}$$

is an fz- dominating set in $\mathcal{G} + \mathcal{H}$. Then, $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \gamma_{fz}(G) - n|\mu_2|$ where n is the number of vertices $u \in \mathcal{G}$ having $\mu_1(u) \geq |\mu_2|$.

Observation 3.5. *If $\gamma_{fz}(\mathcal{H}) \leq M \leq \gamma_{fz}(\mathcal{G})$, then $\mu' \subset \mu$ in $\mathcal{G} + \mathcal{H}$ defined by*

$$\begin{aligned} \mu'(u) &= \mu_2'(u) \quad \textit{if } u \in \mathcal{H} \\ &= max\{0, \mu_1'(u) - \gamma_{fz}(\mathcal{H})\} \quad \textit{if } u \in \mathcal{G} \end{aligned}$$

is an fz- dominating set in $\mathcal{G} + \mathcal{H}$. Then, $\gamma_{fz}(\mathcal{G} + \mathcal{H}) \leq \gamma_{fz}(G) - n\gamma_{fz}(\mathcal{H})$ where n is the number of vertices $u \in \mathcal{G}$ having $\mu_1(u) \geq \gamma_{fz}(\mathcal{H})$.

4 fz- domination in corona of fuzzy graphs

The corona of $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{K}_1 = (u, \mu_2(u))$ is the fuzzy graph $\mathcal{G} \circ \mathcal{K}_1$ obtained by attaching a copy of $\mathcal{K}_1$ to each vertex $v_i \in V_1$ such that $\sigma(v_i, u_i) = \mu_1(v_i) \wedge \mu_2(u_i)$ where u_i represents the vertex in the copy of $\mathcal{K}_1$ corresponding to $v_i \in V_1$.

Observation 4.1. *The following two results are obvious.*

1. $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq \gamma_{fz}(\mathcal{G})$
2. $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$

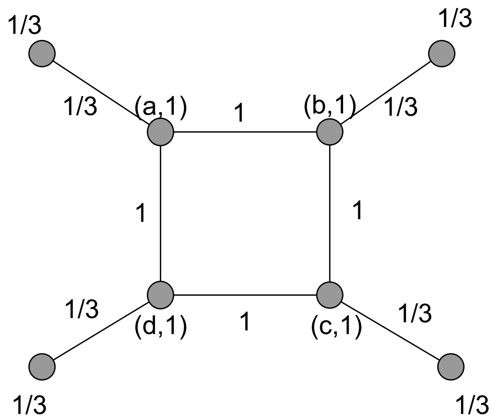

Figure 4: Fuzzy graph $\mathcal{G} \circ \mathcal{K}_1$

Remark 4.1. *The example below shows that equality may occur in observation 4.1(a).*
Consider $\mathcal{G} \circ \mathcal{K}_1$ in figure 4. $\mu' = \{(a, \frac{1}{3}), (b, \frac{1}{3}), (c, \frac{1}{3}), (d, \frac{1}{3})\}$ is a minimum fz-dominating set of $\mathcal{G}$ and $\gamma_{fz}(\mathcal{G}) = \frac{4}{3}$. μ' is an fz-dominating set of $\mathcal{G} \circ \mathcal{K}_1$ also.
Therefore, $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \frac{4}{3} = \gamma_{fz}(\mathcal{G})$.
On the other hand from observation 4.1(a), $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq \gamma_{fz}(\mathcal{G})$.
Thus we get $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = \gamma_{fz}(\mathcal{G})$.

Theorem 4.1. $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \gamma_{fz}(\mathcal{G}) + n\mu_2(u)$ *where* $n = |V_1|$.

Proof. Let μ be the fuzzy subset of $\mathcal{G} \circ \mathcal{K}_1$ and μ_1' be a minimum fz-dominating set of $\mathcal{G}$. Let $\mu' \subset \mu$ be such that

$$\begin{aligned} \mu'(v) &= \mu_1'(v) \quad \text{if } v \in V_1 \\ &= \mu_2(v) \quad \text{otherwise} \end{aligned}$$

Then μ' is an fz-dominating set of $\mathcal{G} \circ \mathcal{K}_1$ and

$$\begin{aligned} |\mu'| &= |\mu'_1| + n|\mu_2| \\ &= \gamma_{fz}(\mathcal{G}) + n\mu_2(u) \end{aligned}$$

Therefore $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq \gamma_{fz}(\mathcal{G}) + n\mu_2(u)$ □

Theorem 4.2. *If $\mu_2(u) \geq \mu_1(v)$ for all $v \in V_1$, then*

$$\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$$

Proof. It is clear that $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$
Consider μ' where

$$\begin{aligned} \mu'(v) &= 0 \quad \text{if } v \in V_1 \\ &= \mu_2(v) \quad \text{if } v = u \end{aligned}$$

Then, μ' is fz-dominating set of $\mathcal{G} \circ \mathcal{K}_1$.
Therefore, $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \leq |\mu'| = n\mu_2(u)$.
Hence $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$ □

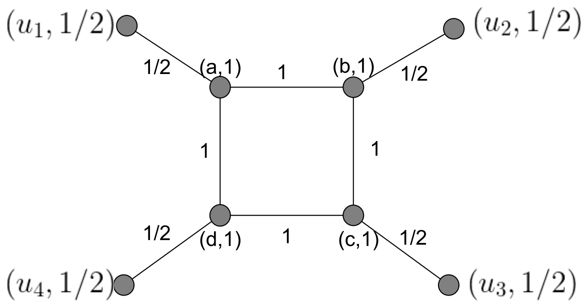

Figure 5: $\mathcal{G}' \circ \mathcal{K}'_1$

Remark 4.2. *The condition $\mu_2(u) \geq \mu_1(v)$ for all $v \in V_1$ is not necessary to get $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) = n\mu_2(u)$. For example, consider $\mathcal{G}' \circ \mathcal{K}'_1$ given in figure 5.*
Here, $\mu_2(u) < \mu_1(v)$ for all $v \in V_1$. Now $\gamma_{fz}(\mathcal{G} \circ \mathcal{K}_1) \geq n\mu_2(u)$ implies that $\gamma_{fz}(\mathcal{G}' \circ \mathcal{K}'_1) \geq 2$.
Also $\mu' = \{(a, \frac{1}{2}), (b, \frac{1}{2}), (c, \frac{1}{2}), (d, \frac{1}{2})\}$ is fz-dominating set of $\mathcal{G}' \circ \mathcal{K}'_1$.
Hence $\gamma_{fz}(\mathcal{G}' \circ \mathcal{K}'_1) = 2 = n\mu_2(u)$

5 fz- domination in Cartesian product

Let $\mathcal{G}_1 = (V_1, \mu_1, \sigma_1)$ and $\mathcal{G}_2 = (V_2, \mu_2, \sigma_2)$. The Cartesian product is the fuzzy graph $\mathcal{G}_1 \Box \mathcal{G}_2 = (V, \mu_1 \times \mu_2, \sigma_1 \times \sigma_2)$ where $V = V_1 \times V_2$,

$$(\mu_1 \times \mu_2)(a, b) = \mu_1(a) \wedge \mu_2(b)$$

and

$$\begin{aligned}(\sigma_1 \times \sigma_2)\Big((a_1, b_2), (a_2, b_2)\Big) &= \mu_1(a_1) \wedge \sigma_2(b_1, b_2) \quad \text{if } a_1 = a_2 \\ &= \sigma_1(a_1, a_2) \wedge \mu_2(b_1) \quad \text{if } b_1 = b_2 \\ &= 0 \quad \text{otherwise}\end{aligned}$$

Theorem 5.1. *For any two nontrivial fuzzy graphs $\mathcal{G}$ and $\mathcal{H}$,*

$$\gamma_{fz}(\mathcal{G} \Box \mathcal{H}) \leq min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(\mathcal{H})\},$$

m, n are the number of vertices with nonzero membership values in $\mathcal{G}$ and $\mathcal{H}$ respectively.

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Proof. Let $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ where

$$V_1 = \{(u_1, \mu_1(u_1)), (u_2, \mu_1(u_2)), ..., (u_m, \mu_1(u_m))\}$$

and

$$V_2 = \{(v_1, \mu_2(v_1)), (v_2, \mu_2(v_2)), ..., (v_n, \mu_2(v_n))\}$$

$\mathcal{G} \Box \mathcal{H} = (V, \mu, \sigma)$ where $V = V_1 \times V_2$, $\mu(u, v) = \mu_1(u) \wedge \mu_2(v)$ and

$$\begin{aligned}\sigma\Big((u_i, v_j), (u'_i, v'_j)\Big) &= \sigma_1(u_i, u'_i) \quad \text{if } v_j = v'_j \\ &= \sigma_2(v_j, v'_j) \quad \text{if } u_i = u'_i \\ &= 0 \quad \text{otherwise}\end{aligned}$$

Let $\mathcal{G}_j$ denotes the fuzzy sub-graph of $\mathcal{G} \Box \mathcal{H}$ induced by $V_1 \times v_j \subset V_1 \times V_2$. Then,

$$V(\mathcal{G}_j) = \{(u_1, v_j), (u_2, v_j), ..., (u_m, v_j)\}$$

$$\mu(u_i, v_j) = \mu_1(u_i) \wedge \mu_2(v_j) \leq \mu_1(u_i)$$

and

$$\sigma\Big((u_i, v_j), (u'_i, v_j)\Big) = min\{\sigma_1(u_i, u'_i), \mu_2(v_j)\} \leq \sigma(u_i, u'_i)$$

Claim: $\gamma_{fz}(\mathcal{G}_j) \leq \gamma_{fz}(\mathcal{G})$. Define μ'_j on $\mathcal{G}_j$ as $\mu'_j(u_i, v_j) = \mu'_1(u_i) \wedge \mu_2(v_j)$ Consider $(u_i, v_j) \in \mathcal{G}_j$. μ'_1 is an fz-dominatng set of $\mathcal{G}$ implies that

$\mu_1(u_i) \leq \mu_1'(u_i) + \sum_{u_k \in \mathcal{G}} \sigma_1(u_k, u_i) \wedge \mu_1'(u_k)$.
Hence,

$$\begin{aligned}\mu_1(u_i) \wedge \mu_2(v_j) &\leq \mu_1'(u_i) \wedge \mu_2(v_j) + \sum_{u_k \in \mathcal{G}} \sigma_1(u_k, u_i) \wedge \mu_1'(u_k) \wedge \mu_2(v_j) \\ &\leq \mu_j'(u_i, v_j) + \sum_{u_k \in \mathcal{G}} \sigma((u_k, v_j), (u_i, v_j)) \wedge \mu_j'(u_k, v_j)\end{aligned}$$

That is,

$$\mu(u_i, v_j) \leq \mu_j'(u_i, v_j) + \sum_{u_k \in \mathcal{G}} (\sigma((u_k, v_j), (u_i, v_j)) \wedge \mu_j'(u_k, v_j)$$

Thus we get μ_j' is an fz- dominating set of $\mathcal{G}_j$ for $j = 1, 2, ..., n$
Also $|\mu_j'| \leq |\mu_1'|$ shows that $\gamma_{fz}(\mathcal{G}_j) \leq \gamma_{fz}(\mathcal{G})$ for $j = 1.2...., n$
Hence

$$\gamma_{fz}(\mathcal{G} \Box \mathcal{H}) \leq n\gamma_{fz}(g)$$

Similarly,

$$\gamma_{fz}(\mathcal{G} \Box \mathcal{H}) \leq m\gamma_{fz}(\mathcal{H})$$

Thus we get,

$$\gamma_{fz}(\mathcal{G} \Box \mathcal{H}) \leq min\{n\gamma_{fz}(\mathcal{G}), m\gamma_{fz}(H)\}$$

□

In the previous theorem, equality might hold. For $\mathcal{G}$,$\mathcal{H}$ and $\mathcal{G} \Box \mathcal{H}$ given in Fig. 6, $\gamma_{fz}(\mathcal{G}) = 0.2$, $\gamma_{fz}(\mathcal{H}) = 0.2$ and $\gamma_{fz}(\mathcal{G} \Box \mathcal{H}) = 0.4$ so that $\gamma_{fz}(\mathcal{G} \Box \mathcal{H}) = min\{n\gamma_{fz}(g), m\gamma_{fz}(H)\}$

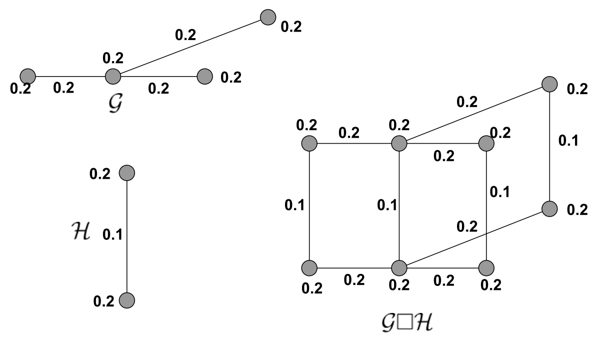

Figure 6: Fuzzy graph $\mathcal{G}$,$\mathcal{H}$ and $\mathcal{G} \Box \mathcal{H}$

V. G. Vizing presented the following conjecture regarding the Cartesian product of crisp graphs in 1968.

$$\gamma(G \Box H) \geq \gamma(G)\gamma(H), \quad \text{for every pair of finite crisp graphs } G \text{ and } H.$$

Possibly the most significant unsolved issue in the field of domination theory is Vizing's conjecture. Here, we investigate the applicability of Vizing's like inequality to fz-dominantion in fuzzy graphs.

Vizing's conjecture is said to be satisfied by a fuzzy graph $\mathcal{G}$, if $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) \geq \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$ for every fuzzy graph $\mathcal{H}$.

Definition 5.1. *A fuzzy graph $\mathcal{H} = (V_1, \mu_1, \sigma_1)$ is known as a partial fuzzy subgraph of $\mathcal{G} = (V, \mu, \sigma)$ induced by V_1 if $V_1 \subset V$, $\mu_1(u) = \mu(u)$ if $u \in V_1$, 0 otherwise and $\sigma_1(u, v) = \sigma(u, v) \wedge \mu(u) \wedge \mu(v)$ for all $u, v \in V$.*

Definition 5.2. *The spanning fuzzy subgraph of $\mathcal{G} = (V, \mu, \sigma)$ is the partial fuzzy subgraph $\mathcal{G}' = (V_1, \mu', \sigma')$ where $V = V_1$ and $\mu = \mu'$*

If $\mathcal{G}'$ is a spanning fuzzy subgraph of the fuzzy graph $\mathcal{G}$, then $\gamma_{fz}(\mathcal{G}') \geq \gamma_{fz}(\mathcal{G})$).

Theorem 5.2. *If $\mathcal{G}$ satisfies Vizing's Conjecture and $\mathcal{G}'$ is a spanning fuzzy subgraph of $\mathcal{G}$ such that $\gamma_{fz}(\mathcal{G}') = \gamma_{fz}(\mathcal{G})$, then $\mathcal{G}'$ also satisfies Vizing's Conjecture.*

Proof. $\mathcal{G}'\Box\mathcal{H}$ is a spanning fuzzy sub- graph of $\mathcal{G}\Box\mathcal{H}$ for every fuzzy graph $\mathcal{H}$. Hence

$$\begin{aligned} \gamma_{fz}(\mathcal{G}'\Box\mathcal{H}) &\geq \gamma_{fz}(\mathcal{G}\Box\mathcal{H}) \\ &\geq \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H}) \quad = \gamma_{fz}(\mathcal{G}')\gamma_{fz}(\mathcal{H}) \end{aligned}$$

□

The example below illustrates that in general this inequality does not hold for fz-domination in fuzzy graphs.

Example 5.1. *Consider the fuzzy graphs $\mathcal{G} = (V_1, \mu_1, \sigma_1)$ and $\mathcal{H} = (V_2, \mu_2, \sigma_2)$ given in figure 7. For $\mathcal{G}$, $V_1 = \{(a, 1), (b, 1), (c, 1)\}$, $\sigma_1(a, b) = \sigma_1(b, c) = 1, \sigma_1(a, c) = 0$. For $\mathcal{H}$, $V_2 = \{(u, 1), (v, 1), (w, 1)\}$, $\sigma_2(u, v) = \sigma_2(v, w) = 1, \sigma_2(u, w) = 0$. $\mu_1' = \{(a, 0.8), (b, 0.6), (c, 0.8)\}$ is a minimum fz-dominating set of $\mathcal{G}$.*
Hence $\gamma_{fz}(\mathcal{G}) = 2.2$. Similarly $\gamma_{fz}(\mathcal{H}) = 2.2$
Now $\mu' = \{((a, u), 0.6), ((a, v), 0.4), ((a, w), 0.6)\}, ((b, u), 0.4), ((b, v), 0.2), ((b, w), 0.4), ((c, u), 0.6), ((c, v), 0.4), ((c, w), 0.6)$ is a minimum fz-dominating set of $\mathcal{G}\Box\mathcal{H}$. Hence $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) = 4.2$
Here

$$\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) < \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$$

There are fuzzy graphs for which

1. $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) < \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$
2. $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) = \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$
3. $\gamma_{fz}(\mathcal{G}\Box\mathcal{H}) > \gamma_{fz}(\mathcal{G})\gamma_{fz}(\mathcal{H})$

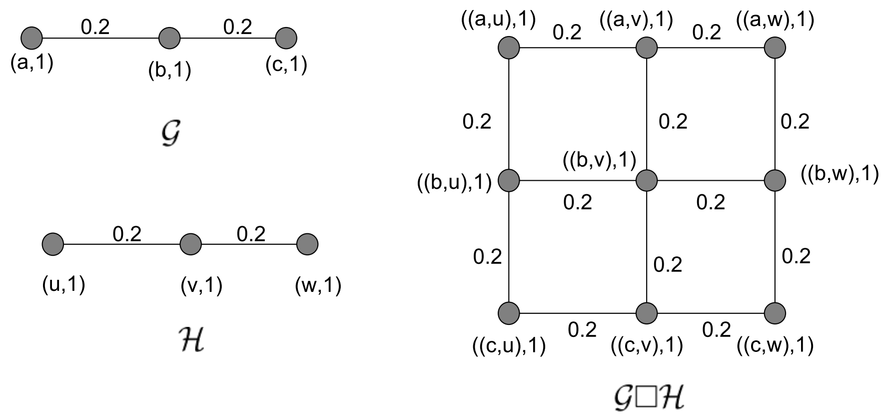

Figure 7: Fuzzy Graphs $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{G}\Box\mathcal{H}$

6 Conclusions

Graph operations are techniques for creating new graphs from ones that already exist, and they are crucial in the design and analysis of large networks. In this article, we investigate various characteristics of the fz-domination number of fuzzy graphs under the influence of some graph operations. It is possible to derive bounds for the fz-domination number of the union, join, corona, and Cartesian product of fuzzy graphs. Through examples, the sharpness of these bounds are demonstrated and the factors that contribute to the sharpness are examined.

References

S. A. K. Bhutani and L. Sathikala. On (r,s)-fuzzy domination in fuzzy graphs. *New Mathematics and Natural Computation 12(01):1-10*, 2016.

N. Gani and V. Chandrasekaran. Domination in fuzzy graph. *Advances in Fuzzy Sets and Systems*, 1, 01 2006.

E. C. G. S. Hedetniemi and C. Mynhardt. Properties of minimal dominating functions of graphs. *Technical Report, DMS-547-IR*, 1990.

S. H. S. Hedetniemi and T. Wimer. Linear time resourse allocation algorithms for trees. *Technical Report URL-014, Department of Mathematics, Clemson University*, 1987.

A. Lekha and K. S. Parvathy. Fuzzy domination in fuzzy graphs. *Journal of Intelligent and Fuzzy Systems*, 2022. doi: 10.3233/JIFS-220987.

O. T. Manjusha and M. S. Sunitha. Strong domination in fuzzy graphs. *Fuzzy Information and Engineering, 7(3):369-377*, 2015.

J. N. Mordeson and P. Chang-Shyh. Operations on fuzzy graphs. *Information Sciences*, 79(3):159–170, 1994. ISSN 0020-0255.

J. N. Mordeson and P. S. Nair. *Fuzzy Graphs and Fuzzy Hypergraphs*. Physica-Verlag, 2000.

A. Rosenfeld. Fuzzy graphs. In *Fuzzy Sets and their Applications to Cognitive and Decision Processes*, pages 77–95. Academic Press, 1975.

A. Somasundaram. Domination in products of fuzzy graphs. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems Vol. 13, No. 2 (2005) 195-204, World Scientific Publishing Company*, 2005.

A. Somasundaram and S. Somasundaram. Domination in fuzzy graphs – i. *Pattern Recognition Letters*, 19(9):787–791, 1998. ISSN 0167-8655.