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Approximation and moduli of continuity for a function belonging to Hölder's class $H^{\alpha}[0,1)$ and solving Lane-Emden differential equation by Boubaker wavelet technique

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Abstract

In this paper, Boubaker wavelet is considered. The Boubaker wavelets are orthonormal. The series of this wavelet is verified for the function $f(t) = t\ \ \forall\ t\ \in$[0,1). The convergence analysis of solution function of Lane-Emden differential equation has been studied. New Boubaker wavelet estimator $E_{2^k,M}(f)$ for the approximation of solution function f belong to Hölder's class $H^{\alpha}[0,1)$ of order $0 < \alpha \leq 1$, has been developed. Furthermore, the moduli of continuity of $\big(f - S_{2^k,M}(f)\big)$ of solution function f of Lane-Emden differential equation has been introduced and it has been estimated for solution function f$\in H^{\alpha}[0,1)$ class. These estimator and moduli of continuity are new and best possible in wavelet analysis. Boubaker wavelet collocation method has been proposed to solve Lane-Emden differential equations with unknown Boubaker coefficients. In this process, Lane-Emden differential equations are reduced into a system of algebraic equations and these equations are solved by collocation method. Three Lane-Emden type equations are solved to demonstrate the applicability of the proposed method. The solutions obtained by the proposed method are compared with their exact solutions. The

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absolute errors are negligible. Thus, this shows that the method described in this paper is applicable and accurate.
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1 Introduction

Wavelet theory is a newly emerging area of research in a mathematical sciences. It has applications in engineering disciplines; such as signal analysis for wave representation and segmentation etc. Wavelets allow the accurate representation of a variety of signals and operators. Wavelets are assumed as a basis function $\{\psi_{n,m}(\cdot)\}$ continuously in time domain. Special feature of wavelets basis is that all functions $\{\psi_{n,m}(\cdot)\}$ are constructed from a single mother wavelet $\psi(\cdot)$ which is small pulse. Many practical and physical problems in the field of science and engineering are formulated as intial and boundry value problems. Approximation of functions by the wavelet method has been discussed by many researchers like Devorce[2], Morlet[5], Meyer[4], Debnath[3], Lal and Satish[11]. The wavelet functions have been applied for finding approximate solutions for some problems arising in numerous branches of science and engineering.

In this paper, Boubaker wavelet has been studied. This wavelet is defined by the orthogonal Boubaker polynomials. It has several interesting and useful properties.

The main aims of present paper are as follows:

(i) To define Boubaker wavelet and to verify Boubaker wavelet series by examples.

(ii) To study the properties of Boubaker wavelet coefficient in expansion of characteristic function.

(iii) To estimate the approximation of solution function f of Lane-Emden differential equations belonging to Hölder's class $H^\alpha[0,1)$ by the Boubaker wavelet series.

(iv) To estimate the moduli of continuity of $\big(f - S_{2^k,M}(f)\big)$ of solution function f of Lane-Emden differential equations belonging to Hölder's class $H^\alpha[0,1)$ by the Boubaker wavelet series.

(v) To solve Lane-Emden differential equation by Boubaker wavelet series by

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collocation method.
The paper is organized as follows: In Section-2, Boubaker wavelet,Boubaker wavelet approximation and moduli of continuity of Function of Hölder's class $H^\alpha[0,1)$ are defined. The Boubaker wavelet series is verified by example. In Section-3 ,theorem concerning the convergence analysis of Boubaker wavelet has been discussed. In Section-4, theorem concerning Boubaker wavelet coefficient in the expansion of characteristic function and approximation in $H^\alpha[0,1)$ have been obtained. In Section-5, theorem concerning the moduli of continuity of $\big(f - S_{2^k,M}(f)\big)$ have been determined. In Section-6, Boubaker wavelet method for solution of differential equation has been discussed . In Section-7, Lane-Emden differential equations have been solved using Boubaker wavelet series by collocation method. Finally, the main conclusions are summarized in Section-8.

2 Definitions and Preliminaries

2.1 Boubaker wavelet

Wavelet functions are constructed from dilation and translation of a definite function, named mother wavelet ψ . $\psi_{b,a}$ may be defined as

$$\psi_{b,a}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \quad a,b \in \mathbb{R}, a \neq 0 \qquad \text{(Daubechies[6])} \qquad (1)$$

where a and b are dilation and translation parametres respectively while t is normalized time. By taking $a = \frac{1}{2^k}$, $b = \frac{n}{2^k}$ and $\psi(t) = \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}B_m(t)$, where $B_m(t)$ is Boubaker polynomial of degree m, in equation (1), it reduces into form

$$\psi^{(B)}_{n,m}(t) = \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}B_m(2^k t - n).$$

In precise,

$$\psi^{(B)}_{n,m}(t) = \begin{cases} \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}B_m(2^k t - n), & if \ \frac{n}{2^k} \leqslant t < \frac{n+1}{2^k}, \\ 0, & otherwise. \end{cases}$$

where $n = 0, 1, 2..., 2^k - 1,\ k = 0, 1, 2, 3, ...,$ while m represents the order of orthogonal Boubaker polynomial (shiralashetti et al.[8]). It has four parameters m, n, k & t. The orthogonal Boubaker polynomial $B_m(t)$ of order m satisfies the following conditions:

$$B_0(t) = 1,\ B_1(t) = \frac{1}{2}(2t-1), B_2(t) = \frac{1}{6}(6t^2-6t+1)\ ;$$
$$B_3(t) = \frac{1}{20}(20t^3-30t^2+12t-1),\ \ B_4(t) = \frac{1}{70}(70t^4-140t^3+90t^2-20t+1)\ ;$$
$$B_5(t) = \frac{1}{252}(252t^5-630t^4+560t^3-210t^2+30t-1);$$
$$B_6(t) = \frac{1}{924}(924t^6-2772t^5+3150t^4-1680t^3+420t^2-42t+1);$$

2.2 Boubaker Wavelet Approximation

A function $f \in L^2[0,1)$ may be expanded in Boubaker wavelet series as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t), \tag{2}$$

where the coefficients $c_{n,m}$ are given by

$$c_{n,m} =< f(t), \psi_{n,m}^{(B)}(t) > . \tag{3}$$

If the series (2) is truncated, then

$$(S_{2^k,M}f)(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t) = C^T\Psi(t) \tag{4}$$

where $(S_{2^k,M}f)$ is the $(2^k, M)^{th}$ partial sum of series (2) and C, $\Psi(t)$ are $2^kM \times 1$ matrices given by

$$C = [c_{0,0}, c_{0,1}, ..., c_{0}, M-1, c_{1,0}, ..., c_{1,M-1}, ..., c_{2^k-1,0}, ..., c_{2^k-1,M-1}]^T \tag{5}$$

and

$$\Psi(t) = [\psi_{0,0}(t), \psi_{0,1}(t), ..., \psi_{0,M-1}(t), \psi_{1,0}(t), \psi_{1,M-1}(t), ..., \psi_{2^k-1,0}(t), ..., \psi_{2^k-1,M-1}]^T. \tag{6}$$

The Boubaker wavelet approximation of f by $(S_{2^k,M}f)$ under norm $||\ ||_2$, denoted by $E_{2^k,M}(f)$, is defined by

$$E_{2^k,M}(f) = \text{min}||f-(S_{2^k,M})||_2 \qquad \text{(Zygmund)[10]}).$$

If $E_{2^k,M}(f) \to 0$ as k $\to \infty$, M$\to\infty$,then $E_{2^k,M}(f)$ is best approximation of f order $(2^k, M)$ (Zygmund[10]).

2.3 Moduli of continuity

The moduli of continuity of a function $f \in L^2[0,1)$ is defined as

$$\begin{aligned} W(f,\delta) &= \sup_{0\le h\le\delta} \|f(\cdot+h)-f(\cdot)\|_2 \\ &= \sup_{0\le h\le\delta}\left(\int_0^1 |f(t+h)-f(t)|^2 dt\right)^{\frac{1}{2}} \end{aligned}$$

It is remarkable to note that $W(f,\delta)$ is a non-decreasing function of δ and $W(f,\delta)\to 0$ as $\delta \to 0$, (Chui [1]).

2.4 Function of Hölder's class $H^\alpha[0,1)$

A function $f \in H^\alpha[0,1)$ if f is continuous and satisfies the inequality

$$f(x)-f(y)=O(|x-y|^\alpha), \forall x,y \in [0,1) \text{ and } 0<\alpha\le 1 \qquad \text{(Das)[9]).}$$

2.5 Example

The example of this section illustrates the validity of the Boubaker wavelet series as follows:
Consider the function $f:[0,1)\to\mathbb{R}$ defined by $f(t)=t \ \ \forall\, t \in$[0,1).
Let

$$f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t). \tag{7}$$

$$\begin{aligned} c_{n,m} &= <f(t),\psi_{n,m}^{(B)}(t)> = \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)\psi_{n,m}^{(B)}(t)dt \\ &= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} t\sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}B_m(2^kt-n)\,dt \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}2^{\frac{k}{2}}\int_0^1 \frac{v+n}{2^k}B_m(v)\frac{dv}{2^k},\ \ 2^kt-n=v \\ &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_m(v)dv. \end{aligned}$$

By above expansion ,taking $m=0$,

$$c_{n,0} = \frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_0(v)dv = \frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)dv = \frac{2n+1}{2^{\frac{3k+2}{2}}}. \tag{8}$$

Next

$$\begin{aligned}
c_{n,1} &= \frac{2\sqrt{3}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_1(v)dv = \frac{\sqrt{3}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)(2v-1)dv \\
&= \frac{\sqrt{3}}{3.2^{\frac{3k+2}{2}}}. \qquad (9)\\
c_{n,2} &= \frac{6\sqrt{5}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_2(v)dv = \frac{\sqrt{5}}{2^{\frac{3k}{2}}}\int_0^1 (v+n)(6v^2-6v+1)dv = 0
\end{aligned}$$

For, $m \geq 2$

$$\begin{aligned}
c_{n,m} &= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\int_0^1 (v+n)B_m(v)dv \\
&= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\left(\int_0^1 vB_m(v)dv + \int_0^1 nB_m(v)dv\right) \\
&= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\left(\int_0^1 \frac{1}{2}(2v-1)B_m(v)dv\right. \\
&+ \left.\int_0^1 \frac{1}{2}B_m(v)dv + \int_0^1 nB_m(v)dv\right) \\
&= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}\left(\int_0^1 B_1(v)B_m(v)dv\right. \\
&+ \left.\frac{1}{2}\int_0^1 B_0(v)B_m(v)dv + n\int_0^1 B_0(v)B_m(v)dv\right) \\
&= \sqrt{(2m+1)}\frac{(2m!)}{(m!)^2}\frac{1}{2^{\frac{3k}{2}}}(0+0+0) = 0 \text{ , } B_m(v) \text{ is orthogonal .} \\
c_{n,m} &= 0 \ \forall n \geq 2^k, \text{ by defintion of } \psi^{(B)}_{n,m}.
\end{aligned}$$

Then , $$f(t) = \sum_{n=0}^{2^k-1} c_{n,0}\psi^{(B)}_{n,0}(t) + \sum_{n=0}^{2^k-1} c_{n,1}\psi^{(B)}_{n,1}(t) + \sum_{n=2^k}^{\infty} c_{n,m}\psi^{(B)}_{n,m}(t), \qquad (10)$$

Next,

$$\begin{aligned}
||f||_2^2 &= < f, f > \\
&= < \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(B)}_{n,m}(t), \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(B)}_{n,m}(t) > \\
&= \left\langle \sum_{n=0}^{2^k-1} c_{n,0}\psi^{(B)}_{n,0}(t) + \sum_{n=0}^{2^k-1} c_{n,1}\psi^{(B)}_{n,1}(t) + \sum_{n=2^k}^{\infty} c_{n,m}\psi^{(B)}_{n,m}(t),\right. \\
&\left. \sum_{n=0}^{2^k-1} c_{n,0}\psi^{(B)}_{n,0}(t) + \sum_{n=0}^{2^k-1} c_{n,1}\psi^{(B)}_{n,1}(t) + \sum_{n=2^k}^{\infty} c_{n,m}\psi^{(B)}_{n,m}(t)\right\rangle \\
&= \sum_{n=0}^{2^k-1} c^2_{n,0}||\psi^{(B)}_{n,0}||_2^2 + \sum_{n=0}^{2^k-1} c^2_{n,1}||\psi^{(B)}_{n,1}||_2^2 + \sum_{n=2^k}^{\infty} c^2_{n,m}||\psi^{(B)}_{n,m}||_2^2,
\end{aligned}$$

$$= \sum_{n=0}^{2^k-1} c_{n,0}^2 + \sum_{n=0}^{2^k-1} c_{n,1}^2 + 0, \{\psi_{n,m}\}_{n,m\in\mathbb{Z}} \text{ being orthonormal}$$

$$= \sum_{n=0}^{2^k-1} \frac{(2n+1)^2}{2^{(3k+2)}} + \sum_{n=0}^{2^k-1} \frac{1}{3.2^{3k+2}} = \frac{1}{3}, \text{ by eqns (8) and (9).}$$

$$\text{Also,} \quad ||f||_2^2 = <f,f>= \int_0^1 |f(t)|^2 dt = \int_0^1 t^2 dt = \frac{1}{3}.$$

Hence, the Boubaker wavelet expansion (7) is verified for $f(t) = t$.

3 Convergence analysis

In this section, the convergence analysis of solution of Lane-Emden differential equation has been studied.

3.1 Theorem

If f be a exact solution of of Lane-Emden differential equation and its Boubaker wavelet series is

$$f(\cdot) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(\cdot) \tag{11}$$

then its $(2^k, M)^{th}$ partial sums $(S_{2^k,M}f)(\cdot) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(\cdot)$ converges to $f(\cdot)$ as M→∞, k→ ∞.

Proof of Theorem 3.1

$$\text{Now,} \quad <f,f> = \left\langle \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}, \sum_{n'=0}^{\infty}\sum_{m'=0}^{\infty} c_{n',m'}\psi_{n',m'}^{(B)} \right\rangle$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{m'=0}^{\infty} c_{n,m}\overline{c_{n',m'}} < \psi_{n,m}^{(B)}, \psi_{n',m'}^{(B)} >$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\overline{c_{n,m}} < \psi_{n,m}^{(B)}, \psi_{n,m}^{(B)} >$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_{n,m}|^2 ||\psi_{n,m}^{(B)}||^2$$

$$= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2||\psi_{n,m}^{(B)}||^2 + \sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty}|c_{n,m}|^2||\psi_{n,m}^{(B)}||^2$$

$$= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2 + 0, \text{ by defintion of } \{\psi_{n,m}\}_{n,m\in\mathbb{Z}}$$

$$\sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2 = ||f||_2^2 < \infty, f \in L^2[0,1). \tag{12}$$

For $M > N$, using eqn (12),

$$||(S_{2^k,M}f) - (S_{2^k,N}f)||_2^2 = \left|\left|\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{N-1}c_{n,m}\psi_{n,m}^{(B)}(t)\right|\right|_2^2$$

$$= \left|\left|\sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t)\right|\right|_2^2$$

$$= \left\langle \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t), \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\psi_{n,m}^{(B)}(t)\right\rangle$$

$$= \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}c_{n,m}\overline{c_{n,m}} < \psi_{n,m}^{(B)}(t), \psi_{n,m}^{(B)}(t) >$$

$$= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}|c_{n,m}|^2||\psi_{n,m}^{(B)}(t)||^2$$

$$= \sum_{n=0}^{2^k-1}\sum_{m=N}^{M-1}|c_{n,m}|^2 \to 0 \text{ as } M \to \infty, N \to \infty.$$

Hence, $\left\{(S_{2^k,M}f)\right\}_{M\in\mathbb{N}}$ is a Cauchy sequence in $L^2[0,1)$, $L^2[0,1)$ is a Banach space and hence $\left\{(S_{2^k,M}f)\right\}_{M\in\mathbb{N}}$ converges to a function $g(t) \in L^2[0,1)$.
Now we need to prove that $g(t) = f(t)$.
For this

$$< g(t) - f(t), \psi_{n_0,m_0}^{(B)}(t) > = < g(t), \psi_{n_0,m_0}^{(B)}(t) > - < f(t), \psi_{n_0,m_0}^{(B)}(t) >$$

$$= < \lim_{M\to\infty}(S_{2^k,M}f)(t), , \psi_{n_0,m_0}^{(B)}(t) > -c_{n_0,m_0}$$

$$= \lim_{M\to\infty}\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}c_{n,m} < \psi_{n,m}^{(B)}(t), \psi_{n_0,m_0}^{(B)}(t) > -c_{n_0,m_0}$$

$$= c_{n_0,m_0} < \psi_{n_0,m_0}(t), \psi_{n_0,m_0}(t) > -c_{n_0,m_0}$$

$$= c_{n_0,m_0} - c_{n_0,m_0} \quad = 0.$$

Thus $< g(t) - f(t), \psi^{(B)}_{n,m}(t) > = 0 \;\forall\; n \geqslant n_0, m \geqslant m_0$.
Then $g(t) = f(t)$.
Hence, $\sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m}\psi^{(B)}_{n,m}(t)$ converges to $f(t)$ as $k \to \infty, M \to \infty$.

4 Approximation analysis

In this Section, approximation f by $(S_{2^k,M}f)$ is estimated as follows.

4.1 Theorem

Let a function $f = \chi_{\left[\frac{n_0}{2^k}, \frac{n_0+1}{2^k}\right)}$, where n_0 is positive integer less than equal to 2^k and Boubaker wavelet expansion of f is

$$f(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{\infty} c_{n,m}\psi^{(B)}_{n,m}(t) \tag{13}$$

then the coefficients $c_{n,m}$ satisfy

$$c_{n,m} = \begin{cases} O\left(\frac{((2m)!\sqrt{(2m+1)}\ 2^{-k/2})}{(m!)^2}\right), & if \;\; n = n_0, \\ 0, & n \neq n_0, \end{cases}$$

4.2 Theorem

Let the solution function f of Lane-Emden differential equation be a uniformly continuous defined in $[0, 1)$ such that

$$|f(t_1) - f(t_2)| \leq |t_1 - t_2|^{\alpha} \frac{1}{2^{m-1}m^{\frac{3}{2}}}, \;\; \forall t_1, t_2 \;\; \in [0, 1), m \geq 1 \tag{14}$$

and its Boubaker wavelet series

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi^{(B)}_{n,m}(t) \tag{15}$$

having $(2^k, M)^{th}$ partial sums

$$(S_{2^k,M}f)(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m}\psi^{(B)}_{n,m}(t) \tag{16}$$

then Boubaker wavelet approximation $E_{2^k,M}(f)$ satisfies

$$E_{2^k,M}(f) = \min||f - (S_{2^k,M}f)||_2 \quad = O\left(\frac{1}{2^{k\alpha}\sqrt{M}}\right).$$

Proof of theorem 4.1 For

$$f = \chi_{\left[\frac{n_0}{2^k}, \frac{n_0+1}{2^k}\right)},$$

and

$$\begin{aligned}
c_{n_0,m} &= < f(t), \psi^{(B)}_{n_0,m}(t) > \\
&= \int_{\frac{n_0}{2^k}}^{\frac{n_0+1}{2^k}} f(t)\psi^{(B)}_{n_0,m}(t)dt \\
&= \int_{\frac{n_0}{2^k}}^{\frac{n_0+1}{2^k}} \chi_{\left[\frac{n_0}{2^k}, \frac{n_0+1}{2^k}\right)}(t)\sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}B_m(2^kt - n_0)\,dt \\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}\int_{\frac{n_0}{2^k}}^{\frac{n_0+1}{2^k}} B_m(2^kt - n_0)\,dt \\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}\int_0^1 B_m(v)\frac{dv}{2^k}, \quad 2^kt - n_0 = v \\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{\frac{k}{2}}}\int_0^1 B_m(v)\,dv \\
|c_{n_0,m}| &\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{-\frac{k}{2}}\int_0^1 |B_m(v)|\,dv \\
&\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{-\frac{k}{2}}, \quad \int_0^1 |B_m(v)|dv \leq 1. \qquad (17)
\end{aligned}$$

Then

$$c_{n,m} = \begin{cases} O\left(\frac{((2m)!\sqrt{(2m+1)}\ 2^{-k/2})}{(m!)^2}\right), & if \ \ n = n_0, \\ 0, & n \neq n_0, \end{cases}$$

Thus, theorem 4.1 is completely established.

Proof of theorem 4.2

$$\begin{aligned}
f(t)-(S_{2^k,M}f)(t) &= \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t)-\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&= \sum_{n=0}^{2^k-1}(\sum_{m=0}^{M-1}+\sum_{m=M}^{\infty})c_{n,m}\psi_{n,m}^{(B)}(t)-\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&= \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)+\sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&- \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(B)}(t)\\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t).
\end{aligned}$$

Next

$$\begin{aligned}
c_{n,m} &= \ < f(t),\psi_{n,m}^{(B)}(t) >\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(t)\psi_{n,m}^{(B)}(t)dt\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \{f(t)-f(\frac{n}{2^k})\}\psi_{n,m}^{(B)}(t)dt+\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(\frac{n}{2^k})\psi_{n,m}^{(B)}(t)dt\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \{f(t)-f(\frac{n}{2^k})\}\psi_{n,m}^{(B)}(t)dt+f(\frac{n}{2^k})\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \psi_{n,m}^{(B)}(t)dt\\
&= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \{f(t)-f(\frac{n}{2^k})\}\psi_{n,m}^{(B)}(t)dt\ ,\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} \psi_{n,m}^{(B)}(t)dt=0, m\geq 1.
\end{aligned}$$

Then

$$\begin{aligned}
|c_{n,m}| &\leq \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} |f(t)-f(\frac{n}{2^k})|\ |\psi_{n,m}^{(B)}(t)|dt\\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}2^{\frac{k}{2}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} |f(t)-f(\frac{n}{2^k})|\ |B_m(2^kt-n)|dt\\
&= \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{\frac{k}{2}}}\int_0^1 |f(\frac{u+n}{2^k})-f(\frac{n}{2^k})||B_m(u)|du, 2^kt-n=u\\
&\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{\frac{k}{2}}}\frac{1}{2^{k\alpha}}\frac{1}{2^{(m-1)}m^{\frac{3}{2}}}\int_0^1 |B_m(u)|du
\end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{(2m+1)}\frac{(2m)!}{(m!)^2}\frac{1}{2^{k\alpha+\frac{k}{2}}}\frac{1}{2^{(m-1)}m^{\frac{3}{2}}}, \int_0^1 |B_m(u)|du \leq 1 \\
&\leq \frac{1}{2^{k(\alpha+\frac{1}{2})}m}, \qquad \frac{(m!)^2}{(2m)!} \leqslant \frac{1}{2^{m-1}}. \qquad (18)
\end{aligned}
$$

Next,

$$
\begin{aligned}
||f-(S_{2^k,M}f)||_2^2 &= \int_0^1 |f(t)-(S_{2^k,M}f)(t)|^2 dt \\
&= \int_0^1 \left(\sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(B)}(t)\right)^2 dt \\
&= \int_0^1 \left(\sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}^2(\psi_{n,m}^{(B)}(t))^2\right. \\
&\quad \left. + \sum\sum_{0\leqslant\ n\neq n'\leq\ 2^k-1}\ \sum\sum_{M\leqslant\ m\neq m'\leq\infty} c_{n,m}c_{n',m'}\psi_{n,m}^{(B)}(t)\psi_{n',m'}^{(B)}(t)\right) dt \\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} c_{n,m}^2 \int_0^1 (\psi_{n,m}^{(B)}(t))^2\, dt \\
&\quad + \sum\sum_{0\leqslant\ n\neq n'\leq\ 2^k-1}\ \sum\sum_{M\leqslant\ m\neq m'\leq\infty} c_{n,m}c_{n',m'} \int_0^1 \psi_{n,m}^{(B)}(t)\psi_{n',m'}^{(B)}(t)\, dt \\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} |c_{n,m}|^2 ||\psi_{n,m}^{(B)}(t)||_2^2 \\
&= \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} |c_{n,m}|^2\ ,\ \{\psi_{n,m}\}_{n,m\in\mathbb{Z}} \text{ being orthonormal in [0,1)} \\
&\leq \sum_{n=0}^{2^k-1}\sum_{m=M}^{\infty} \frac{1}{2^{k(2\alpha+1)}m^2},\ \text{ by eqn (18).} \\
&= \frac{2^k}{2^{k(2\alpha+1)}}\sum_{m=M}^{\infty}\frac{1}{m^2} \\
&\leq \frac{2^k}{2^{k(2\alpha+1)}}\left(\frac{1}{M^2}+\int_M^{\infty}\frac{dm}{m^2}\right), \text{by Cauchy's intergal test} \\
&= \frac{1}{2^{2k\alpha}}\left(\frac{1}{M^2}+\frac{1}{M}\right) \\
&\leq \frac{2}{2^{2k\alpha}M}
\end{aligned}
$$

$$
\text{Then } E_{2^k,M}(f) = \min||f-(S_{2^k,M}f)||_2 \leqslant \frac{\sqrt{2}}{2^{k\alpha}\sqrt{M}} = O\left(\frac{1}{2^{k\alpha}\sqrt{M}}\right)
$$

Thus, theorem 4.2 is completely established.

5 Moduli of continuity

The moduli of continuity of $\big(f - (S_{2^k,M}f)\big)$ have been determined in this section as follows :

5.1 Theorem

If the solution function f of Lane-Emden differential equation satisfies eqns (14), (15) & (16), then moduli of continuity of $\big(f - (S_{2^k,M}f)\big)$ is given by

$$\begin{aligned} W\bigg(\big(f - (S_{2^k,M}f)\big), \frac{1}{2^k}\bigg) &= \sup_{0 \leq h \leq \frac{1}{2^k}} ||\big(f - (S_{2^k,M}f)\big)(\cdot + h) - \big(f - (S_{2^k,M}f)\big)(\cdot)||_2 \\ &= O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg) \end{aligned}$$

Proof of theorem (5.1)
Following the proof of theorem (4.2) ,

$$||f - (S_{2^k,M}f)||_2 \quad = O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg).$$

Then

$$\begin{aligned} W\bigg(\big(f - (S_{2^k,M}f)\big), \frac{1}{2^k}\bigg) &= \sup_{0 \leq h \leq \frac{1}{2^k}} ||\big(f - (S_{2^k,M}f)\big)(t + h) - \big(f - (S_{2^k,M}f)\big)(t)||_2 \\ &\leq ||\big(f - (S_{2^k,M}f)\big)||_2 + ||\big(f - (S_{2^k,M}f)\big)||_2 \\ &= 2||\big(f - (S_{2^k,M}f)\big)||_2 \\ &= 2.O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg). \\ W\bigg(\big(f - (S_{2^k,M}f)\big), \frac{1}{2^k}\bigg) &= O\bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\bigg) \end{aligned}$$

6 Boubaker wavelet method for solution of differential equations

In this Section, the solution of Lane-Emden differential equations are obtained by applying Boubaker wavelet collocation method.

Consider the Lane-Emden differential of the form

$$f^{''}(t) + \frac{\alpha}{t} f^{'}(t) + f(t) = h(t), \text{ where } t \in [0,1) \quad (\text{Wazwaz})[7]). \tag{19}$$

$$f(0) = a, f^{'}(0) = b \tag{20}$$

The solution of any differential equation can be expanded as Boubaker wavelet series as follows

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}^{(B)}(t)$$

Now $f(t)$ is approximated by truncated series

$$(S_{2^k,M} f)(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(t) \tag{21}$$

then the following residual is obtained by substituting $(S_{2^k,M} f)$ from eqn(21) into eqn(19)

$$\begin{aligned} R(t) &= t \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{''(B)}(t) + \alpha \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{'(B)}(t) + t \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(t) \\ &- t \times h(t). \end{aligned}$$

The collocation method yields

$$R(t_i) = 0, \ i = 1, 2, 3, ..., 2^k M - 2.$$

Moreover using the intial conditions eqn (20),

$$\sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{(B)}(0) = a, \quad \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{'(B)}(0) = b. \tag{22}$$

Hence $2^k M$ system of equations are derived in the unknown coefficients $c_{n,m}$ which can be computed. This procedure is applied for differential equations of heigher order.

7 Illustrated Examples

In this Section, three Lane-Emden differential equations have been solved by using the procedure discussed in previous section-6. Illustrated examples are as follows:

Example (1)

Consider the following Lane-Emden differential equation

$$f^{''}(t)+\frac{2}{t}f^{'}(t)+f(t)=1+12t+t^3,\ f(0)=1,\ f^{'}(0)=0,\ 0\leq t<1 \quad (23)$$

The exact solution of eqn (23) is $f(t)=t^3+1$.
Now the differential equation has been solved by applying the procedure described in Section-6, using Boubaker wavelet method by taking $M=5, k=0$. Consider

$$\begin{aligned} f(t) &= \sum_{m=0}^{4} c_{0,m}\psi_{0,m}^{(B)}(t) \\ &= c_{0,0}\psi_{0,0}^{(B)}+c_{0,1}\psi_{0,1}^{(B)}+c_{0,2}\psi_{0,2}^{(B)}c_{0,3}\psi_{0,3}^{(B)}+c_{0,4}\psi_{0,4}^{(B)} \quad (24)\\ f(t) &= c_{0,0}+c_{0,1}\sqrt{3}(2t-1)+c_{0,2}\sqrt{5}(6t^2-6t+1) \\ &+ c_{0,3}\sqrt{7}(20t^3-30t^2+12t-1) \\ &+ c_{0,4}3(70t^4-140t^3+90t^2-20t+1) \quad (25) \end{aligned}$$

Differentiate eqn (25) with respect to t,

$$\begin{aligned} f^{'}(t) &= c_{0,1}(2\sqrt{3})+c_{0,2}\sqrt{5}(12t-6)+c_{0,3}\sqrt{7}(60t^2-60t+12) \\ &+ c_{0,4}3(280t^3-420t^3+180t-20) \quad (26) \end{aligned}$$

$$f^{''}(t)=c_{0,2}(12\sqrt{5})+c_{0,3}\sqrt{7}(120t-60)+c_{0,4}3(840t^2-840t+180)$$

Substitute these values of $f(t), f^{'}(t)$ and $f^{''}(t)$ in given differential eqn (23)

$$\begin{aligned} &c_{0,0}+c_{0,1}\big[\frac{4\sqrt{3}}{t}+\sqrt{3}(2t-1)\big]+c_{0,2}\big[12\sqrt{5}+\frac{2\sqrt{5}}{t}(12t-6) \\ &+\sqrt{5}(6t^2-6t+1)\big]+c_{0,3}\big[\sqrt{7}(120t-60)+\frac{2\sqrt{7}}{t}(60t^2-60t+12) \\ &+\sqrt{7}(20t^3-30t^2+12t-1)\big]+c_{0,4}\big[3(840t^2-840t+180) \\ &+\frac{6}{t}(280t^3-420t^2+180t-20)+3(70t^4-140t^3+90t^2-20t)+1\big] \\ &=1+12t+t^3 \quad (27) \end{aligned}$$

Using intial condition, $f(0)=1$ and $f^{'}(0)=0$ in eqns (25) and (26)

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}+3c_{0,4}=1 \quad (28)$$

$$2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}+12\sqrt{7}c_{0,3}-60c_{0,4}=0 \quad (29)$$

Now collocate the eqn (27) at $t_1 = 0.5$,$t_2 = 0.7$ and $t_3 = 0.9$, which are obtained by $x_i = \frac{i-\frac{1}{2}}{2^k M} = \frac{i-\frac{1}{2}}{5}$,$i = 2, 4, 5$ respectively.
A system of three linear equations are derived.

$$c_{0,0} + 13.8564c_{0,1} + 25.7147c_{0,2} - 31.7490c_{0,3} - 88.875c_{0,4} = 7.125 \quad (30)$$

$$c_{0,0} + 10.5902c_{0,1} + 41.5844c_{0,2} + 57.79832c_{0,3} - 21.7675c_{0,4} = 9.743 \quad (31)$$

$$c_{0,0} + 9.0836c_{0,1} + 51.7127c_{0,2} + 166.0120c_{0,3} + 351.9676c_{0,4} = 12.529 \quad (32)$$

Solving these eqns (30),(31) and (32) with (28) and (29)
$c_{0,0} = 1.2499999999$, $c_{0,1} = 0.2598076211$, $c_{0,2} = 0.1118033988$
$c_{0,3} = 0.0188982236$, $c_{0,4} = -0.0000000000$.
Substitute all these values of $c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}$ in eqn (25)

$$\begin{aligned} f(t) &= 1.2499999999 + 0.2598076211\sqrt{3}(2t-1) \\ &+ 0.1118033988\sqrt{5}(6t^2 - 6t + 1) \\ &+ 0.0188982236\sqrt{7}(20t^3 - 30t^2 + 12t - 1) \\ &- 0.0000000000(70t^4 - 140t^3 + 90t^2 - 20t + 1) \end{aligned} \quad (33)$$

Comparison of exact and Boubaker wavelet solutions are given in table (1) for $k = 0, M = 5$.

Table (1)

t	Exact solution	Approximate solution	Absolute error($\times 10^{-15}$)
0.1	1.001000000000000	1.001000000000000	0
0.2	1.008000000000000	1.0080000000000000	0
0.3	1.027000000000000	1.0270000000000000	0
0.4	1.064000000000000	1.0640000000000000	0
0.5	1.125000000000000	1.1250000000000000	0
0.6	1.216000000000000	1.2160000000000000	0
0.7	1.343000000000000	1.3430000000000000	0
0.8	1.512000000000000	1.5120000000000000	0
0.9	1.729000000000000	1.7290000000000000	0.222044604925031

Table(1):Comparison table of exact and Boubaker wavelet solutions.

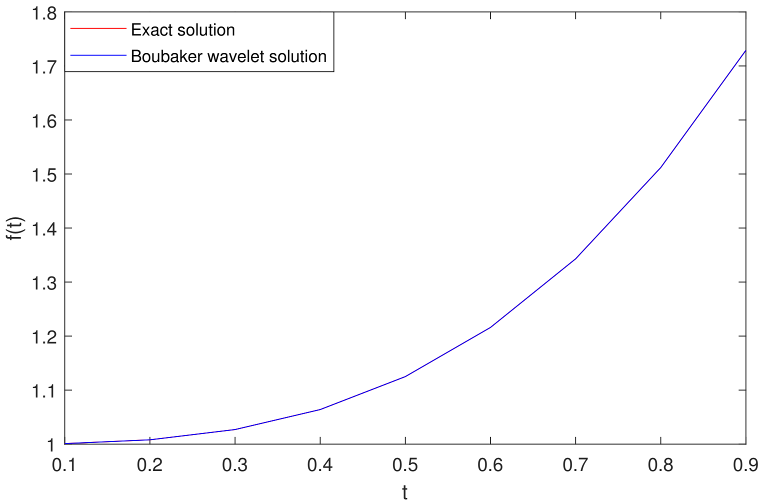

Fig.(1):The graphs of Boubaker wavelet and Exact solutions.

Example (2)

Consider the following Lane-Emden differential equation

$$f^{''}(t)+\frac{2}{t}f^{'}(t)+f(t)=6+12t+t^2+t^3,\ y(0)=0,\ y^{'}(0)=0,\ 0\leq t<1 \quad (34)$$

The exact solution of eqn (34) is $f(t)=t^3+t^2$.
Following the procedure adopted in Example (1);

$$\begin{aligned}
f(t) &= \sum_{m=0}^{4} c_{0,m}\psi_{0,m}^{(B)}(t)\\
&= c_{0,0}\psi_{0,0}^{(B)}+c_{0,1}\psi_{0,1}^{(B)}+c_{0,2}\psi_{0,2}^{(B)}+c_{0,3}\psi_{0,3}^{(B)}+c_{0,4}\psi_{0,4}^{(B)}\\
f(t) &= c_{0,0}+c_{0,1}\sqrt{3}(2t-1)+c_{0,2}\sqrt{5}(6t^2-6t+1)\\
&+ c_{0,3}\sqrt{7}(20t^3-30t^2 12t-1)\\
&+ c_{0,4}3(70t^4-140t^3+90t^2-20t+1) \qquad (35)\\
f^{'}(t) &= c_{0,1}(2\sqrt{3})+c_{0,2}\sqrt{5}(12t-6)+c_{0,3}\sqrt{7}(60t^2-60t+12)\\
&+ c_{0,4}3(280t^3-420t^3+180t-20) \qquad (36)\\
f^{''}(t) &= c_{0,2}(12\sqrt{5})+c_{0,3}\sqrt{7}(120t-60)+c_{0,4}3(840t^2-840t+180)
\end{aligned}$$

Substitute these values of $f(t), f^{'}(t)$ and $f^{''}(t)$ in given differential eqn (34)

$$c_{0,0}+c_{0,1}\big[\frac{4\sqrt{3}}{t}+\sqrt{3}(2t-1)\big]+c_{0,2}\big[12\sqrt{5}+\frac{2\sqrt{5}}{t}(12t-6)$$
$$\sqrt{5}(6t^2-6t+1)\big]+c_{0,3}\big[\sqrt{7}(120t-60)+\frac{2\sqrt{7}}{t}(60t^2-60t+12)$$
$$\sqrt{7}(20t^3-30t^2+12t-1)\big]+c_{0,4}\big[3(840t^2-840t+180)$$
$$+\frac{6}{t}(280t^3-420t^2+180t-20)+3(70t^4-140t^3+90t^2-20t)+1\big]$$
$$=6+12t+t^2+t^3 \qquad (37)$$

Using intial condition, $f(0)=0$ and $f^{'}(0)=0$ in eqns (35) and (36)

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}+3c_{0,4}=0 \qquad (38)$$

$$2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}+12\sqrt{7}c_{0,3}-60c_{0,4}=0 \qquad (39)$$

Now collocating the equations (37) at $t_1=0.5$,$t_2=0.7$ and $t_3=0.9$,

$$c_{0,0}+13.856405c_{0,1}+25.71478c_{0,2}-31.74901c_{0,3}-88.875c_{0,4}=12.375 \qquad (40)$$

$$c_{0,0}+10.59025c_{0,1}+41.58447c_{0,2}+57.79832c_{0,3}-21.76757c_{0,4}=15.233 \qquad (41)$$

$$c_{0,0}+9.08364c_{0,1}+51.71279c_{0,2}+166.01207c_{0,3}+351.96766c_{0,4}=18.339 \qquad (42)$$

Solving these eqns (40),(41) and (42) with (38) and (39) ,
$c_{0,0}=0.5833333333$, $c_{0,1}=0.5484827557$, $c_{0,2}=0.1863389981$
$c_{0,3}=0.0188982236$, $c_{0,4}=-0.0000000000$.
Substitute all these values of $c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4}$ in eqn (35).

$$\begin{aligned} f(t) &= 0.5833333333+0.5484827557\sqrt{3}(2t-1)\\ &+ 0.1863389981\sqrt{5}(6t^2-6t+1)\\ &+ 0.0188982236\sqrt{7}(20t^3-30t^2+12t-1)\\ &- 0.0000000000(70t^4-140t^3+90t^2-20t+1) \end{aligned} \qquad (43)$$

Comparison of exact and Boubaker wavelet solutions are given in table (2) for $k=0, M=5$.

Table (2)

t	Exact solution	Approximate solution	Absolute error ($\times 10^{-15}$)
0.1	0.011000000000000	0.011000000000000	0.017347234759768
0.2	0.048000000000000	0.048000000000000	0.020816681711722
0.3	0.117000000000000	0.117000000000000	0.027755575615629
0.4	0.224000000000000	0.224000000000000	0
0.5	0.375000000000000	0.375000000000000	0
0.6	0.576000000000000	0.576000000000000	0.222044604925031
0.7	0.833000000000000	0.833000000000000	0.111022302462516
0.8	1.152000000000000	1.152000000000000	-0.222044604925031
0.9	1.539000000000000	1.539000000000000	-0.222044604925031

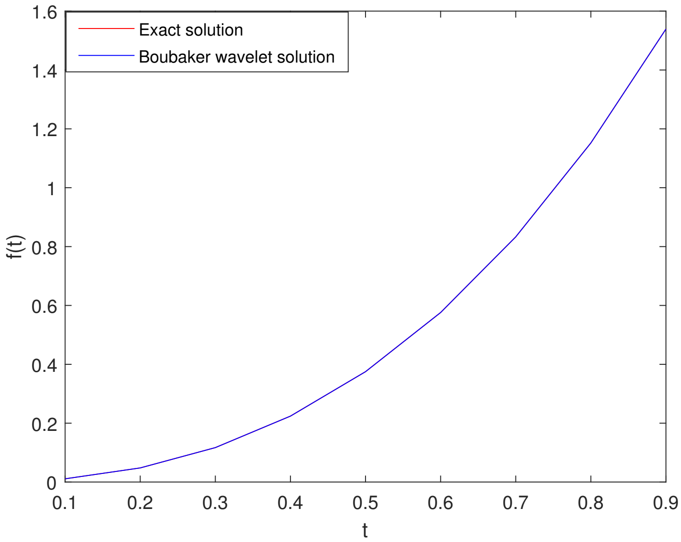

Fig.(2):The graphs of Boubaker wavelet and Exact solutions.

Example 3

Consider the following Lane-Emden differential equation

$$f^{''}(t) + \frac{2}{t}f^{'}(t) = 2(2t^2 + 3)f(t),\ f(0) = 1,\ f^{'}(0) = 0,\ 0 \le t < 1 \tag{44}$$

The exact solution of eqn (44) is $f(t) = e^{t^2}$.
Following the procedure adopted in Example(1);

$$\begin{aligned} f(t) &= \sum_{m=0}^{4} c_{0,m}\psi_{0,m}^{(B)}(t) \\ &= c_{0,0}\psi_{0,0}^{(B)} + c_{0,1}\psi_{0,1}^{(B)} + c_{0,2}\psi_{0,2}^{(B)} + c_{0,3}\psi_{0,3}^{(B)} + c_{0,4}\psi_{0,4}^{(B)} \end{aligned}$$

$$\begin{aligned} f(t) &= c_{0,0} + c_{0,1}\sqrt{3}(2t-1) + c_{0,2}\sqrt{5}(6t^2 - 6t + 1) \\ &+ c_{0,3}\sqrt{7}(20t^3 - 30t^2 + 12t - 1) \\ &+ c_{0,4}3(70t^4 - 140t^3 + 90t^2 - 20t + 1) \qquad (45) \\ f^{'}(t) &= c_{0,1}(2\sqrt{3}) + c_{0,2}\sqrt{5}(12t - 6) + c_{0,3}\sqrt{7}(60t^2 - 60t + 12) \\ &+ c_{0,4}3(280t^3 - 420t^3 + 180t - 20) \qquad (46) \\ f^{''}(t) &= c_{0,2}(12\sqrt{5}) + c_{0,3}\sqrt{7}(120t - 60) + c_{0,4}3(840t^2 - 840t + 180) \end{aligned}$$

Substitute these values of $f(t)$, $f^{'}(t)$ and $f^{''}(t)$ in given differential eqn (44)

$$\begin{aligned}
&-2(2t^2+3)c_{0,0}+c_{0,1}\big[\frac{4\sqrt{3}}{t}-2(2t^2+3)\sqrt{3}(2t-1)\big]\\
&+c_{0,2}\big[12\sqrt{5}+\frac{2\sqrt{5}}{t}(12t-6)-2(2t^2+3)\sqrt{5}(6t^2-6t+1)\big]\\
&+c_{0,3}\big[\sqrt{7}(120t-60)+\frac{2\sqrt{7}}{t}(60t^2-60t+12)\\
&+2(2t^2+3)\sqrt{7}(20t^3-30t^2+12t-1)\big]\\
&+c_{0,4}\big[3(840t^2-840t+180)+\frac{6}{t}(280t^3-420t^2+180t-20)\\
&-6(2t^2+3)(70t^4-140t^3+90t^2-20t)+1\big]=0 \qquad (47)
\end{aligned}$$

Using intial condition, $f(0)=1$ and $f^{'}(0)=0$ in eqns (45) and (46)

$$c_{0,0}-\sqrt{3}c_{0,1}+\sqrt{5}c_{0,2}-\sqrt{7}c_{0,3}+3c_{0,4}=1 \qquad (48)$$

$$2\sqrt{3}c_{0,1}-6\sqrt{5}c_{0,2}+12\sqrt{7}c_{0,3}-60c_{0,4}=0 \qquad (49)$$

Now collocating the equation (47) at $t_1=0.5$,$t_2=0.7$ and $t_3=0.9$,

$$-7c_{0,0}+13.85640c_{0,1}+34.65905c_{0,2}-31.74901c_{0,3}-97.8750c_{0,4}=0 \qquad (50)$$

$$-7.96c_{0,0}+4.38258c_{0,1}+46.79361c_{0,2}+68.22893c_{0,3}-18.73013c_{0,4}=0 \qquad (51)$$

$$-9.24c_{0,0}-5.10531c_{0,1}+41.18002c_{0,2}+163.84467+02c_{0,3}+359.12542c_{0,4}=0 \qquad (52)$$

Solving these eqns (50), (51) and (52) with (48) and (49),
$c_{0,0}=1.38821917722$, $c_{0,1}=0.388372764863$, $c_{0,2}=0.158389887721$
$c_{0,3}=0.02903271870$, $c_{0,4}=0.002368327161$
Substitute all these values of $c_{0,0},c_{0,1},c_{0,2},c_{0,3},c_{0,4}$ in eqn (45)

$$\begin{aligned}
f(t) &= 1.3882191+0.38837276\sqrt{3}(2t-1)+0.1583898\sqrt{5}(6t^2-6t+1)\\
&+ 0.02903271870\sqrt{7}(20t^3-30t^2+12t-1)\\
&+ 0.002368327161(70t^4-140t^3+90t^2-20t+1) \qquad (53)
\end{aligned}$$

Comparison of exact and Boubaker wavelet solutions are given in table (3) for $k=0,M=5$

Table (3)

t	Exact solution	Approximate solution	Absolute error
0.1	1.010050167084168	1.005192015148452	0.004858151935716
0.2	1.040810774192388	1.023531157694065	0.017279616498324
0.3	1.094174283705210	1.060057300954230	0.034116982750980
0.4	1.173510870991810	1.121003955135684	0.052506915856126
0.5	1.284025416687741	1.213798267334502	0.070227149353240
0.6	1.433329414560340	1.347061021536102	0.086268393024238
0.7	1.632316219955379	1.530606638615246	0.101709581340133
0.8	1.896480879304952	1.775443176336035	0.121037702968916
0.9	2.247907986676472	2.093772329351916	0.154135657324556

Table(3):Comparision table of exact and Boubaker wavelet solutions.
Comparison of exact and Boubaker wavelet solutions are given in table (4) for $k = 0, M = 6$.

Table (4)

t	Exact solution	Approximate solution	Absolute error
0.1	1.010050167084168	1.000547638755526	0.009502528328642
0.2	1.040810774192388	1.011676778238292	0.029133995954096
0.3	1.094174283705210	1.043773805588921	0.050400478116290
0.4	1.173510870991810	1.103730511655493	0.069780359336317
0.5	1.284025416687741	1.196980758433975	0.087044658253766
0.6	1.433329414560340	1.329537146508637	0.103792268051703
0.7	1.632316219955379	1.510027682492480	0.122288537462899
0.8	1.896480879304952	1.751732446467655	0.144748432837297
0.9	2.247907986676472	2.074620259425891	0.173287727250581

Table(4):Comparison table of exact and Boubaker wavelet solutions.

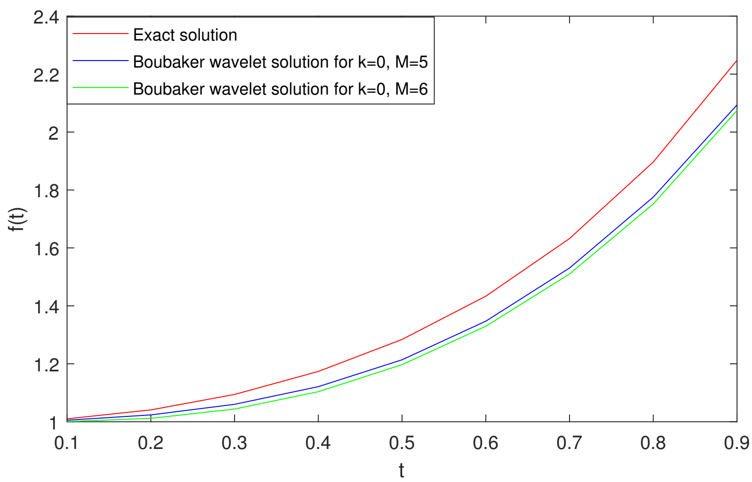

Fig.(3):The graphs of Boubaker wavelet and Exact solutions.

8 Discussion and Conclusion

1. Boubaker wavelets have applications in approximation theory, moduli of continuity and solution of Lane-Emden differential equations. The Boubaker wavelet expansion of solution of Lane-Emden Differential equation is verified and its convergence analysis has been studied. The estimator $E_{2^k,M}(f)$ of $\big(f - S_{2^k,M}(f)\big)$ has been developed. Furthermore, moduli of continuity

$$W\big((f - (S_{2^k,M}f)), \frac{1}{2^k}\big) \quad \leq \quad 2\, E_{2^k,M}(f)$$

This shows that moduli of continuity $W\Bigg(\big(f - (S_{2^k,M}f)\big), \frac{1}{2^k}\Bigg)$ is sharper and better than the approximation $E_{2^k,M}(f)$ of $\big(f - S_{2^k,M}(f)\big)$. Hence the moduli of continuity $W\Bigg(\big(f - (S_{2^k,M}f)\big), \frac{1}{2^k}\Bigg)$ has been also estimated in this research paper.

2. A method has been proposed to solve Lane-Emden differential equation by Boubaker wavelet collocation method. To illustrate the effectiveness and accuracy of the proposed method, three Lane-Emden differential equations have been solved by proposed method, It is observed that the exact solutions of considered differential equations are atmost same to their solutions obtained by proposed method. This is a significant achivement of the research paper in wavelet analysis.

3. Our results are concerned with Boubaker wavelet estimator $E_{2^k,M}(f)$, moduli of continuity $W\Bigg(\big(f - (S_{2^k,M}f)\big), \frac{1}{2^k}\Bigg)$ and the solutions of Lane-Emden diffential equations by this method.

4. (i) By theorem 4.2,

$$E_{2^k,M}(f) = O\Bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\Bigg) \to 0 \text{ as } k \to \infty, M \to \infty.$$

.

(ii) As per theorem 5.1,

$$W\Bigg(\big(f - S_{2^k,M}(f)\big), \frac{1}{2^k}\Bigg) = O\Bigg(\frac{1}{2^{k\alpha}\sqrt{M}}\Bigg) \to 0 \text{ as } k \to \infty, M \to \infty.$$

Thus $E_{2^k,M}(f)$ and $W\Bigg(\big(f - S_{2^k,M}(f)\big), \frac{1}{2^k}\Bigg)$ are best possible estimation in wavelet analysis

5. Solution of Lane-Emden differential equation by Boubaker wavelet series by collocation method is approximately same as exact solution of Lane-Emden differential equation. Only a few number of Boubaker wavelet basis is needed to achieve the heigh accuracy. This is significant achivement in wavelet analysis.

6. Limitations and possible future development:

(i) A non-linear Lane-Emden equation can not solved by Boubaker wavelets without using collocation method

(ii) In general, Boubaker wavelets in one variable are ineffective to solve a problem expressed in partial differential equations of two or more variables.

(iii) It is known that $H^\alpha[0,1) \nsubseteq H_2^\alpha[0,1)$.To find the approximate solution of Lane-Emden differential equation in class $H_2^\alpha[0,1)$.

(iv) To define two dimensional Boubaker wavelets and to find the solution of the partial differential equation by this method.

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