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HYPERCOMPOSITIONAL STRUCTURES FROM THE COMPUTER THEORY

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Abstract This paper presents the several types of hypercompositional structures that have been introduced and used for the approach and solution of problems in the theory of languages and automata.

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Certain properties of the Automata, as well as some essential elements of the structure of the formal languages gave the initiative of introducing the theory of hypercompositional structures into the above theories [5]. This paper will present the structures that have been used for this purpose along with some of the characteristic properties that they have. The use of those structures can be found in the papers that appear as references in the following text.

In [5], it has been proved that the set of the words A^* over an alphabet A, can be organized into a join hypergroup, which was named **B-hypergroup**, through the introduction of the hypercomposition:

$$a+b = \{a, b\} \text{ for every } a, b \in A^*$$

It is worth mentioning that this hypercomposition can be found in a paper by L. Konguetsof, written as early as the 60's. We have introduced again this hypercomposition though, motivated by the theory of Languages and we have named it **B-hypercomposition,** after the Binary result it gives. The deriving structures (**B-hypergroup, Dilated B-hypergroup, B-hyperringoid** etc.) which have already been studied in depth, have produced many and interesting results in both theories.

The join hypergroup is a commutative hypergroup (H,+), which also satisfies the axiom:

(J) $(a : b) \cap (c : d) \neq \varnothing \Rightarrow (a + d) \cap (c + b) \neq \varnothing$ for every a, b, c, d $\in$ H where $a : b = \{x \in H \mid a \in x+b\}$, is the induced hypercomposition from + [2]. We remind that the axiom (J) as well as the join space, i.e. a commutative hypergroup enriched with (J) and certain additional axioms, have been introduced by W. Prenowitz for the study of Geometry with methods and tools from the Hypercompositional Algebra (e.g. see [17]).

Moreover, the set of the words A^* is a semigroup with composition the concatenation of the words. It has been proved though that the concatenation is bilateral distributive to the B-hypercomposition [5]. Thus, there appeared a new hypercompositional structure, the **hyperringoid**.

Definition 1. A triplet (Y,+,.) is called **hyperringoid**, if

i. (Y,+) is a hypergroup

ii. (Y,.) is a semigroup

iii. the composition is bilateral distributive to the hypercomposition

If (Y,+) is a join hypergroup, then the hyperringoid is named **join**.

An important join hyperringoid for the theory of languages is the B-hyperringoid, in which the (Y,+) is a B-hypergroup.

The study of the theory of languages and automata through the theory of the hypercompositional structures has also led to the introduction of a new hypergroup, the **fortified join hypergroup (FJH).** This new hypergroup has been introduced in order to satisfy the need of considering a non scalar neutral element in the join hypergroup and therefore being able to describe the "null word", the use of the symbol <SOS> (Start Of String) in the realization of the automaton etc.

Definition 2. Fortified Join Hypergroup is a join hypergroup which also satisfies the axioms:

FJ_1 There exists a unique neutral element, denoted by 0, the zero element of H, such

that for every $x \in H$ holds: $x \in x+0$ and $0+0 = 0$

and

FJ_2 For every $x \in H \setminus \{0\}$, there exists one and only one element $x' \in H \setminus \{0\}$, the opposite

or symmetrical of x, denoted by -x, such that: $0 \in x+x'$ Also $-0 = 0$.

From the above axioms, it is obvious that the FJH places itself between the canonical [13] and the join hypergroup, since, as it is known, if a join hypergroup has a scalar neutral element, it is a canonical one [1], [3].

The relevant analysis of the properties of the FJHs [6] has revealed that they consist of two kinds of elements, the **canonical (c-elements)** and the **attractive** ones **(a-**

elements). This distinction appears due to the different behavior of the elements in their hypersum with the zero element. So, for the canonical elements x, x+0 equals to {x}, i.e. they act like the elements of the canonical hypergroup, while, for the attractive ones, x+0 = {x, 0} i.e. they attract the neutral element in the result of the hypersum x+0. Moreover, since 0+0 is not a biset, we have included 0 among the canonical elements. Furthermore the property -(x-x) = x-x is not always valid in a FJH. Thus, there derives another distinction of the elements, namely the **normal** ones, i.e. the elements that satisfy the above equality, and the **abnormal** ones, i.e. the elements that do not satisfy it. The following Proposition gives a list of the fundamental properties of the elements of this new hypergroup.

Proposition. *In a FJH the following are valid:*

1. *if x is a c-element, then -x is also a c-element.*
2. *if x is an a-element, then -x is also an a-element.*
3. *The sum of two a-elements consists only of a-elements (and the 0, if they are opposite) and also it always contains the two addends.*
4. *The sum of two non opposite c-elements consists of c-elements, while the sum of two opposite c-elements contains all the a-elements.*
5. *The sum of an a-element with a non zero c-element is the c-element.*
6. *All the c-elements are normal elements.*
7. *If y is normal, or if $x \notin y-y$, then $-(x : y) = (-x) : (-y)$*

For the relevant proofs of the above see [6].

Moreover, it has been proved that in the FJHs the reversibility holds under conditions. More precisely, $z \in x+y \Rightarrow y \in z-x$, except if $z = x \neq y$ where $x \in x+y \Rightarrow x \in x-y$, while generally $y \notin x-x$. This gives as a result that for every $x \neq y$ holds $x-y = (x:y) \cup (-y):(-x)$, while $x-x \subseteq x:x$. If one of the x, y is a c-element then $x-y = x:y = (-y):(-x)$ [6].

Apart from the different kinds of elements, as described above, the FJH has a variety of subhypergroups [11]. Initially, very significant in every join hypergroup are the intersections $x:y \cap z:w$ which appear in the first part of the join axiom. Thus, if h is a subhypergroup of a join hypergroup H and if x, y, z, w ∈ h, then the following can happen:

i> $[(x:y) \cap (z:w)] \cap (H \setminus h) \neq \varnothing$

ii> $[(x:y) \cap (z:w)] \subseteq h$

If <ii> is valid for every x, y, z, w ∈ h, then h is called **join subhypergroup** of H. Since it has been proved that a subhypergroup h of a commutative hypergroup H is closed in H if and only if $x:y \subseteq h$ for every x, y ∈ h [4], one can easily see that the join subhypergroups are the closed ones. Now if H is a FJH for which -x ∈ h for every x ∈ h, then h is called **symmetrical subhypergroup** of H. It can be proved that every join subhypergroup of a FJH is a symmetrical one.

A hyperringoid now, with additive hypergroup a FJH, is called **join hyperring.** The join hyperring has properties, some of which are very different from the properties of Krasner's hyperring [12], i.e. hyperringoid in which the additive hypergroup is a canonical one. So, for instance, in a join hyperring, the property $(-x)(-y) = xy$ is not always valid. In [10], example 3.1. one can see a join hyperring for which $(-x)(-y) = -xy$. Also, its canonical and attractive elements give special properties in the multiplication. Thus, while the product of two c-elements is always a c-element, the result of the product of a c-element with an a-element is always the zero element. Furthermore the product of two a-elements is also the zero element, if the join hyperring contains a nonzero c-element.

Another structure which has been used for the approach of the theory of automata through the theory of the hypercompositional structures is the **join polysymmetrical hypergroup.**

Definition 3. Join Polysymmetrical Hypergroup is a join hypergroup which also satisfies the axioms:

JP_1 There exists a unique neutral element, denoted by 0, the zero element of H, such
that for every $x \in H$ holds: $x \in x+0$ and $0+0 = 0$

and

JP_2 For every $x \in H \setminus \{0\}$, there exists at least one element $x' \in H \setminus \{0\}$, the opposite or
symmetrical of x, such that: $0 \in x+x'$ The set of the opposite elements of x is
denoted by S(x) and named **symmetrical** (set) of x. Also S(0) = 0.

Those hypergroups have appeared and used for the minimization of the automaton [5] and their study has revealed the significant properties that they have [6]. For instance join polysymmetrical hypergroups are the P-hypergroups i.e. hypergroups that are defined from an abelian group (G,+), a subset P which contains the neutral element of G and hypercomposition "$:^P$" defined as follows:

$$x :^P y = x+y+P, \text{ for every } x, y \in G$$

In addition to the above hypergroups that have been introduced from the study of the theory of automata and languages, notable in the above study, is the role of the canonical hypergroup, with its different types (superiorly canonical and strongly canonical [14], [15]) as well as the canonical polysymmetrical hypergroup [16]. Those hypergroups have been used in order to describe the structure of an automaton. More precisely, through the introduction of the notion of the **order** of a state there appear the **attached order hypergroups** of an automaton, which belong to he above categories [9].

Furthermore, in the study of the theory of languages and automata, except from the special types of hypergroups, we have also used the hypergroup itself. The introduction of the following hypercomposition into the set S of the states of an automaton, has made S a hypergroup:

$$s_1 + s_2 = \begin{cases} \{s \in S \mid s = s_1 w \text{ and } s_2 = sy, \text{ with } w,y \in A^*\}, & \text{if there exist } z \in A^*, \text{ such that } s_2 = s_1 z \\ \{s_1, s_2\}, & \text{if there does not exist } z \in A^*, \text{ such that } s_2 = s_1 z \end{cases}$$

This hypergroup has been named **attached hypergroup of the paths.** With a proper generalization of this hypergroup in the case of the operation of the automaton, we have obtained the **attached hypergroup of the operation.** Using this last hypergroup, an algorithm has been developed, which, among other information, gives all the possible states that an automaton can be found at any clock pulse, during its operation, as well as all the possible paths that it may have passed through up to any clock pulse [8].

Lastly, in an automaton, the word (of the language it accepts) causes the system to move from state to state. Therefore, this is an action from a set of operators, which is a subset of A^* on the set of the states of the automaton. This has led to the introduction of two more hypercompositional structures, the **hypermoduloid** and the **supermoduloid.**

Definition 4. If M is a hypergroup and Y is a hyperringoid of operators over M, such that for every $\kappa, \lambda \in Y$ and $s, t \in M$, the axioms:

i. $(s\kappa)\lambda = s(\kappa\lambda)$
ii. $(s+t)\lambda = s\lambda + t\lambda$
iii. $s(\lambda+\kappa) \subseteq s\lambda + s\kappa$

hold, then M is called **hypermoduloid** over Y. If Y is a set of hyperoperators, that is, if there exists an external hyperoperation from MxY to *P*(M) satisfying axiom <i>, then M is called **supermoduloid** over Y.

The hypermoduloids are being used in the study of the deterministic automata, while the supermoduloids are being used in the case of the non deterministic automata [7].

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