The sum of the series of reciprocals of the quadratic polynomial with different negative integer roots

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Abstract

This contribution, which is a follow-up to author's paper [1] and [2] dealing with the sums of the series of reciprocals of some quadratic polynomials, deals with the series of reciprocals of the quadratic polynomials with different negative integer roots. We derive the formula for the sum of this series and verify it by some examples evaluated using the basic programming language of the CAS Maple 16.

Keywords: sequence of partial sums, telescoping series, harmonic number, computer algebra system Maple.

1 Introduction and basic notions

Let us recall the basic terms, concepts and notions. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each n as the sum of the sequence $\{a_k\}$ from a_1 to a_n, i.e.

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a *sum s* and write $\sum_{k=1}^{\infty} a_k = s$.

The *telescoping series* is any series where nearly every term cancels with a preceding or following term, so its partial sums eventually only have a fixed number of terms after cancellation. Telescoping series are not very common in mathematics but are interesting to study. The method of changing series whose terms are rational functions into telescoping series is that of transforming the rational functions by the method of partial fractions.

For example, the series $\sum_{k=1}^{\infty} \dfrac{1}{k^2 + k}$ has the general nth term

$$a_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}.$$

After removing the fractions we get the equation $1 = A(n+1) + Bn$. Solving it for A and B we obtain $a_n = 1/n - 1/(n+1)$. After that we arrange the terms of the nth partial sum $s_n = a_1 + a_2 + \cdots + a_n$ in a form where can be seen what is cancelling. Then we find the limit of the sequence of the partial sums s_n in order to find the sum s of the infinite telescoping series as $s = \lim_{n\to\infty} s_n$. In our case we get

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

So we have $s = \lim_{n\to\infty} \left(1 - \dfrac{1}{n+1}\right) = 1$.

The *nth harmonic number* is the sum of the reciprocals of the first n natural numbers:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

The values of the sequence $\{H_n - \ln n\}$ decrease monotonically towards the limit $\gamma \doteq 0.57721566$, which is so-called the *Euler-Mascheroni constant*. Basic information about harmonic numbers can be found e.g. in the web-sites [3] or [4], interesting information are included e.g. in the paper [5]. First ten values of the harmonic numbers are presented in this table:

n	1	2	3	4	5	6	7	8	9	10
H_n	1	$\dfrac{3}{2}$	$\dfrac{11}{6}$	$\dfrac{25}{12}$	$\dfrac{137}{60}$	$\dfrac{49}{20}$	$\dfrac{363}{140}$	$\dfrac{761}{280}$	$\dfrac{7129}{2520}$	$\dfrac{7381}{2520}$

2 The sum of the series of reciprocals of the quadratic polynomial with different negative integer roots

Now, we deal with the series formed by reciprocals of the normalized quadratic polynomial $(k - a)(k - b)$, where $a < b < 0$ are integers. Let us consider the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)},$$

and determine its sum $s(a, b)$.

We express the nth term a_n of this series as the sum of two partial fractions

$$a_n = \frac{1}{(n-a)(n-b)} = \frac{A}{n-a} + \frac{B}{n-b}.$$

From the equality of two linear polynomials $1 = A(n - b) + B(n - a)$ for $n = a$ we get $A = 1/(a - b)$ and for $n = b$ we get $B = 1/(b - a) = -1/(a - b)$. So we have

$$a_n = \frac{1}{a-b}\left(\frac{1}{n-a} - \frac{1}{n-b}\right) = \frac{1}{b-a}\left(\frac{1}{n-b} - \frac{1}{n-a}\right). \tag{1}$$

For the nth partial sum of the given series so we get

$$s_n = \frac{1}{b-a}\left[\left(\frac{1}{1-b} - \frac{1}{1-a}\right) + \left(\frac{1}{2-b} - \frac{1}{2-a}\right) + \cdots \right.$$
$$\left. \cdots + \left(\frac{1}{n-1-b} - \frac{1}{n-1-a}\right) + \left(\frac{1}{n-b} - \frac{1}{n-a}\right)\right].$$

The first terms that cancel each other will be obviously the terms for which for the suitable index ℓ it holds $1/(1-a) = 1/(\ell-b)$. Therefore the last term from the beginning of the expression of the nth partial sum s_n, which will not cancel, will

Radovan Potůček

be the term $1/(-a)$, so that the first terms from the beginning of the expression the sum s_n, which will not cancel, will be the terms generating the sum

$$\frac{1}{1-b} + \frac{1}{2-b} + \cdots + \frac{1}{-a}.$$

Analogously, the last terms that cancel each other will be the terms for which for the suitable index m it holds $1/(n-b) = 1/(m-a)$. Therefore the first term from the ending of the expression of the nth partial sum s_n, which will not cancel, will be the term $1/(n+1-b)$, so that the last terms from the ending in the expression of the sum s_n, which will not cancel, will be the terms generating the sum

$$-\frac{1}{n+1-b} - \frac{1}{n+2-b} - \cdots - \frac{1}{n-a}.$$

After cancelling all the inside terms with the opposite signs we get the nth partial sum

$$s_n = \frac{1}{b-a}\left(\frac{1}{1-b} + \frac{1}{2-b} + \cdots + \frac{1}{-a} - \frac{1}{n+1-b} - \frac{1}{n+2-b} - \cdots - \frac{1}{n-a}\right).$$

Because for integer c it holds $\lim\limits_{n\to\infty} \dfrac{1}{n+c} = 0$, then the searched sum, where $a < b < 0$, is

$$s(a,b) = \lim_{n\to\infty} s_n = \frac{1}{b-a}\left(\frac{1}{1-b} + \frac{1}{2-b} + \cdots + \frac{1}{-a}\right) =$$
$$= \frac{1}{b-a}\left[\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{-a} - \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{-b}\right)\right],$$

so we get

Theorem 2.1. The series $\sum\limits_{k=1}^{\infty} \dfrac{1}{(k-a)(k-b)}$, where $a < b < 0$ are integers, has the sum

$$s(a,b) = \frac{1}{b-a}\big(H_{-a} - H_{-b}\big), \tag{2}$$

where H_n is the nth harmonic number.

Corolary 2.1. For the sum $s(a,b)$ above it obviously hold:

1. $s(a,b) = s(b,a)$,

2. $s(a, a+1) = H_{-a} - H_{-a-1} = \dfrac{1}{-a}$,

62

3. $s(a, a+i) = \frac{1}{i}\left(H_{-a} - H_{-a-i}\right) = \frac{1}{i}\left(\frac{1}{-a-i+1} + \frac{1}{-a-i+2} + \cdots + \frac{1}{-a}\right)$,

$i \in \mathbb{N}$.

Remark 2.1. *Let us note, that the formula* (2) *holds also in the case* $a < b = 0$. *Because* H_0 *is defined as* 0, *it has the form*

$$s(a, 0) = \frac{1}{0-a}\left(H_{-a} - H_0\right) = \frac{H_{-a}}{-a}. \tag{3}$$

Example 2.1. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{\left(k-(-5)\right)\left(k-(-2)\right)} = \sum_{k=1}^{\infty} \frac{1}{(k+2)(k+5)},$$

where $a = -5$, $b = -2$, *has the* n*th partial sum*

$$s_n = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{n+3} - \frac{1}{n+4} - \frac{1}{n+5}\right).$$

By the relation $s(-5, -2) = \lim\limits_{n\to\infty} s_n$, *since* $\lim\limits_{n\to\infty} \frac{1}{n+c} = 0$, *or by Theorem* 2.1 *we get its sum*

$$s(-5, -2) = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{1}{3}\left(H_5 - H_2\right) = \frac{1}{3}\left(\frac{137}{60} - \frac{3}{2}\right) = \frac{47}{180} = 0.26\overline{1}.$$

Example 2.2. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{\left(k-(-5)\right)k} = \sum_{k=1}^{\infty} \frac{1}{k(k+5)},$$

where $a = -5$, $b = 0$, *has the* n*th partial sum*

$$s_n = \frac{1}{5}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} - \frac{1}{n+5}\right).$$

By the relation $s(-5, 0) = \lim\limits_{n\to\infty} s_n$, *since* $\lim\limits_{n\to\infty} \frac{1}{n+c} = 0$, *or by Theorem* 2.1, *or by the Remark* 2.1 *we get its sum*

$$s(-5, 0) = \frac{1}{5}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{H_5}{5} = \frac{137/60}{5} = \frac{137}{300} = 0.45\overline{6}.$$

3 Numerical verification

We solve the problem to determine the values of the sum

$$s(a,b) = \sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)}$$

for $a = -1, -2, \ldots, -9$ and $b = a+1, a+2, \ldots, -8$. We use on the one hand an approximative direct evaluation of the sum

$$s(a,b,t) = \sum_{k=1}^{t} \frac{1}{(k-a)(k-b)},$$

where $t = 10^8$, using the basic programming language of the computer algebra system Maple 16, and on the other hand the formula (2) for evaluation the sum $s(a,b)$. We compare $45 = 9 + 8 + \cdots + 1$ pairs of these ways obtained sums $s(a,b,10^8)$ and $s(a,b)$ to verify the formula (2). We use following simple procedures hnum, rp2abneg and two **for** statements:

```
hnum:=proc(h)
    local i,s; s:=0;
    if h=0 then s:=0 else
    for i from 1 to h do
        s:=s+1/i;
    end do;
    end if;
end proc:

rp2abneg:=proc(a,b,n)
    local k,sab,sumab; sumab:=0;
    sab:=(hnum(-a)-hnum(-b))/(b-a);
    print("n=",n,"s(",a,b,")=",evalf[20](sab));
    for k from 1 to n do
        sumab:=sumab+1/((k-a)*(k-b));
    end do;
    print("sum(",a,b,")=",evalf[20](sumab),
        "diff=",evalf[20](abs(sumab-sab)));
end proc:

for i from -1 by -1 to -9 do
    for j from i+1 by -1 to -8 do
        rp2abneg(i,j,100000000);
    end do;
end do;
```

64

The approximative values of the sums $s(a, b)$ rounded to 3 decimals obtained by these procedures are written into the following table:

$s(a,b)$	$a=-1$	$a=-2$	$a=-3$	$a=-4$	$a=-5$	$a=-6$	$a=-7$	$a=-8$	$a=-9$
$b=0$	1.000	0.750	0.611	0.521	0.457	0.408	0.370	0.340	0.314
$b=-1$	×	0.500	0.417	0.361	0.321	0.290	0.266	0.245	0.229
$b=-2$	×	×	0.333	0.292	0.261	0.238	0.219	0.203	0.190
$b=-3$	×	×	×	0.250	0.225	0.206	0.190	0.177	0.166
$b=-4$	×	×	×	×	0.2000	0.183	0.170	0.159	0.149
$b=-5$	×	×	×	×	×	0.167	0.155	0.145	0.136
$b=-6$	×	×	×	×	×	×	0.143	0.134	0.126
$b=-7$	×	×	×	×	×	×	×	0.125	0.118
$b=-8$	×	×	×	×	×	×	×	×	0.111

Computation of 45 couples of the sums $s(a, b, 10^8)$ and $s(a, b)$ took over 18 minutes. The absolute errors, i.e. the differences $\left|s(a, b) - s(a, b, 10^8)\right|$, have here place value about 10^{-8}.

4   Conclusion

We dealt with the sum of the series of reciprocals of the quadratic polynomials with different negative integer roots a and b, i.e. with the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)},$$

where $a < b < 0$ are integers. We derived that the sum $s(a, b)$ of this series is given by the formula

$$s(a, b) = \frac{1}{b - a}\left(H_{-a} - H_{-b}\right),$$

where H_n is the nth harmonic number. We verified this result by computing 45 various sums by using the CAS Maple 16.

We also stated that this formula holds also for $a < b = 0$, when it has the form

$$s(a, 0) = \frac{1}{0 - a}\left(H_{-a} - H_0\right) = \frac{H_{-a}}{-a}.$$

The series of reciprocals of the quadratic polynomials with different negative integer roots so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a simple formula.

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