MODERATE-DENSITY BURST ERROR CORRECTING LINEAR CODES

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ABSTRACT

Lower and upper bounds for the existence of linear codes which correct burst of length b (fixed) and whose weight lies between certain limits have been presented.

Keywords : Error detecting codes, error correcting codes, burst errors, moderate-density burst, lower and upper bounds.

1. INTRODUCTION

It is well known that during the process of transmission errors occur predominantly in the form of a burst. However, it does not generally happen that all the digits inside any burst length get corrupted. Also when burst length is large then the actual number of errors inside the burst length is also not very less. Keeping this in view, we study codes which detect/correct moderate-density burst errors.

In the literature, various kinds of burst errors have been studied, viz. open loop bursts (c.f. Peterson and Weldon, Jr. (1972), p.109), closed-loop bursts [Campopiono, 1962], C.T. bursts [Chien and Tang, (1965)], low-density bursts [Dass, 1975], etc. One important kind of bursts errors which has not drawn much attention is burst of specified length (fixed) [Dass, 1982]. In this paper, we derive lower and upper bounds for linear codes that detect/correct Moderate-density bursts of length b (fixed) for some positive integer b.

In what follows we shall consider a linear code to be a subspace of n-tuples over GF(q). The weight of a vector shall be considered in the Hamming's sense [Hamming, 1950] and we shall mean by a burst of length b (fixed), is an n-tuple whose only nonzero components are confined to b consecutive positions, the first of which is nonzero and the number of its starting positions is the first (n-b+1) components.

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2. BOUNDS FOR CODES CORRECTING MODERATE-DENSITY BURSTS

In this section, we consider codes correcting moderate-density burst errors. We first obtain a lower bound on the number of check digits which is a necessary conditions for the existence of codes capable of correcting bursts of length b (fixed) with weight lying between w_1 and w_2 ($0 \le w_1 \le w_2 \le b$).

Before this we prove the following Lemma.

Lemma 1. If J (n, b, w_1, w_2) denotes the number of n-tuples over GF(q) which form bursts of length b (fixed) with weight lying w_1 and w_2 ($0 \le w_1 \le w_2 \le b$) then

$$J(n, b, w_1, w_2) = (n-b+1) \sum_{\substack{i=w_1-1 \\ w_1 \neq 0}}^{w_2-1} \binom{b-1}{i} (q-1)^{i+1} \qquad (1)$$

Proof. The Lemma follows immediately from the fact that the number of bursts of length b (fixed) with weight i is

$$\binom{b-1}{i-1} (q-1)^i (n-b+1). . \square$$

Theorem 1. The number of parity check symbols in an (n, k) linear code that corrects all bursts of length b (fixed) of weight lying between w_1 and w_2 ($0 \le w_1 \le w_2 \le b$) is at least.

$$\log_q [1 + J(n, b, w_1, w_2)]. \qquad (2)$$

Proof. Since the code has q^{n-k} cosets in all, and all the error patterns are to be in different cosets of the standard array, therefore, in view of Lemma 1, we must have

$$q^{n-k} \ge 1 + J(n, b, w_1, w_2). \qquad (3)$$

The result now follows by taking logarithm on both sides. □

Remarks. If we put $w_1 = 0$ and $w_2 = b$ in the above result then weight constraints imposed on the burst becomes redundant and we get

$$q^{n-k} \ge 1 + [(n-b+1)(q-1)]\, q^{b-1},$$

which gives the number of parity check digits in an (n, k) linear code over GF(q) that corrects all bursts of length b (fixed), a result due to Dass [1980].

Now, if we take, $w_1 = 0$ and $w_2 = w$ in (3) we get

$$q^{n-k} \ge 1 + (n-b+1)(q-1)\, [1+(q-1)]^{(b-1,\, w-1)},$$

which gives the number of check digits required for linear codes correcting bursts of length b (fixed) with weight w or less ($w \le b$) a result which is again due to Dass [1983].

Moreover, when $w_1 = w$ and $w_2 = b$ we obtain

$$q^{n-k} \ge 1 + (n-b+1) \sum_{\substack{i=w-1 \\ w \ne 0}}^{b-1} \binom{b-1}{i} (q-1)^{i+1},$$

which gives the number of check digits required in an (n, k) linear code that corrects all bursts of length b (fixed) with weight w or more ($w \le b$) which coincides with the result due to the authors [2000].

Now, we first obtain a sufficient condition giving an upper bound for the existence of a code capable of detecting moderate-density burst errors, and then in the theorem following this result we shall obtain an upper bound for codes correcting such errors.

Theorem 2. Given non-negative integers, w_1, w_2 and b such that $0 \le w_1 \le w_2 \le b$, a sufficient condition that there exists an (n, k) linear code that has no burst of length b (fixed) whose weight lies between w_1 and w_2, as a code word is

$$q^{n-k} > 1 + \sum_{\substack{i=w_1-1 \\ w_1 \ne 0}}^{w_2-1} \binom{b-1}{i} (q-1)^{i} . \qquad (4)$$

Proof. The existence of such a code will be proved by constructing a suitable $(n-k) \times n$ parity check matrix H for the desired code. For this we first construct an $(n-k) \times n$ matrix H' and then H will be obtained by reversing altogether the columns of H'.

We select any non-zero (n–k)-tuple as the first column of H'. Subsequent columns are added to H' in such a way that after having selected j–1 columns $h_1, h_2, \ldots, h_{j-1}$ suitably a nonzero (n–k)-tuple is chosen as the j-th column such that it is not a linear combination of any p columns ($w_1-1 \le p \le w_2-1$) from the immediately preceding b–1 columns $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}$. Such a condition will ensure that a burst of length b (fixed) with weight lying between w_1, w_2 cannot be a code word in the code whose parity-check matrix is H to be obtained from H' as prescribed earlier. In other words,

$$h_j \ne a_1 h_{j-b+1} + a_2 h_{j-b+2} + \ldots + a_{b-1} h_{j-1}, \qquad (5)$$

where number of nonzero a_i's lies between w_1-1 and w_2-2. Since $a_i \in GF(q)$, the possible number of linear combinations on the R.H.S. of (5) including the case when all the α_i's are zero is

$$1 + \sum_{\substack{i=w_1-1 \\ w_1 \ne 0}}^{w_2-1} \binom{b-1}{i} (q-1)^{i} .$$

Therefore, a column h_j can be added to H′ provided that this number is less than the total number of (n–k)-tuples.

At worst, all these linear combinations might yield a distinct sum, therefore, h_j can always be added to H′ provided that

$$q^{n-k} > 1 + \sum_{\substack{i=w_1-1 \\ w_1 \neq 0}}^{w_2-1} \binom{b-1}{i} (q-1)^i \ . \qquad (6)$$

It is important note that this relation is independent of j, therefore we can go on adding the columns as long as we wish but for the code of length j we shall stop after choosing j columns. So for j = n we shall added upto n columns.

By reversing the order of columns of the matrix $H' = [h_1, h_2, \ldots, h_n]$, we get the required parity check matrix $H = [H_1, H_2, \ldots, H_n]$ where $h_I = H_{n-i+1}$ (i.e. $h_n = H_1$, $H_{n-1} = H_2, \ldots, h_1 = H_n$].

Thus we obtain the inequality as stated in (4). □

Examples 1. Consider the following 5x7 matrix of a (7, 2) code over GF (2).

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This matrix has been constructed by the synthesis procedure outlined in the proof of theorem 2 by taking b = 4, $w_1 = 2$ and $w_2 = 3$. The code words of this code are 0000000, 0111101, 1000111, 1111010 which are not bursts of length 4 with weight lying between $w_1 = 2$, $w_2 = 3$.

Next we derive sufficient condition for codes correcting moderate-density bursts of length b (fixed).

Theorem 3. A sufficient condition for the existence of an (n, k) linear code over GF(q) which corrects all burst of length b (fixed) with weight lying between w_1 and w_2 ($0 \leq w_1 \leq w_2 \leq b$) is

$$q^{n-k} > 1 + \left[\sum_{\substack{i=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-1}{i}(q-1)^i\right]\left[(n-2b+1)\sum_{\substack{i=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-1}{i}(q-1)^{i+1}\right] + \left[\sum_{i=0}^{p}\binom{b-1}{i}(q-1)^i\right]$$
$$+\sum_{k=1}^{b-1}\left[\sum_{\substack{r_1=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-k}{r_1}(q-1)^{r_1+1}\sum_{\substack{r_2,r_3:\\ r_2+r_3\geq w_1-2}}\binom{k-1}{r_2}\binom{b-k-1}{r_3}(q-1)^{r_2+r_3}\right]$$
$$+\sum_{k=1}^{b-1}\left[\sum_{\substack{r_2,r_2,r_3:\\ r_1+r_2+r_3\leq 2w_2-2\\ w_1-k-1\leq r_1\leq w_1-2}}\binom{b-k}{r_1}\binom{k-1}{r_2}\binom{b-k-1}{r_3}(q-1)^{r_1+r_2+r_3+1}\right]$$

(7)

where $p = 2w_2-1$, when $b \geq 2w_1$, $q=2$

$= 2b - 2w_1-1$ when $b < 2w_1$, $q = 2$

and $w_1-k \leq r_1 \leq w_2-1$, $w_1-k-1 \leq w_2-1$, $0 \leq r_2 \leq 2w_2-3$, $r_1 + r_2 + r_3 \leq 2w_2-2$.

Proof. The existence of such a code shall be proved as in the previous theorem.

A nonzero (n-k)-tuple is chosen as the first column of H′. Subsequent columns are added such that after having selected j–1 columns, $h_1, h_2, \ldots, h_{j-1}$ suitably a column h_j is added provided that it is not a linear combination of any number of columns lying between w_1-1 and w_2-1 among the immediately preceding b–1 columns h_{j-b+1}, $h_{j-b+2}, \ldots, h_{j-1}$ together with any number of columns lying between w_1 and w_2 among any b consecutive columns out of all the j–1 columns selected so far. In other words, h_j can be added provided that

$$h_j \neq (\alpha_1 h_{j-b+1} + \alpha_2 h_{j-b+2} + \ldots + \alpha_{b-1} h_{j-1}) + (\beta_i h_i + \beta_{i+1} h_{i+1} + \ldots + \beta_{i+b} h_{i+b-1}) \qquad (8)$$

where h_j's are any b consecutive columns from all the j–1 previously chosen columns and the number of nonzero β_i's lies between w_1 and w_2 whereas the number of nonzero α_i's lies between w_1-1 and w_2-1 along with the case when all the α_i's are zero.

To compute the number of all possible linear combinations corresponding to R.H.S. of (8) for all possible choices of α_j and β_i we analyse three different cases as follows.

Case 1. When h_j's are completely confined to the first j–b columns.

The number of ways that the coefficients α_j's can be selected is

$$\sum_{\substack{i=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-1}{i}(q-1)^i \ . \qquad (9)$$

Further the number of ways that the coefficients of β_i's which form a burst of length b (fixed) with weight lying between w_1 and w_2 in a vector of length j-b can be selected is (refer Lemma 1).

$$J(j-b, b, w_1, w_2) = (j-2b+1) \sum_{\substack{i=w_1-1 \\ w_1 \neq 0}}^{w_2-1} \binom{b-1}{i} (q-1)^{i+1} \text{ .} \tag{10}$$

Therefore, the total number of choices of coefficients in this case is

$$\left[\sum_{\substack{i=w_1-1 \\ w_1 \neq 0}}^{w_2-1} \binom{b-1}{i} (q-1)^{i} \right] \left[(j-2b+1) \sum_{\substack{i=w_1-1 \\ w_1 \neq 0}}^{w_2-1} \binom{b-1}{i} (q-1)^{i+1} \right] \text{ .} \tag{11}$$

Case II. When h_j's are completely confined to the immediately preceding b–1 columns.

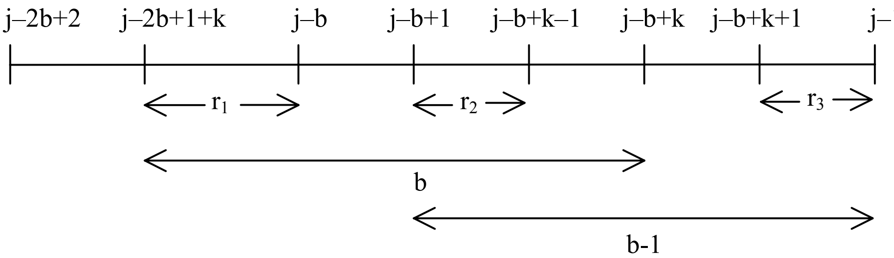
In this case the number of linear combinations corresponding to R.H.S. of (8) is

$$\sum_{i=0}^{p} \binom{b-1}{i} (q-1)^{i} \text{ ,} \tag{12}$$

where $p = 2w_2-1$, when $b \geq 2w_1$, $q=2$

$= 2b-2w_1-1$ when $b < 2w_1$, $q=2$

Case III. When h_j's are neither completely confined to the first (j–b) columns nor to the last b–1 columns.



Let the burst starts from (j–2b+1+k)-th position which can continue upto (j–b+k)-th position, ($1 \leq k \leq b-1$). We select at least w_1–1 and at the most w_2–1 nonzero components corresponding to j–2b+1+k, j–2b+2+k, …, j–b+k–1 columns together with nonzero components lying between w_1–1 and w_2–1 corresponding to j–b+1, j–b+2, …, j–1 columns. Let r_1, r_2 and r_3 be the number of nonzero components corresponding to columns lying between (j–2b+1+k)-th to (j–b)-th, (j–b+1)-th to (j–b+k–1)-th and (j–b+k+1)-th to (j–1)-th column respectively, such that

$w_1-k \leq r_1 \leq w_2-1,\ w_1-k-1 \ \leq r_3 \leq w_2-1,\ 0\leq r_2 \leq 2w_2-3,\ r_1 + r_2 + r_3 \leq 2w_2-2 \qquad (13)$

Therefore total possible number of distinct choices of coefficients is

$$+\sum_{k=1}^{b-1}\left[\sum_{\substack{r_1=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-k}{r_1}(q-1)^{r_1+1}\sum_{\substack{r_2,r_3:\\ r_2+r_3\geq w_1-2}}^{w_2-1}\binom{k-1}{r_2}\binom{b-k-1}{r_3}(q-1)^{r_2+r_3}\right]$$

$$+\sum_{k=1}^{b-1}\left[\sum_{\substack{r_2,r_2,r_3:\\ r_1+r_2+r_3\leq 2w_2-2\\ w_1-k-1\leq r_1\leq w_1-2}}\binom{b-k}{r_1}\binom{k-1}{r_2}\binom{b-k-1}{r_3}(q-1)^{r_1+r_2+r_3+1}\right] \tag{14}$$

Thus total possible number of distinct linear combinations corresponding to (8), which cannot be equal to h_j including zero vector is

$$1+\left[\sum_{\substack{i=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-1}{i}(q-1)^{i}\right]\left[(n-2b+1)\sum_{\substack{i=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-1}{i}(q-1)^{i+1}\right]+\left[\sum_{i=0}^{p}\binom{b-1}{i}(q-1)^{i}\right]$$

$$+\sum_{k=1}^{b-1}\left[\sum_{\substack{r_1=w_1-1\\ w_1\neq 0}}^{w_2-1}\binom{b-k}{r_1}(q-1)^{r_1+1}\sum_{\substack{r_2,r_3:\\ r_2+r_3\geq w_1-2}}\binom{k-1}{r_2}\binom{b-k-1}{r_3}(q-1)^{r_2+r_3}\right]$$

$$+\sum_{k=1}^{b-1}\left[\sum_{\substack{r_1,r_2,r_3:\\ r_1+r_2+r_3\leq 2w_2-2\\ w_1-k-1\leq r_1\leq w_1-2}}\binom{b-k}{r_1}\binom{k-1}{r_2}\binom{b-k-1}{r_3}(q-1)^{r_1+r_2+r_3+1}\right] \tag{15}$$

Therefore, the j-th column can be added to H′ provided that

$$q^{n-k} > M, \tag{16}$$

where M denotes expression (15).

For the existence of an (n, k) desired code relation (16) should hold for j = n so that it is possible to add upto nth column to form an (n–k) x n matrix. Thus we have constructed the matrix $H' = [h_i]$, (h_i denotes the i-th column from which we obtain the required parity check matrix $H = [H_i]$, (H_i denotes the i-th column) by reversing its column altogether, i.e. $h_i \rightarrow H_{n-i+1}$. This proves the result. □

Remarks 1. If we take $w_1 = 0$, $w_2 = b$ in (16) the weight constraints becomes redundant. Hence the bound gives $q^{n-k} > q^{b-1} [q^{b-1} (n-2b+1) (q-1)+1]$ which is a result due to Dass [1980].

2. If we put $w_1 = 0$, $w_2 = w$, in (16) we get the bound obtained by Dass [1982], which is a sufficient condition for the existence of low-density burst correcting code that corrects all bursts of length b (fixed).

Example 2. Consider the following matrix 6x9 of a (9,3) code over GF(2) which can correct all bursts of length 4 (fixed) with weight 2 or 3.

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

This matrix is constructed by the synthesis procedure outline in the proof of theorem 3.

It can be seen from the table 1 that syndromes of all the correctable error patterns are distinct and therefore the null space of this matrix gives the desired code.

Table 1

Error Pattern	Syndrome	Error Pattern	Syndrome
110000000	000011	111000000	000111
011000000	000110	011100000	001110
001100000	001100	001110000	011100
000110000	011000	000111000	111000
000011000	110000	000011100	001111
000001100	011111	000001110	011110
101000000	000101	110100000	001101
010100000	001010	011010000	011010
001010000	010100	001101000	110100
000101000	101000	000110100	100111
000010100	101111	000011010	110001
000001010	100001	000001101	010101
100100000	001001	101100000	001011
010010000	010010	010110000	010110
001001000	100100	001011000	101100
000100100	110111	000101100	010111
000010010	010001	000010110	101110
000001001	101010	000001011	101011

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