Ratio Mathematica 11
(1996), 81 - 103

ON A FIRST ORDER UNILATERAL EVOLUTIVE PERIODIC PROBLEM

Angela Gallo , Anna Maria Piccirillo

SUNTO - Si considera il problema periodico connesso ad una disequazione variazionale di evoluzione del primo ordine e si analizza la questione dell'esistenza della soluzione.

ABSTRACT - We consider the periodic problem related to a first order evolutive variational equation and we study existence of solution.

INTRODUCTION

Let Ω_l and Ω_2 be open sets of R^n with $\Omega = \Omega_1 \cap \Omega_2 \neq \emptyset$, $V_l\,(l=1,2)$ a closed subspace of $H^{m_l}(\Omega_l)\,(m_l \in N_0)$.

We note that

$$L^2(0,T;V_l) \subseteq L^2\left(0,T;L^2(\Omega_l)\right) \subseteq L^2\left(0,T;V_l'\right) \quad \left(0<T<+\infty,\ V_l' = dual\ space\ of\ V_l\right),$$

with continuous and dense embedding. We add that the reflexive Banach space

$$S_1 = \left\{v_1 \in L^2(0,T;V_1)\,/\,v_1' \in L^2\left(0,T;V_1'\right)\right\},\ \|v_1\|_{S_1} = \|v_1\|_{L^2(0,T;V_1)} + \left\|v_1'\right\|_{L^2\left(0,T;V_1'\right)}$$

Research performed under the financial support of M.U.R.S.T.
Authors' Address: Dipartimento di Matematica e Applicazioni.
Facoltà di Ingegneria, via Claudio, 21 - 80125 Napoli, Italy

is such that [8]

$$S_1 \subseteq C^0\left([0,T],\, L^2(\Omega_1)\right)$$

with continuous embedding.

Let us denote by:

$(\cdot\,,\cdot)\,,\,|\cdot|$ the inner product and the norm in $L^2(\Omega)$,

$(\cdot\,,\cdot)_l\,,\,|\cdot|_l$ the inner product and the norm in $L^2(\Omega_l)$,

$\|\cdot\|_l$ the norm in V_l ,

$\langle\cdot\,,\cdot\rangle_l$ the pairing between V_l and its dual V_l'.

Let be given $f_l \in L^2\left(0,T;V_l'\right)$ and the family of operators $A_l(t) \in \pounds\left(V_l,V_l'\right)\left(t \in [0,T]\right)$ such that the bilinear forms on V_l

$$a_l(t,y,z) = \left\langle A_l(t)y,z\right\rangle_l$$

are simmetric, uniformly coercive and continuous respect to t,

$$a_l(t,y,y) \geq c'\|y\|_l^2 \quad \forall t \in [0,T] \text{ and } \forall y \in V_l \ , \qquad (c',c'' = const. > 0)$$

$$\left|a_l(t,y,z)\right| \leq c''\|y\|_l \ \ \|z\|_l \quad \forall t \in [0,T] \ \text{ and } \ \forall y,z \in V_l \ ,$$

and we have

$$a_l(\cdot,y,z) \in C^{0,1}\left([0,T]\right) \ \forall y,z \in V_l \ ,$$

$$\left|a_l'(t,y,z)\right| \leq c'''\,\|y\|_l\,\|z\|_l \text{ a.e. on }]0,T[\text{ and } \forall y,z \in V_l \qquad (c''' = const. > 0) \quad ,$$

where $a_l'(t,y,z) = \dfrac{d}{dt}a_l(t,y,z)$.

Let be given $\psi \in L^2\left(0,T;L^2(\Omega)\right)$ such that the closed convex set

$$K = \left\{(v_1,v_2) \in \prod_{l=1}^{2} L^2(0,T;V_l) \,/\, v_1(t)\,\psi(t) \leq v_2(t) \text{ on } \Omega \text{ a.e. on }]0,T[\right\}$$

is not empty, then we consider the following

PROBLEM (P). Find $(u_1,u_2)\in\left[S_1\times L^2(0,T;V_2)\right]\cap K$ such that $u_1(0)=u_1(T)$ and moreover

$$\int_0^T\left\langle u_1'(t),v_1(t)-u_1(t)\right\rangle_1 dt+\sum_1^2{}_l\int_0^T\left\{a_l\left(t,u_l(t),v_l(t)-u_l(t)\right)+\right.$$
$$\left.-\left\langle f_l(t),v_l(t)-u_l(t)\right\rangle_l\right\}dt\ge 0\qquad\forall(v_1,v_2)\in K$$

or

$$\left\langle u_1'(t),v_1(t)-u_1(t)\right\rangle_1+\sum_1^2{}_l\left\{a_l\left(t,u_l(t),v_l(t)-u_l(t)\right)-\left\langle f_l(t),v_l(t)-u_l(t)\right\rangle_l\right\}\ge 0$$
$$a.e.\ on\ ]0,T[\ and\ \forall(v_1,v_2)\in K.$$

The problem we will analyze has been suggested by a paper by Toscano [10]. But derivatives respect to t of both the components of the unknown value (u_1,u_2) appear in it.

In our analysis we state two existence theorems for the solution of the problem (P) (Theor. 1 and 3); the uniqueness of the above solution is moreover easy to check.

The first theorem is obtained making use either of an existence and unicity theorem of the weak solution of a first order evolution variational inequality or of the fact, that owing to a hypothesis on ψ, each element of the convex set is approximable with proper regular functions.

The proof of the second theorem is based substantially on a result relative to the penalized problem (Theor. 2). To demonstrate the latter we need, first, to prove the existence of the periodic solution of an appropriate system of first order non-linear scalar differential equations. That, differently from standard cases, involves some difficulty originating from the absence of the term $\frac{du_2}{dt}$.

Finally we will examine the case $V_1=H_0^1(\Omega)$ $V_2=H^1(\Omega)$ and $A_1(t)$ and $A_2(t)$ the latter being second order integro-differential operators. We will also point out a property of regularity "respect to x" of the solution (Theor. 4), approximating u_1 with the solutions of a family of linear variational equations dependent on a parameter.

1. In order to prove a first existence theorem for our problem (P), we assume:

$i_1)$ there exists a non-negative constant γ_l such tat

$$a_l(T,y,y)-a_l(0,y,y)\geq \gamma_l\|y\|_l^2 \quad \forall y\in V_l \ ,$$

$i_2)$ $f_1=g_1+g_2 \quad with \quad g_1\in H^1\left(0,T;V_1'\right), \quad g_2\in L^2\left(0,T;L^2(\Omega_1)\right),$

$g_1(T)-g_1(0)\in L^2(\Omega_1) \quad if\ \gamma_1=0 \ ,$

$i_3)$ $f_2\in H^1\left(0,T;V_2'\right), \quad f_2(0)=f_2(T) \quad if\ \ \gamma_2=0\ ,$

$i_4)$ $\psi\in H^1\left(0,T;L^2(\Omega)\right), \ \psi(t)\leq 0 \ on\ \Omega\ a.e.on\ ]0,T[, \ \psi(0)=\psi(t)\ .$

Let us denote by L the operator from

$$D(L)=\left\{(v_1,v_2)\in S_1\times L^2(0,T;V_2)\,/\,v_1(0)=v_1(T)\right\}$$

so defined

$$\forall v=(v_1,v_2)\in D(L) \ \ and \ \ \forall w=(w_1,w_2)\in\prod_{l=1}^{2}L^2(0,T;V_l)$$

$$\langle Lv,w\rangle=\int_0^T\left\langle v_1'(t),w_1(t)\right\rangle_1 dt$$

Obviously the operator L is into $\left[\prod_{l=1}^{2}L^2\left(0,T;V_l'\right)\right]$ and it is linear and monotone.

We note that

$$(1)\qquad \overline{K\cap D(L)}=K \quad in\ \prod_{l=1}^{2}L^2(0,T;V_l)\ .$$

Indeed, if $(v_1,v_2)\in K$, for any $\varepsilon>0$ let $v_{l\varepsilon}\in H^1(0,T;V_l)$ be the solution of the problem

$$\varepsilon v'_{l\varepsilon}(t)+v_{l\varepsilon}(t)=v_{l\varepsilon}(t) \quad a.e. \ on \ ]0,T[, \quad v_{l\varepsilon}(0)=v_{l\varepsilon}(T) \ ;$$

we obtain [3]:

$$v_{l\varepsilon}\to v_l \quad \text{in} \quad L^2(0,T;V_l) \quad \text{for} \quad \varepsilon\to 0$$

and by means of $i_4)$

$$(v_{1\varepsilon},v_{2\varepsilon})\in K$$

since, taking into account that

$$v_{l\varepsilon}(t)=e^{-\frac{t}{\varepsilon}}v_{l\varepsilon}(0)+\frac{1}{\varepsilon}\int_0^t e^{\frac{s-t}{\varepsilon}}v_l(s)\,ds \quad \forall t\in[0,T],$$

$$v_{l\varepsilon}(0)=\frac{1}{\varepsilon\left(1-e^{-\frac{T}{\varepsilon}}\right)}\int_0^T e^{\frac{s-T}{\varepsilon}}v_l(s)\,ds,$$

we get

$$(v_{1\varepsilon}(t)-v_{2\varepsilon}(t),\varphi)\le(-\psi(t),\varphi) \quad \forall t\in[0,T] \text{ and } \forall\varphi\in C_0^\infty(\Omega) \text{ with } \varphi\ge 0.$$

THEOREM 1. Assumed that hypoteses $i_1)-i_4)$ hold, the problem (P) admits one (and only one) solution (u_1,u_2) and it results $u_1\in H^1(0,T;L^2(\Omega_1))$.

Proof. At first since L is monotone operator and the bilinear forms a_1 and a_2 are uniformly coercive and continuous respect to t, using also the considerations given for (1) then [4] there exists one $(u_1,u_2)\in K$ such that

$$\text{(2)} \qquad \int_0^T\left\{\left\langle v_1'(t),v_1(t)-u_1(t)\right\rangle_1+\sum_1^2{}_l\left[a_l(t,v_l(t),v_l(t)-u_l(t))+\right.\right.$$
$$\left.\left.-\left\langle f_l(t),v_l(t)-u_l(t)\right\rangle_l\right]\right\}dt\ge 0 \quad \forall(v_1,v_2)\in K\cap D(L).$$

By virtue of (1), we can state to result by proving that

(3) $$u_1' \in L^2\left(0,T; L^2(\Omega_1)\right), \qquad u_1(0) = u_1(T),$$

since, using (2) with $v_l = \lambda u_l + (1-\lambda)v_l \left(\lambda \in [0,1[\right)$ and taking the limit as $\lambda \to 1$, we have

$$\int_0^T \left\{ \left(u_1'(t), v_1(t) - u_1(t)\right)_1 + \sum_1^2{}_l \left[a_l\left(t, u_l(t), v_l(t) - u_l(t)\right) - \left\langle f_l(t), v_l(t) - u_l(t)\right\rangle_l \right] \right\} dt \geq 0.$$

If $\gamma_1 > 0$, being $u_{l\varepsilon}$ the element of $H^1\left(0,T;V_l\right)$ solution of the problem

(4) $$\varepsilon u_{l\varepsilon}'(t) + u_{l\varepsilon}(t) = u_l(t) \quad \text{a.e. on }]0,T[, \; u_{l\varepsilon}(0) = u_{l\varepsilon}(T),$$

inequality (2),written with $v_l = u_{l\varepsilon}$, produces the upper limitation

$$\begin{aligned} &\int_0^T \left| u_{1\varepsilon}'(t) \right|_1^2 dt + \frac{\gamma_1}{2} \left\| u_{1\varepsilon}(0) \right\|_1^2 + \frac{\gamma_2}{2} \left\| u_{2\varepsilon}(0) \right\|_2^2 \leq \\ &\leq \frac{c'''}{2} \sum_1^2{}_l \int_0^T \left\| u_{l\varepsilon}(t) \right\|_l^2 dt + \sum_1^2{}_l \int_0^T \left\langle f_l(t), u_{l\varepsilon}'(t) \right\rangle_l dt \end{aligned}$$

and then, because of i_2,), i_3) , we are lead to:

$$\left\| u_{l\varepsilon}' \right\|_{L^2\left(0,T;L^2(\Omega_1)\right)} \leq c \qquad (c = const. > 0 \; indep. \; of \; \varepsilon)$$

from which the (3).

Let us study the case $\gamma_1 = 0$, showing at first that

(5) $$u_1 \in C^0\left([0,T], L^2(\Omega_1)\right), \qquad u_1(0) = u_1(T).$$

Let $\{f_{1n}\}$ be a sequence $\in L^2\left(0,T;L^2(\Omega_1)\right)$ such that $f_{1n} \to f_1$ in $L^2\left(0,T;V_1'\right)$. Proceeding similarly to what reasoned with the case $\gamma_1 > 0$, we state the existence and uniqueness of $(\overline{u}_{1n},\overline{u}_{2n})\in K$, with $\overline{u}_{1n}'\in L^2\left(0,T;L^2(\Omega_1)\right)$ and $\overline{u}_{1n}(0)=\overline{u}_{1n}(T)$, such that for every $(v_1,v_2)\in K$

$$(6)\qquad \int_0^T \left\{\left(\overline{u}_{1n}'(t),v_1(t)-\overline{u}_{1n}'(t)\right)_1+\sum_1^2{}_l\, a_l\left(t,\overline{u}_{ln}(t),v_l(t)-\overline{u}_{ln}(t)\right)+\right.$$
$$\left.-\left(f_{1n}(t),v_1(t)-\overline{u}_{1n}(t)\right)_1-\left\langle f_2(t),v_2(t)-\overline{u}_{2n}(t)\right\rangle_2\right\}dt\geq 0$$

or

$$\left(\overline{u}_{1n}'(t),v_1(t)-\overline{u}_{1n}(t)\right)_1+\sum_1^2{}_l\, a_l\left(t,\overline{u}_{ln}(t),v_l(t)-\overline{u}_{ln}(t)\right)\geq$$
$$\geq\left(f_{1n}(t),v_1(t)-\overline{u}_{1n}(t)\right)_1+\left\langle f_2(t),v_2(t)-\overline{u}_{2n}(t)\right\rangle_2 \qquad a.e.\ on\ ]0,T[$$

so that $\forall\, m,n\in N$

$$\left(\overline{u}_{1m}'(t)-\overline{u}_{1n}'(t),\overline{u}_{1m}(t)-\overline{u}_{1n}(t)\right)_1+\sum_1^2{}_l\, a_l\left(t,\overline{u}_{lm}(t)-\overline{u}_{ln}(t),\overline{u}_{lm}-\overline{u}_{ln}(t)\right)\leq$$
$$\leq\left(f_{1m}(t)-f_{1n}(t),\overline{u}_{1m}(t)-\overline{u}_{1n}(t)\right)_1 \qquad a.e.\ on\ ]0,T[$$

from which

$$\frac{d}{dt}\left|\overline{u}_{1m}(t)-\overline{u}_{1n}(t)\right|_1^2+c'\left\|\overline{u}_{1m}(t)-\overline{u}_{1n}(t)\right\|_1^2+2c'\left\|\overline{u}_{2m}(t)-\overline{u}_{2n}(t)\right\|_2^2\leq$$
$$\leq\frac{1}{c'}\left\|f_{1m}(t)-f_{1n}(t)\right\|_{V_1'}^2 \qquad a.e.\ on\ ]0,T[.$$

Then the following upper limitations hold:

$$c'\int_0^T\left\|\overline{u}_{1m}(t)-\overline{u}_{1n}(t)\right\|_1^2dt+2c'\int_0^T\left\|\overline{u}_{2m}(t)-\overline{u}_{2n}(t)\right\|_2^2dt\leq$$
$$\leq\frac{1}{c'}\int_0^T\left\|f_{1m}(t)-f_{1n}(t)\right\|_{V_1'}^2dt,$$

$$\forall\, t\in[0,T]\qquad \left|\overline{u}_{1m}(t)-\overline{u}_{1n}(t)\right|_1^2\leq\frac{1}{c'}\int_0^T\left\|f_{1m}(t)-f_{1n}(t)\right\|_{V_1'}^2dt+$$
$$+\left|\overline{u}_{1m}(0)-\overline{u}_{1n}(0)\right|_1^2,$$

$$\left(e^{c'T}-1\right)\left|\overline{u}_{1m}(0)-\overline{u}_{1n}(0)\right|_1^2\leq\frac{e^{c'T}}{c'}\int_0^T\left\|f_{1m}(t)-f_{1n}(t)\right\|_{V_1'}^2dt,$$

from which we deduce the existence of an element $(\overline{u}_1,\overline{u}_2) \in K$ with $\overline{u}_1 \in C^0\left([0,T], L^2(\Omega_1)\right)$ and $\overline{u}_1(0)=\overline{u}_1(T)$, such that

$$\overline{u}_{ln} \to \overline{u}_l \quad in\ L^2(0,T;V_l),$$
$$\overline{u}_{1n} \to \overline{u}_1 \quad in\ C^0\left([0,T],L^2(\Omega_1)\right).$$

Then , taking into account that (6) is equivalent to the upper limitation

$$\int_0^T \Big\{ \left\langle v_1'(t), v_1(t)-\overline{u}_{1n}(t)\right\rangle_1 + \sum_{1}^{2}{}_l\, a_l\left(t,\overline{u}_{ln}(t), v_l(t)-\overline{u}_{ln}(t)\right)+$$
$$-\left(f_{1n}(t), v_1(t)-\overline{u}_{1n}(t)\right)_1 - \left\langle f_2(t), v_2(t)-\overline{u}_{2n}(t)\right\rangle_2 \Big\} dt \geq 0 \;\; \forall (v_1,v_2)\in K \cap D(L),$$

we get

$$\overline{u}_l = u_l$$

and (5) follows as a consequence. We have to verify the first of (3). To this aim, considered the solution of the problem (4):

$$(7)\; u_{l\varepsilon}(t) = \frac{e^{-\frac{t}{\varepsilon}}}{\varepsilon\left(1-e^{-\frac{T}{\varepsilon}}\right)} \int_0^T e^{\frac{s-T}{\varepsilon}} u_l(s)\, ds + \frac{1}{\varepsilon} \Big/ \int_0^t e^{\frac{s-t}{\varepsilon}} u_l(s)\, ds \quad \forall t \in [0,T]$$

,

it is enough to establish that

$$(8) \qquad \|u_{l\varepsilon}\|_{L^2\left(0,T;L^2(\Omega_l)\right)} \leq c \qquad (c = \text{const.} > 0 \text{ indep. of } \varepsilon).$$

Indeed, setting in (2) $v_l = u_{l\varepsilon}$, we obtain:

$$\frac{1}{2}\int_0^T \left|u_{1\varepsilon}(t)\right|_1^2 dt + \frac{\gamma_2}{2}\left\|u_{2\varepsilon}(0)\right\|_2^2 \leq \frac{c'''}{2}\sum_1^2 {}_l \int_0^T \left\|u_{l\varepsilon}(t)\right\|_l^2 dt +$$
$$+\left(g_1(T)-g_1(0), u_{1\varepsilon}(0)\right)_1 - \int_0^T \left\langle g_1'(t), u_{1\varepsilon}(t)\right\rangle_1 dt +$$
$$+\frac{1}{2}\int_0^T \left|g_2(t)\right|_1^2 dt + \int_0^T \left\langle f_2(t), u_{2\varepsilon}'(t)\right\rangle_2 dt$$

from which (8) follows by $i_3)$ and by the upper limitation

$$\left|u_{1\varepsilon}(0)\right|_1 \leq \left\|u_1\right\|_{C^0\left([0,T], L^2(\Omega_1)\right)}$$

by virtue of (7) and the first of (5).

2. Now let we assume:

$i_1')$ there exists a non-negative constant γ_1 such that

$$a_1(T,y,y) - a_1(0,y,y) \geq \gamma_1 \|y\|_1^2 \quad \forall y \in V_1,$$

bilinear form a_2 is independent by t, in other words

$$a_2(t,y,z) = a_2(y,z) \quad \forall y,z \in V_2\ ,$$

$i_2')$ $f_1 = g_1 + g_2$ with $g_1 \in H^1\left(0,T;V_1'\right)$, $g_2 \in L^\infty\left(0,T;L^2(\Omega_1)\right)$,

$g_1(T) - g_1(0) \in L^2(\Omega_1)$ *if* $\gamma_1 = 0$,

$i_3')$ $f_2 \in C^{0,1}\left([0,T], V_2'\right)$, $f_2(0) = f_2(T)$,

$i_4')$ $\psi \in C^{0,1}\left([0,T], H^{m_1}(\Omega_1)\right)$, $\psi'(t) \in V_1$ *a.e. on* $]0,T[$,

$\psi(0) = \psi_1 + \psi_2$ *with* $\psi \in V_1$ *and* $\psi_2 \leq 0$ *on* Ω, $\psi(0) = \psi(T)$

Concerning penalized problem, the following existence and unicity theorem holds.

THEOREM 2. Under the hypothesis $i_1'\big) - i_4'\big)$, for any $\varepsilon > 0$ there exists one and only one $(u_{1\varepsilon}, u_{2\varepsilon}) \in \prod_{l=1}^{2} L^2(0,T;V_l)$, with $u_{1\varepsilon}' \in L^2\big(0,T;L^2(\Omega_1)\big)$, which is solution of problem

(9)

$$\left(u_{1\varepsilon}'(t), y\right)_1 + a_1\left(t, u_{1\varepsilon}(t), y\right) + a_2\left(u_{2\varepsilon}(t), z\right) + \frac{1}{\varepsilon}\left(\left[u_{1\varepsilon}(t) + \psi_n(t) - u_{2\varepsilon}(t)\right]^+, y - z\right) =$$
$$= \left\langle f_1(t), y\right\rangle_1 + \left\langle f_2(t), z\right\rangle_2 \quad \text{a.e. on }]0,T[\quad \forall (y,z) \in V_1 \times V_2,$$

(10) $\quad u_{1\varepsilon}(0) = u_{1\varepsilon}(T)$,

and it results

(11) $\quad u_{2\varepsilon} \in C^0([0,T], V_2), \quad u_{2\varepsilon}(0) = u_{2\varepsilon}(T),$

(12) $\quad \left\|u_{1\varepsilon}'\right\|_{L^2(0,T;L^2(\Omega_1))} + \left\|u_{1\varepsilon}\right\|_{L^2(0,T;V_1)} + \left\|u_{2\varepsilon}\right\|_{L^2(0,T;V_2)} \le c$

being c a positive independent of ε constant.

Proof. It is easily seen the unicity of $(u_{1\varepsilon}, u_{2\varepsilon})$. As far as the existence we at first observe that for any $z \in L^2(\Omega)$ and for any $t \in [0,T]$, since the operator B_ε on V_2 into V_2' such that

$$\left\langle B_\varepsilon y, w\right\rangle_2 = a_2(y, w) - \frac{1}{\varepsilon}\left([z - y]^+, w\right) \qquad \forall y, w \in V_2$$

is strictly monotone, bounded, hemicontinuous and coercive, variational equation

$$w \in V_2 \,/\, a_2(w, y) = \frac{1}{\varepsilon}\left([z - w]^+, y\right) + \left\langle f_2(t), y\right\rangle_2 \qquad \forall y \in V_2$$

admits unique solution [7]. Denoted by $\tau_\varepsilon(t,z)$ this solution, we note that

$\alpha_1\big)$ There exists a positive dependent by ε constant, $c(\varepsilon)$, such that

$$\left\|\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right\|_2 \le c(\varepsilon)\left[|z'-z''|+\left\|f_2(t')-f_2(t'')\right\|_{V_2'}\right]$$

for each $t',\ t'' \in [0,T]$ and $z',z'' \in L^2(\Omega)$.

Indeed, owing the upper limitation

$$\left(\left[z'-\tau_\varepsilon(t',z')\right]^+-\left[z''-\tau_\varepsilon(t'',z'')\right]^+,\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right)\le$$
$$\le\left(\left[z'-\tau_\varepsilon(t',z')\right]^+-\left[z''-\tau_\varepsilon(t'',z'')\right]^+,z'-z''\right),$$

we obtain:

$$a_2\left(\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z''),\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right)\le$$
$$\le\frac{1}{\varepsilon}\left(\left[z'-\tau_\varepsilon(t',z')\right]^+-\left[z''-\tau_\varepsilon(t'',z'')\right]^+,z'-z''\right)+$$
$$+\left\langle f_2(t')-f_2(t''),\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right\rangle_2$$

as well as

$$c'\left\|\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right\|_2^2\le\frac{1}{\varepsilon}\int_\Omega\left\{|z'-z''|+\left|\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right|\right\}|z'-z''|dx+$$
$$+\left\|f_2(t')-f_2(t'')\right\|_{V_2'}\left\|\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right\|_2\le$$
$$\le\frac{1}{\varepsilon}\int_\Omega|z'-z''|^2dx+\frac{1}{\varepsilon}\frac{\sigma}{2}\left\|\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right\|_2^2+\frac{1}{\varepsilon}\frac{1}{2\sigma}\int_\Omega|z'-z''|^2dx+$$
$$+\frac{1}{2\sigma}\left\|f_2(t')-f_2(t'')\right\|_{V_2'}^2+\frac{\sigma}{2}\left\|\tau_\varepsilon(t',z')-\tau_\varepsilon(t'',z'')\right\|_2^2\quad\forall\sigma>0.$$

Let $\{z_j\}$ be a base of V_1 such that, denote by V_{1n} the space spanned by $\{z_1,\dots z_n\}$, $\psi\in V_{11}$. Let us moreover denote by P_n the projector onto V_{1n} in V_1, we set

$$\forall t\in[0,T]\quad \psi_{1n}(t)=\int_0^t P_n\psi(z)d\tau+\psi(0)\ ,$$

therefore

(13) $\psi_{1n}(0) = \psi(0) = \psi_{1n}(T)$,

(14) $\psi_{1n} \to \psi \quad in \quad C^0\left([0,T], H^{m1}(\Omega_1)\right),$

(15) $\psi'_{1n} \to \psi' \quad in \quad L^2\left(0,T;V_1\right).$

Now we'll prove that the system

(16)
$$\sum_1^n {}_i\left(z_i, z_j\right) g_i'(t) = -\sum_1^n {}_i a_1\left(t, z_i, z_j\right) g_i(t) +$$
$$-\frac{1}{\varepsilon}\left(\left[\sum_1^n {}_i g_i(t) z_i + \psi(t) - \tau_\varepsilon\left(t, \sum_1^n {}_i g_i(t) z_i + \psi_{1n}(t)\right)\right]^+, z_j\right) +$$
$$+\left\langle g_1(t), z_j\right\rangle_1 + \left(g_2(t), z_j\right)_2 \qquad \text{a.e. on } \left]0,T\right[\quad \forall j \in \{1,\dots,n\},$$

verifies the following statement:

$\alpha_2)$ The system (16) admits one and only one solution $\left(g_1,\dots,gn\right) \in C^{0,1}\left([0,T], R^n\right)$ such that

$$w_{1n}(0) = w_{1n}(T) \ ,$$

where $w_{1n}(t) = \sum_1^n i\, g_i(t) z_i \qquad \forall t \in [0,T]$, and it results

(17) $\left|w_{1n}(0)\right|_1 \le \delta \qquad (\delta = \text{const.} > 0 \text{ indep. of } \varepsilon, n).$

We'll limit ourselves to establish the existence of $\left(g_1,\dots,g_n\right)$ because its uniqueness is readly verifies.

Let $u_0 = \sum_1^n {}_i \xi_i z_i \in V_{1n}$ be given; the property $\alpha_1)$ and the hypotheses made on data imply that the (16) has one and only one solution $\left(g_1,\dots,g_n\right)$ $\in C^{0,1}\left([0,T], R^n\right)$ so that

$$g_i(0) = \xi_i \quad \forall i \in \{1,\dots,n\}.$$

Setting for any $t \in [0,T]$ $w_{1n} = \sum_1^n ig_i(t) z_i$ and $w_{2n}(t) = \tau_\varepsilon \left(t, \psi_{1n}(t) + w_{1n}(t)\right)$, it follows:

$w_{1n} \in C^{0,1}\left([0,T], V_1\right)$, $w_{1n}(0) = u_0$ and in virtue by $\alpha_1)$ $w_{2n} \in C^{0,1}\left([0,T], V_2\right)$,

$$(18)\ \left(\omega_{1n}'(t), y\right)_1 + a_1\left(t, \omega_{1n}(t), y\right) + a_2\left(w_{2n}(t), z\right) + \frac{1}{\varepsilon}\left(\left[\omega_{1n}(t) + \psi_{1n}(t) - w_{2n}(t)\right]^+, y - z\right) =$$

$$= \left\langle f_1(t), y\right\rangle_1 + \left\langle f_2(t), y\right\rangle_2 \qquad \text{a.e. on }]0,T[\text{ and } \forall (y,z) \in V_{1n} \times V_2.$$

Observing that in virtue by $i_4')$ it results

$$\forall t \in [0,T] \qquad \omega_{1n}(t) + \psi_{1n}(t) - \psi_2 \in V_{1n}, \qquad \psi_2 \le 0 \text{ on } \Omega,$$

we insert $y = w_{1n}(t) + \psi_{1n}(t) - \psi_2$ and $z = w_{2n}(t)$ in (18) and obtain

$$(19)\ \frac{d}{dt}\left|w_{1n}(t) + \psi_{1n}(t) - \psi_2\right|_1^2 + c_1\left[\left\|w_{1n}(t) + \psi_{1n}(t) - \psi_2\right\|_1^2 + \left\|w_{2n}(t)\right\|_2^2\right] \le$$

$$\le c_2\left[\left\|\psi'(t)\right\|_1^2 + \left\|f_1(t)\right\|_{V_1'}^2 + \left\|f_2(t)\right\|_{V_2'}^2 + 1\right] \quad \text{a.e. on }]0,T[$$

$(c_i = \text{const.} > 0$ indep. of $\varepsilon, n, u_0)$

From (19) along with (13) we arrive to

$$(20) \qquad e^{c_1 T}\left|w_{1n}(T) + \psi_1\right|_1^2 \le \left|u_0 + \psi_1\right|_1^2 + c_3 \qquad (c_3 = \text{const.} > 0 \text{ indep. of } \varepsilon, n, u_0).$$

Let $\mathfrak{M}\left(-\psi_1, \eta\right)$ be closed ball of $L^2\left(\Omega_1\right)$ of center $-\psi_1$ and radius $\eta \ge \sqrt{\dfrac{c_3}{e^{c_1 T} - 1}}$ and, for any $u_0 \in V_{1n} \, \mathfrak{M}\left(-\psi_1, \eta\right)$, let w_{1n} be the solution of (16) $\mathfrak{M}$ such that $w_{1n}(0) = u_0$. Then

$$F: u_0 \in V_{1n} \cap \mathfrak{M}\left(-\psi_1, \eta\right) \to w_{1n}(T)$$

is, by virtue of (20=, an operator into $V_{1n} \cap \beta(-\psi_1, \eta)$ and moreover, how it is easily seen, not expencive i.e.

$$|Fu_0 - F\bar{u}_0|_1 \leq |u_0 - \bar{u}_0|_1 \qquad \forall u_0, \bar{u}_0 \in V_{1n} \cap \mathfrak{M}(-\psi_1, \eta)$$

Consequently [5] there exists at least $u_0 \in V_{1n} \cap \beta(-\psi_1, \eta)$ that is fixed for F; hence the statement $\alpha_2)$ holds.

Obviously for $w_{2n}(t) = \tau_\varepsilon(t, w_{1n}(t) + \psi_{1n}(t)) \quad \forall t \in [0,T]$ we have

$$w_{2n}(0) = w_{2n}(T) \tag{21}$$

also in view of $i_3')$, (13).

The relation (18) together with (13), (14), (17) produces the upper limitations:

$$\|w_{1n}\|_{C^0([0,T],L^2(\Omega_1))} \leq c \tag{22}$$

(c = const. > 0 indep. of ε, n)

$$\|w_{1n}\|_{L^2(0,T;V_1)} \leq c\,. \tag{23}$$

Using (18) with $y = w_{1n}'(t) + \psi_{1n}'(t)$ and $z = w_{2n}'(t)$, and taking into account $i_1')$, (13), (21), we get

$$\int_0^T \left|w_{1n}'(t) + \psi_{1n}'(t)\right|_1^2 dt + \frac{\gamma_1}{2}\|w_{1n}(0)\|_1^2 = \int_0^T a_1'(t, w_{1n}(t), w_{1n}(t))\, dt +$$

$$- \int_0^T a_1(t, w_{1n}(t), \psi_{1n}'(t))\, dt + \int_0^T (\psi_{1n}'(t), w_{1n}'(t) + \psi_{1n}'(t))_1\, dt +$$

$$+ \int_0^T \langle f_1(t), w_{1n}'(t) + \psi_{1n}'(t)\rangle_1\, dt + \int_0^T \langle f_2(t), w_{2n}'(t)\rangle_2\, dt$$

and from here

(24) $$\left\| w'_{1n} \right\|_{L^2\left(0,T;L^2(\Omega_1)\right)} \leq c \quad \text{(c = const. > 0 indep. of } \varepsilon\text{; n)}$$

because of $\left.i'_2\right)$, $\left.i'_3\right)$, (14), (15), (21), (22), (23).

From the upper limitations (23), (24) we deduce that there exists $u_{l\varepsilon} \in L^2\left(0,T;V_l\right)$, with $u'_{1\varepsilon} \in L^2\left(0,T;L^2(\Omega_1)\right)$, and $0 \in L^2\left(0,T;L^2(\Omega)\right)$ such that, to within a subsequence, as $n \to +\infty$

(25) $$\begin{aligned} w_{ln} &\to u_{l\varepsilon} && \text{weakly in } L^2\left(0,T;V_l\right), \\ w_{1n} &\to u_{1\varepsilon} && \text{weakly in } L^2\left(0,T;L^2(\Omega_1)\right), \\ \left[w_{1n}+\psi_{1n}-w_{2n}\right]^+ &\to 0 && \text{weakly in } L^2\left(0,T;L^2(\Omega)\right). \end{aligned}$$

Of course (10), (12), hold. Starting from (18) and based on (25) we derive

$$\left(u'_{1\varepsilon}(t),y\right)_1 + a_1\left(t,u_{1\varepsilon}(t),y\right) + a_2\left(u_{2\varepsilon}(t),z\right) + \frac{1}{\varepsilon}\left(0(t),y-z\right) =$$
$$= \left\langle f_1(t),y\right\rangle_1 + \left\langle f_2(t),z\right\rangle_2 \quad \text{a.e. on }]0,T[\quad \forall\left(y,z\right) V_1 \times V_2$$

as well as

$$\lim^{n} \frac{1}{\varepsilon}\int_0^T \left(\left[w_{1n}(t)+\psi_{1n}(t)-w_{2n}(t)\right]^+, \left[w_{1n}(t)+\psi_{1n}(t)-w_{2n}(t)\right]-\left[u_{1\varepsilon}(t)+\psi(t)-u_{2\varepsilon}(t)\right]\right)dt \leq$$
$$\leq \int_0^T \left\langle f_1(t),u_{1\varepsilon}(t)\right\rangle_1 dt + \int_0^T \left\langle f_2(t),u_{2\varepsilon}(t)\right\rangle_2 dt - \int_0^T a_1\left(t,u_{1\varepsilon}(t),u_{1\varepsilon}(t)\right) dt +$$
$$-\int_0^T a_2\left(u_{2\varepsilon}(t),u_{2\varepsilon}(t)\right) dt - \frac{1}{\varepsilon}\int_0^T \left(0(t),u_{1\varepsilon}(t)-u_{2\varepsilon}(t)\right) dt =$$
$$= \int_0^T \left(u'_{1\varepsilon}(t),u_{1\varepsilon}(t)\right)_1 dt = 0;$$

the latter estimation implies that [7]

$$0=\left[u_{1\varepsilon}+\psi-u_{2\varepsilon}\right]^{+}$$

taking into account the fact that the operator from $L^2\left(0,T;L^2(\Omega)\right)$ into its dual space:

$$\left\langle C_\varepsilon u,v\right\rangle=\frac{1}{\varepsilon}\int_0^T\left(u^{+}(t),v(t)\right)dt \qquad \forall(u,v)\in\left[L^2\left(0,T;L^2(\Omega)\right)\right]^2$$

is pseudo-monotone. And so equality (9) holds. The relation (11) is directly deduced from $u_{2\varepsilon}(t)=\tau_\varepsilon\left(t,u_{1\varepsilon}(t)+\psi(t)\right) \qquad \forall t\in[0,T]$.

By virtue of theorem 2 we can prove quickly

THEOREM 3. If the hypotheses $i_1^{'})-i_4^{'})$ hold, the problem (P) admits one (and only one) solution $\left(u_1,u_2\right)$ and it results $u_1\in H^1\left(0,T;L^2\left(\Omega_1\right)\right)$.

PROOF. Let $\left(u_{1\varepsilon},u_{2\varepsilon}\right)$ be the solution of the problem (9), (10). Holding (12), there exist $u_I\in L^2\left(0,T;V_I\right)$, with $u_1^{'}\in L^2\left(0,T;L^2\left(\Omega_1\right)\right)$ and $u_I(0)=u_I(T)$, and a positive numerical infinitesimal sequence $\{\varepsilon_n\}$ such that

$$u_{I\varepsilon_n}\to u_I \qquad \text{weakly in } L^2\left(0,T;V_I\right),$$
$$u_{1\varepsilon_n}^{'}\to u_1^{'} \qquad \text{weakly in } L^2\left(0,T;L^2\left(\Omega_1\right)\right).$$

Let $\left(v_1,v_2\right)$ an element of **K** be given, by virtue of (9), (10), (12)

$$\frac{1}{\varepsilon_n}\int_0^T\left|\left[u_{1\varepsilon_n}(t)+\psi(t)-u_{2\varepsilon_n}(t)\right]^{+}\right|^2dt=\frac{1}{\varepsilon_n}\int_0^T\left(\left[u_{1\varepsilon_n}(t)+\psi(t)-u_{2\varepsilon_n}(t)\right]^{+},u_{1\varepsilon_n}(t)-u_{2\varepsilon_n}(t)\right)dt+$$

$$+\frac{1}{\varepsilon_n}\int_0^T\left(\left[u_{1\varepsilon_n}(t)+\psi(t)-u_{2\varepsilon_n}(t)\right]^{+},v_1(t)+\psi(t)-v_2(t)\right)dt+$$

$$-\frac{1}{\varepsilon_n}\int_0^T \left(\left[u_{1\varepsilon_n}(t)+\psi(t)-u_{2\varepsilon_n}(t)\right]^+, v_1(t)-v_2(t)\right)dt \le c$$

(c = const. indep. of ε_n)

$$\int_0^T \left(u'_{1\varepsilon_n}(t), v_1(t)\right)_1 dt + \int_0^T a_1\left(t, u_{1\varepsilon_n}(t), v_1(t)\right)dt + \int_0^T a_2\left(u_{2\varepsilon_n}(t), v_2(t)\right)dt \ge$$

$$\ge \sum_1^2{}_l \int_0^T \left\langle f_l(t), v_l(t) - u_{l\varepsilon_n}(t)\right\rangle_l + \int_0^T a_1\left(t, u_{1\varepsilon_n}(t), u_{1\varepsilon_n}(t)\right)dt +$$

$$+\int_0^T a_2\left(u_{2\varepsilon_n}(t), u_{2\varepsilon_n}(t)\right)dt$$

then of course we have

$$\int_0^T \left|\left[u_1(t)+\psi(t)-u_2(t)\right]^+\right|^2 dt = 0\ ,$$

$$\int_0^T \left(u'_1(t), v_1(t)\right)_1 dt + \int_0^T a_1\left(t, u_1(t), v(t)\right)dt + \int_0^T a_2\left(u_2(t), v_2(t)\right)dt \ge$$

$$\ge \sum_1^2{}_l \int_0^T \left\langle f_l(t), v_l(t) - u_l(t)\right\rangle_l dt$$

$$+\int_0^T a_1\left(t, u_1(t), u_1(t)\right)dt + \int_0^T a_2\left(u_2(t), u_2(t)\right)dt\ ,$$

and then $\left(u_1, u_2\right)$ is the solution of the problem (P).

3. Let us complete the study of problem (P) by analysing a particular case. Let: $\Omega_1 = \Omega_2 = \Omega$ be a $C^{1,1}$ open, bounded, connected set of $\mathbf{R}^n$;

$$V_1 = H_0^1(\Omega), \quad V_2 = H^1(\Omega) \ ;$$

$$\forall t \in [0,T]\,\forall y,z \in H_0^1(\Omega) \qquad \langle A_1(t)y,z\rangle_1 = \sum_{ij}^{n} \int_\Omega a_{ij}(t,x) y_{x_i} z_{x_j}\, dx \ ,$$

$$\forall t \in [0,T]\,\forall y,z \in H^1(\Omega) \qquad \langle A_2(t)y,z\rangle_2 = \sum_{ij}^{n} \int_\Omega a_{ij}(t,x) y_{x_i} z_{x_j}\, dx + \int_\Omega b(t,x) yz\, dx$$

where

$$a_{ij} = a_{ji} \in C^1([0,T]\times\overline{\Omega}), \ \ b \in C^0([0,T]\times\overline{\Omega}), \ \ \frac{\partial b}{\partial t} \in C^0([0,T]\times\overline{\Omega}),$$

$$a_{ij}(0,x) = a_{ij}(T,x), \ \ b(0,x) = b(T,x) \ \ \forall x \in \overline{\Omega} \ ,$$

$$\Sigma_{ij} a_{ij}(t,x)\lambda_i\lambda_j \geq c_0|\lambda|^2 \quad \forall (t,x) \in [0,T]\times\overline{\Omega} \ \mathit{and} \ \ \forall \lambda = (\lambda_1 \ldots \lambda_n) \in \mathbf{R}^n$$

$(c_0 = \text{const.} > 0)$,

$$b(t,x) \geq b_0 \quad \forall (t,x) \in [0,T]\times\overline{\Omega} \qquad (b_0 = \text{const.} > 0) \ ;$$

$$f_1 \in L^2(0,T;L^2(\Omega)), \ f_2 \in L^2(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \ f_2(0) = f_2(T) ;$$

$$\psi = 0$$

In such a situation the problem (P), by virtue of theorem 1, admits only one solution (u_1, u_2) satisfying

THEOREM 4. Under the above stated hypotheses, we have:

$$u_1 \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \ , \tag{26}$$

$$u_2 \in L^2(0,T;H^2_{loc}(\Omega)) \ . \tag{27}$$

Proof. We now prove relation (26). First of all, for every $\varepsilon > 0$ there exists one and only one $u_{1_\varepsilon} \varepsilon\, L^2(0,T;H_0^1(\Omega))$ such that

(28) $$A_1(t)u_{1_\varepsilon}(t)-\left(f_2(t)+b(t)u_2(t)\right)+\frac{u_{1_\varepsilon}(t)-u_1(t)}{\varepsilon}=0$$

in the sense of $D'(\Omega)$ a.e. on $]0,T[$.

Assumpions made for Ω, a_{ij}, b and f_2 assure [7], [9] that $u_{1\varepsilon}$ belongs $L^2\left(0,T;H^2(\Omega)\right)$ and

(29) $$\|u_{1\varepsilon}(t)\|_{H^2(\Omega)}\le c\|A_1(t)u_{1\varepsilon}(t)\|_{L^2(\Omega)} \quad \text{a.e. on }]0,T[$$

(c = const. > 0 indep. of ε)

Let us verify that

(30) $$\left(u_{1\varepsilon},u_2\right)\in \mathbf{K}$$

We have at once

(31) $$\left[u_{1_\varepsilon}(t)-u_2(t)\right]^+\in H_0^1(\Omega) \quad \text{a.e. on }]0,T[$$

Indeed, for each $z\in H^1(\Omega)$ denoted by $\gamma_0 z$ the trace of z on $\partial\Omega$, it results

$$\gamma_0\left[u_{1\varepsilon}(t)-u_2(t)\right]^+ = max\left\{\gamma_0\left[u_{1\varepsilon}(t)-u_2(t)\right],0\right\}=max\left\{-\gamma_0 u_2(t),0\right\}=0$$

since

$$u_1(t)\le u_2(t) \quad \text{on} \quad \Omega\Rightarrow\gamma_0 u_2(t)\ge 0.$$

From (28), (31) we obtain

$$\sum_{1}^{n}{}_{ij}\int_\Omega a_{ij}(t,x)\frac{\partial u_{1\varepsilon}(t)}{\partial x_i}\ \frac{\partial\left[u_{1\varepsilon}(t)-u_2(t)\right]^+}{\partial x_j}+$$

$$-\int_\Omega\left(f_2(t)+b(t)u_2(t)\right)\left[u_{1\varepsilon}(t)-u_2(t)\right]^+dx+$$

$$+\frac{1}{\varepsilon}\int_{\Omega}\left(u_{1\varepsilon}(t)-u_1(t)\right)\left[u_{1\varepsilon}(t)-u_2(t)\right]^+ dx=0$$

a.e. on $]0,T[$

or

$$\sum_{1}^{n}{}_{ij}\int_{\Omega} a_{ij}(t,x)\frac{\partial\left[u_{1\varepsilon}(t)-u_2(t)\right]}{\partial x_i}\ \frac{\partial\left[u_{1\varepsilon}(t)-u_2(t)\right]^+}{\partial x_j}dx+$$

$$+\sum_{1}^{n}{}_{ij}\int_{\Omega} a_{ij}(t,x)\frac{\partial u_2(t)}{\partial x_i}\ \frac{\partial\left[u_{1\varepsilon}(t)-u_2(t)\right]^+}{\partial x_j}dx+$$

$$-\int_{\Omega}\left(f_2(t)+b(t)u_2(t)\right)\left[u_{1\varepsilon}(t)-u_2(t)\right]^+ dx+$$

$$+\frac{1}{\varepsilon}\int_{\Omega}\left(u_{1\varepsilon}(t)-u_1(t)\right)\left[u_{1\varepsilon}(t)-u_2(t)\right]^+ dx=0$$

a.e. on $]0,T[$

from which we deduce

$$\int_{\Omega}\left(u_{1\varepsilon}(t)-u_1(t)\right)\left[u_{1\varepsilon}(t)-u_2(t)\right]^+ dx\le 0 \tag{32}$$

a.e. on $]0,T[$

because

$$\sum_{1}^{n}{}_{ij}\int_{\Omega} a_{ij}(t,x)\frac{\partial\left[u_{1\varepsilon}(t)-u_2(t)\right]}{\partial x_i}\ \frac{\partial\left[u_{1\varepsilon}(t)-u_2(t)\right]^+}{\partial x_j}dx=$$

$$= \sum_{1\,ij}^{n} \int_{\Omega} a_{ij}(t,x) \frac{\partial [u_{1\varepsilon}(t) - u_2(t)]^+}{\partial x_i} \frac{\partial [u_{1\varepsilon}(t) - u_2(t)]^+}{\partial x_j} dx \geq 0$$

a.c. on $]0,T[$

and, since $\left(u_1, u_2 + [u_{1\varepsilon} - u_2]^+\right) \in \mathbf{K}$, moreover we get

$$\sum_{1\,ij}^{n} \int_{\Omega} a_{ij}(t,x) \frac{\partial u_2(t)}{\partial x_i} \frac{\partial [u_{1\varepsilon}(t) - u_2(t)]^+}{\partial x_j} dx +$$

$$- \int_{\Omega} \left(f_2(t) + b(t) u_2(t)\right) [u_{1\varepsilon}(t) - u_2(t)]^+ dx \geq 0$$

a.c. on $]0,T[$.

Let be $\Omega_t^+ = \{x \in \Omega / u_{1\varepsilon}(t,x) > u_1(t,x)\}$, holding (32), we have

$$\int_{\Omega_t^+} \left(u_{1\varepsilon}(t) - u_1(t)\right) [u_{1\varepsilon}(t) - u_2(t)]^+ dx = 0$$

a.c. on $]0,T[$

from which

$$u_{1\varepsilon}(t) \leq u_2(t) \quad on\, \Omega_t^+ \qquad \text{a.e. on }]0,T[$$

and consequently (30) is true.

The relations (28), (30) bring us to the upper limitation

$$\int_0^T \|A_1(t) u_{1\varepsilon}(t)\|_{L^2(\Omega)}^2 dx \leq \int_0^T \left(A_1(t) u_{1\varepsilon}(t), f_2(t) + b(t) u_2(t)\right) dx +$$

$$+\int_0^T \left(f_1(t) - u_1'(t), A_1(t)u_{1\varepsilon}(t) - f_2(t) - b(t)u_2(t)\right) dt$$

therefore

$$(33) \quad \int_0^T \left\|A_1(t)u_{1\varepsilon}(t)\right\|^2_{L^2(\Omega)} dt \le c \qquad (c = \text{const.} > 0 \text{ indep. of } \varepsilon)$$

and then, taking into account (29):

$$\int_0^T \left\|u_{1\varepsilon}(t)\right\|^2_{H^2(\Omega)} dt \le c \qquad (c = \text{const.} > 0 \text{ indep. of } \varepsilon)$$

From here we obtain that (26) holds taking into account that

$$u_{1\varepsilon} \to u_1 \quad in \quad L^2\left(0,T;L^2(\Omega)\right) \quad as \ \varepsilon \to 0$$

guaranteed by (28), (33).

Finally, starting from (26) and using the equation

$$a_2\left(t, u_2(t), \varphi\right) = \left(f_1(t) + f_2(t) - u_1'(t) - A_1(t)u_1(t), \varphi\right) \qquad \text{a.e. on }]0,T[$$
$$\forall \varphi \in C_0^\infty(\Omega)$$

and a well-known theorem of "inner" regularity for elliptic equations we come to (27).

REMARK. Other examples, where there is a greater regularity "respect to x" for the solution of problem (P), are suggested by the elliptic variational inequalities, which have been examined in [1], [2], [6], [11]. In fact the methods followed there can be used, duly changed, in the evolutive case we have considered in this work.

BIBLIOGRAFIA

1. M. BIANCARDI, *Su una disequazione variazione nello spazio* $H^2(\Omega) \times H^1\left(R^n\right)$ *relativa ad un operatore del quarto ordine*, Boll. UMI (6), 3-B (1984), 779-795.
2. M. BIANCARDI, A. ESPOSITO, L.A. MACCHETTA, *Un teorema di regolarità relativo ad un problema unilaterale*, Rend. Ist. Matem. Univ. Trieste, XVIII (1981), 58-70.
3. H. BRÉZIS, *Problèmes unilatéraux,* J. Math. Pures Appl. 51 (1972), 1-168.
4. H. BRÉZIS, *Inéquations variationelle associés a dès opérateurs d'évolution*, in A. Ghizzetti (ed.): Theory and applications of monotone operators, NATO Summer School Venice, Oderisi, 1969.
5. F. BROWDER, *Problémes non linéaires,* Conférence à Montreal, 1965.
6. A. ESPOSITO, *Un problema differenziale unilaterale del quarto ordine in due variabili*, Boll. Un. Mat. Ital., (6), 2-A (1983).
7. J.L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, Gauthier-Villars, 1969.
8. J.L. LIONS, E. MAGENES, *Problèmes aux limites non homogènes et applications*, vol. I, 1967.
9. J. NECAS, *Les mèthodes directes en théorie des équations elliptiques,* Mason, 1967.
10. R. TOSCANO, *Teoremi di esistenza e di regolarità per una classe di disequazioni variazionali di evoluzione del primo ordine,* Rend. Ist. Mat. Univ. Trieste, 17 (1985).
11. R. TOSCANO, A. GALLO, *Proprietà di regolarità della soluzione di una disequazione variazionale connessa a due operatori ellittici del secondo ordine,* Ricerche di Matematica, XXXIII (1984), 20-40.