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PRACTICAL ASPECTS IN SOME NOT CLASSICAL METHODS FOR THE NUMERICAL EVALUATION OF CPV-INTEGRALS

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SUNTO - In questo lavoro presentiamo due metodi non classici per la valutazione numerica di integrali pesati e con singolarità di tipo Cauchy (CPV). Il primo metodo è una quadratura interpolatoria di tipo gaussiano sul semidisco unitario. Il secondo è una integrazione numerica quasi interpolatoria basata sull'approssimazione spline.

ABSTRACT. - Two different not classical approaches are given for the numerical evaluation of weighted Cauchy principal value integrals. The first approach is an interpolatory Gaussian integration over the unit half disc. The second method is a quasi interpolatory integration rule based on spline approximation.

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INTRODUCTION

Frequently real systems modeling, e.g. aerodynamics or structural related problems, leads to integral equations having irrational kernel with real poles of first order in the integration interval.
Precisely, finding solutions of these equations, often yields singular integrals as:

$$I = \int_{-1}^{1} \frac{w(x)}{x-\lambda} f(x)dx \qquad (1)$$

where $w(x) = (1-x^2)^{\alpha-1/2}$ with $\alpha = 0, 1/2, 1$, and $-1<\lambda<1$.
In this work we shall discuss some approaches to the numerical evaluation of integrals (1). In particular we present some not classical methods to approximate (1), in alternative to classical methods which we briefly recall.
This paper is organized as follows: in section 1 we consider classical and not classical Gaussian methods and in section 2 classical and not classical quadrature based on particular approximating splines. Here we present a short survey of them, with the aim to collect the must significant results and make comments on the computational aspects, which may be useful to the reader.

1. GAUSSIAN METHODS

Let us consider integral (1), defined in the Cauchy principal value sense:

$$I = \lim_{\varepsilon \to 0^+} \int_{-1}^{\lambda-\varepsilon} \frac{w(x)}{x-\lambda} f(x)dx + \int_{\lambda+\varepsilon}^{1} \frac{w(x)}{x-\lambda} f(x)dx$$

A classical way for the numerical computation of integrals (1) is the following.
Using the transformation:

$$x = \frac{t+\lambda}{t\lambda+1}$$

we obtain:

$$I = w(\lambda) p.v. \int_{-1}^{1} w(t) g(\lambda,t) t^{-1} dt \qquad (2)$$

where:

$$g(\lambda,t)=\frac{f(\frac{t+\lambda}{t\lambda+1})}{(t\lambda+1)^{2\alpha}}$$

Then we generate quadrature formulas, based on the interpolation of the function g(λ,t) on n distinct points x_i $(i=1, 2,...,n)$ belonging to [-1,1], with a polynomial of degree $n-1$. From the integration of the polynomial the following quadrature formula holds:

$$I\cong\sum_{i=1}^{n}w_i g(\lambda,x_i) \qquad (3)$$

In (3) x_i $(i=1,.2,...,n)$ are called nodes of the quadrature; w_i $(i=1, 2,...,n)$ are the weights depending on $w(t)$, on λ, and on x_i (in particular $w_i = w'_i/x_i$ when $x_i \neq 0$). The formula (3) is exact at least for all polynomials of degree $n-1$ (polynomial order $n-1$).

When the nodes are the zeros of the polynomial of degree n orthogonal with respect to $w(t)$ in [-1,1] (in the case of (1) the orthogonal polynomials are called Chebyshev polynomials of first kind, for α = 1, of second kind, for α = 0, and Legendre polynomials, for α = 1/2) the corresponding quadrature formulas (3) are called Gaussian, and are exact for $g(\lambda,t) \in P_{2n-1}(t)$ (polynomial order $2n-1$).

Many authors proposed formulas of interpolatory type and related algorithms to evaluate the weights (for instance Delves, Hunters, Elliot, Paget, and Monegato). For an accurate bibliography see Monegato [11].

As it is well known, the accuracy— which is of high polynomial order — of the Gaussian formulas is related to the regularity of the function $f(x)$.

Nevertheless the non uniform distribution of the nodes x_i $(i=1, 2,...,n)$ and some cancellation problems affect sometimes adversely the accuracy of this type of quadrature formulas.

Gautschi and Milovanovic in [9] proposed an alternative Gaussian technique which makes the degree of the numerical cancellation in the computation of the quadrature 1-2 orders of magnitude smaller.

This technique approximates integrals (2) by a Gaussian quadrature formula on the semicircle in the complex plane.

With the assumption that $f(z)$ is an analytic function on the closed upper unit half disc, it follows:

$$I=w(\lambda)i(\pi f(\lambda)-\int_0^\pi w(e^{i\theta})g(\lambda,e^{i\theta})d\theta)$$

If $f(z)$ is real for real z, then:

$$I = w(\lambda)\,\mathrm{Im}\int_0^\pi w(e^{i\theta})g(\lambda,e^{i\theta})d\theta$$

To approximate integral (1) Gautschi and Milovanovic in [9] and Milovanovic in [10] proposed the application of the following complex gaussian quadrature formula:

$$I \cong w(\lambda)\,\mathrm{Im}(\sum_{i=1}^{n}\delta_i g(\lambda,\xi_i)) \tag{4}$$

where ξ_i $(i=1, 2, \ldots, n)$ are the zeros of a suitable complex polynomial $\pi_n(z)$ and δ_i $(i=1, 2, \ldots, n)$ are the corresponding weights.
$\{\pi_k(z)\}$ of degree precisely k is the class of the polynomials orthogonal with respect to the non hermitian inner product:

$$\langle p,q\rangle = \int_\Gamma (iz)^{-1} w(z)p(z)q(z)dz$$

where $w(z)$ is as in (1) and Γ is the circular part of the boundary of D_+, where:

$$D_+ = \{z \in C: |z| < 1, \mathrm{Im}(z) > 0\}$$

The comparison between formulas (4) and (3) shows the better behaviour of (4) in terms of rounding errors and consequently better algorithmic stability. Unfortunately formula (4) concentrates again the nodes in the neighbourhood of the extremes of the integration interval.
In [2] and [3] we proposed an extension of formula (4) which provides a better distribution of the quadrature nodes.
Precisely considering the complex polynomial $E_{n+1}(z)$ of degree $n+1$ satisfying the following orthogonality conditions:

$$\int_\Gamma (iz)^{-1} w(z)\pi_n(z)E_{n+1}(z)z^k dz = 0, k = 0,1,\ldots,n$$

to approximate (2) we proposed the application of the following extended complex Gaussian quadrature formula:

$$I \cong w(\lambda)\,\mathrm{Im}(\sum_{i=1}^{n}\sigma_i g(\lambda,\xi_i) + \sum_{j=1}^{n+1}\tau_j g(\lambda,\zeta_j)) \tag{5}$$

where ξ_i $(i=1, 2, ..., n)$ are the zeros of the complex polynomial $\pi_n(z)$, ζ_j $(j=1, 2, ..., n+1)$ are the zeros of the complex polynomial $E_{n+1}(z)$, σ_i, τ_j the corresponding weights.
In [1, 2] we have proved that the zeros of $E_{n+1}(z)$ are non real, simple, contained in the upper unit half disc D_+ and located symmetrically with respect to the imaginary axis; in [3] we have also tested formula (5) and compared it with (4) in the approximation of some integrals of type (1). In particular, for λ belonging to a neighbourhood of the extremes of the integration interval, we showed the better behaviour of (5), which has a suitable distribution of nodes in the upper unit half disc.

3. APPROXIMATING SPLINE METHODS

We point out that the above mentioned "Gaussian " or "extended Gaussian" methods as n increases converge very rapidly, when they do, if applied to integrals (1) with analytical functions. However very frequently the regular behaviour of function $f(x)$ is lost in practical applications.
Moreover, those formulas require that the function be calculated at the zeros of particular polynomials, what is not always convenient.
Facing such problems, recently an alternative approach, to evaluate (1), based on the spline approximation has been selected.
We recall that the spline functions of order p on [a,b] are the piecewise polynomials of degree p in subintervals of [a,b], with conditions of regularity in the connection points, which build the vector of definition of the splines [8].
A typical expression for such approximation is the following:

$$I \cong \sum_{i=1}^{n} w_i(\lambda) f(x_i)$$

where the set of points x_i $(i=1, 2,...,n)$ is the so called mesh and $w_i(\lambda)$ $(i=1, 2,...,n)$ are the weights evaluated through spline approximation.
In an early paper [6] an interpolating cubic spline $(p=3)$, is proposed to approximate integral (1). To have convergence, the integrand function $f(x)$ is required to be continuous with its first derivative, in the integration interval, and the mesh to be equally spaced. Therefore, paper [6] doesn't effectively overcome the two drawbacks of Gaussian formulas.
In a subsequent paper [7] restrictions, as far as the function to be integrated is concerned, are slightly relaxed. However some strong limitations on the mesh are still present. Besides, some restrictions on the spline order and on border conditions are required.

Recently, in paper [12], Rabinowitz gives a new idea. Namely he proposes to apply, to approximate integral (1), a class of approximating splines of order p:

$$S_n(x) = \sum_{i=1}^{n} f(x_i) B_{i,p}(x)$$

where $B_{i,p}(x)$ are the B-splines [8] of order p, forming a basis for the spline space determined by p and by the vector of nodes $\Pi_n = (t_i, \; i = 1, 2, \ldots, n)$, and x_i $(i=1, 2, \ldots, n)$ are arbitrary points subjected only to the condition that x_i lies in the support of $B_{i,p}(x)$.
Consequently, as $f(x)$ is approximated by $S_n(x)$, then:

$$w_i(\lambda) = p.v. \int_{-1}^{1} w(x) \frac{B_{i,p}(x)}{x-\lambda} dx$$

A particularly case occurs when x_i $(i=1, 2, \ldots, n)$ are the so-called Schoenberg points, precisely

$$x_i = \frac{t_{i+1} + t_{i+2} + \ldots + t_{i+p-1}}{p-1}$$

in which case the spline is called variational diminishing Schoenberg spline (SVD).
In [4] we draw that the conditions on $f(x)$ for the convergence of the quadrature formula are the same as for the interpolating spline method [7].
We note that this method does not place any restriction neither on the order of the approximating spline, nor on the mesh. So the nodes of the mesh can have multiplicity greater than one and an arbitrary distribution.
This property is very useful because a suitable distribution of the nodes in the integration interval enables us to perform the irregularities of the function, (what is impossible with Gaussian formulas).
Some examples showing the advantages of the formula based on the SVD splines are also presented in [4, 5].
Moreover a new application of these formulas is presented in [5]. Precisely we have approximated the solution of the particular integro-differential equation known as Prandtl's equation. It must be noted that, as this equation presents the derivative of the unknown function in the CPV integral, then it has been necessary to provide a suitable adaptation of the formulas used in [4].

CONCLUDING REMARKS

In view of the application of quadrature formulas to the solution of integral equations with weighted CPV integrals the following remarks can be useful. If the model can be described by a sufficiently regular function, then gaussian or quasi-gaussian formulas realize computational advantages by a reduction of the order of the system (said collocation system), which provides the values of the unknown function.
On the contrary, if the model is described by a function of low order of regularity, then approximating splines perform better.

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