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Rough sets applied in sublattices and ideals of lattices

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Abstract

The purpose of this paper is the study of rough hyperlattice. In this regards we introduce rough sublattice and rough ideals of lattices. We will proceed by obtaining lower and upper approximations in these lattices.

Keywords: rough set, lower approximation, upper approximation, rough sublattice, rough ideal

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1 Introduction

Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory (for more details see[3, 13]). Lattices are partially ordered sets in which least upper bounds and greatest lower bounds of any two elements exist. A lattice is a set on which two operations are defined, called *join* and *meet* and denoted by $\vee$

and $\wedge$, which satisfy the *idempotent*, *commutative* and *associative* laws, as well as the *absorption* laws:
$a \vee (b \wedge a) = a$,
$a \wedge (b \vee a) = a$.

Lattices are better behaved than partially ordered sets lacking upper or lower bounds.

The concept of rough set was originally proposed by Pawlak [21, 22] as a formal tool for modeling and processing incomplete information in information systems. Since then the subject has been investigated in many papers (see [20, 23, 24]). The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. Some authors, for example, Bonikowaski [5], Iwinski [15], and Pomykala and Pomykala [24] studied algebraic properties of rough sets. The lattice theoretical approach has been suggested by Iwinski [15]. In this paper we concentrates on the relationship between rough sets and lattice theory. We introduce the notion of rough sublattices (resp. ideals) of lattices, and investigate some properties of lower and upper approximations in lattices.

2 Preliminaries

Suppose that U is a non-empty set. A *partition* or *classification* of U is a family P of non-empty subsets of U such that each element of U is contained in exactly one element of P. Recall that an *equivalence relation* on a set U is a reflexive, symmetric, and transitive binary relation on U. Each partition P induces an equivalence relation θ on U by setting:
$x\theta y \Leftrightarrow$ x and y are in the same class of P.

Conversely, each equivalence relation θ on U induces a partition P of U whose classes have the form $[x]_\theta = \{y \in U \mid x\theta y\}$.

Given a non-empty universe U, by $P(U)$ we will denote the power set on U. If θ is an equivalence relation on U then for every $x \in U$, $[x]_\theta$ denotes the equivalence class of θ determined by x. For any $X \subseteq U$, we write X^c to denote the complementation of X in U, that is the set $U \setminus X$.

Definition 2.1. [8] A pair(U, θ); where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an *approximation space*.

Definition 2.2. [8] For an approximation space (U, θ), by a *rough approximation* in (U, θ) we mean a mapping $\mathfrak{A} : P(U) \to P(U) \times P(U)$ defined by for every $X \in P(U)$, $\mathfrak{A}(X) = (\underline{\mathfrak{A}}(X), \overline{\mathfrak{A}}(X))$ where $\underline{\mathfrak{A}}(X) = \{x \in X \mid [x]_\theta \subseteq X\}$, $\overline{\mathfrak{A}}(x) = \{x \in X \mid [x]_\theta \cap X \neq \emptyset\}$. $\underline{\mathfrak{A}}(X)$ is called a *lower rough approximation* of X in (U, θ), where as $\overline{\mathfrak{A}}(X)$ is called *upper rough approximation* of X in (U, θ).

Definition 2.3. [8] Given an approximation space (U, θ) a pair $(A, B) \in P(U) \times P(U)$ is called a *rough set* in (U, θ) iff $(A, B) = \mathfrak{A}(X)$ for some $X \in P(U)$.

For the sake of illustration, let (U, θ) is an approximation space, where:
$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, and an equivalence relation θ with the following equivalence classes:
$E_1 = \{x_1, x_4, x_8\}$,
$E_2 = \{x_2, x_5, x_7\}$,
$E_3 = \{x_3\}$,
$E_4 = \{x_6\}$,
Let $X = \{x_3, x_5\}$, then $\underline{\mathfrak{A}}(X) = \{x_3\}$ and $\overline{\mathfrak{A}}(X) = \{x_2, x_3, x_5, x_7\}$ and so $(\{x_3\}, \{x_2, x_3, x_5, x_7\}) = \mathfrak{A}(X)$ is a rough set.
The reader will find in [18,21-25] a deep study of rough set theory.

Definition 2.4. [7] A subset $X$of U is called $definable$ if $\underline{\mathfrak{A}}(X) = \overline{\mathfrak{A}}(X)$. If $X \subseteq U$ given by a predicate P and $x \in U$, then:
1. $x \in \underline{\mathfrak{A}}(X)$ means that x certainly has property P,
2. $x \in \overline{\mathfrak{A}}(X)$ means that x possibly has property P,
3. $x \in U \setminus \overline{\mathfrak{A}}(X)$ means that x definitely does not have property P.

When $\mathfrak{A}(A) \sqsubseteq \mathfrak{A}(B)$, we say that $\mathfrak{A}(A)$ is a *rough subset* of $\mathfrak{A}(B)$. Thus in the case of rough sets $\mathfrak{A}(A)$ and $\mathfrak{A}(B)$, $\mathfrak{A}(A) \sqsubseteq \mathfrak{A}(B)$ if and only if $\underline{\mathfrak{A}}(A) \subseteq \underline{\mathfrak{A}}(B)$ and $\overline{\mathfrak{A}}(A) \subseteq \overline{\mathfrak{A}}(B)$. This property of rough inclusion has all the properties of set inclusion. The rough complement of $\mathfrak{A}(A)$ denoted by $\mathfrak{A}^c(A)$ is defined by: $\mathfrak{A}^c(A) = (U \setminus \overline{\mathfrak{A}}(A), U \setminus \underline{\mathfrak{A}}(A))$. Also, we can define $\mathfrak{A}(A) \setminus \mathfrak{A}(B)$ as follows:
$\mathfrak{A}(A) \setminus \mathfrak{A}(B) = \mathfrak{A}(A) \sqcap \mathfrak{A}^c(B) = (\underline{\mathfrak{A}}(A) \setminus \overline{\mathfrak{A}}(B), \overline{\mathfrak{A}}(A) \setminus \underline{\mathfrak{A}}(B))$.

Let L be a lattice and $S \subseteq L$, If S is a lattice, then S is called a *sublattice* of L. A sublattice I is called an *ideal* of L, if $a \in L$ and $x \in I$ imply $a \wedge x \in L$

(see[2]).

Let ρ be an equivalence relation on L and x, y, $z \in L$.

(1) ρ is called a *congruence relation* if $x\rho y$ implies $(x \vee z)\rho(y \vee z)$ and $(x \wedge z)\rho(y \wedge z)$.

(2) ρ is called a *complete congruence relation* if $[x]_\rho \vee [y]_\rho = [x \vee y]_\rho$, and $[x]_\rho \wedge [y]_\rho = [x \wedge y]_\rho$.

If ρ is a congruence relation on L, then it is easy to verify that $[x]_\rho \vee [y]_\rho \subseteq [x \vee y]_\rho$, $[x]_\rho \wedge [y]_\rho \subseteq [x \wedge y]_\rho$.

3 Rough ideals of lattices

Throughout this paper L denotes a lattice. Let ρ be an equivalence relation on L and X be a non-empty subset of L. When $U = L$ and θ is the above equivalence relation, then we use the pair (L, ρ) instead of the approximation space (U, θ). Also, in this case we use the symbols $\underline{\mathfrak{A}}_\rho(X)$ and $\overline{\mathfrak{A}}_\rho(X)$ instead of $\underline{\mathfrak{A}}(X)$ and $\overline{\mathfrak{A}}(X)$.

Proposition 3.1. For every approximation space (L, ρ), where ρ is an equivalence relation, and every subsets A, $B \subseteq L$, we have:

(1) $\underline{\mathfrak{A}}_\rho(A) \subseteq A \subseteq \overline{\mathfrak{A}}_\rho(A)$;

(2) $\underline{\mathfrak{A}}_\rho(\emptyset) = \emptyset = \overline{\mathfrak{A}}_\rho(\emptyset)$;

(3) $\underline{\mathfrak{A}}_\rho(L) = L = \overline{\mathfrak{A}}_\rho(L)$;

(4) If $A \subseteq B$, then $\underline{\mathfrak{A}}_\rho(A) \subseteq \underline{\mathfrak{A}}_\rho(B)$, and $\overline{\mathfrak{A}}_\rho(A) \subseteq \overline{\mathfrak{A}}_\rho(B)$;

(5) $\underline{\mathfrak{A}}_\rho(\underline{\mathfrak{A}}_\rho(A)) = \underline{\mathfrak{A}}_\rho(A)$;

(6) $\overline{\mathfrak{A}}_\rho(\overline{\mathfrak{A}}_\rho(A)) = \overline{\mathfrak{A}}_\rho(A)$;

(7) $\overline{\mathfrak{A}}_\rho(\underline{\mathfrak{A}}_\rho(A)) = \underline{\mathfrak{A}}_\rho(A)$;

(8) $\underline{\mathfrak{A}}_\rho(\overline{\mathfrak{A}}_\rho(A)) = \overline{\mathfrak{A}}_\rho(A)$;

(9) $\underline{\mathfrak{A}}_\rho(A) = (\overline{\mathfrak{A}}_\rho(A^c))^c$;

(10) $\overline{\mathfrak{A}}_\rho(A) = (\underline{\mathfrak{A}}_\rho(A^c))^c$;

(11) $\underline{\mathfrak{A}}_\rho(A \cap B) = \underline{\mathfrak{A}}_\rho(A) \cap \underline{\mathfrak{A}}_\rho(B)$;

(12) $\overline{\mathfrak{A}}_\rho(A \cap B) \subseteq \overline{\mathfrak{A}}_\rho(A) \cap \overline{\mathfrak{A}}_\rho(B)$;

(13) $\underline{\mathfrak{A}}_\rho(A \cup B) \supseteq \underline{\mathfrak{A}}_\rho(A) \cup \underline{\mathfrak{A}}_\rho(B)$;

(14) $\overline{\mathfrak{A}}_\rho(A \cup B) = \overline{\mathfrak{A}}_\rho(A) \cup \overline{\mathfrak{A}}_\rho(B)$;

(15) $\underline{\mathfrak{A}}_\rho([x]_\rho) = \overline{\mathfrak{A}}_\rho([x]_\rho)$ for all $x \in L$;

Proof. (15) $\underline{\mathfrak{A}}_\rho([x]_\rho) = \{y \in L \mid [y]_\rho \subseteq [x]_\rho\} = [x]_\rho$, and $\overline{\mathfrak{A}}_\rho([x]_\rho) = \{y \in L \mid [y]_\rho \cap [x]_\rho \neq \emptyset\} = [x]_\rho$. Hence $\underline{\mathfrak{A}}_\rho([x]_\rho) = \overline{\mathfrak{A}}_\rho([x]_\rho)$.

The other parts of the proof is similar to the [17, Theorem 2.1] and [7, Proposition 4.1]. □

The following example shows that the converse of (12) and (13) in Proposition 3.1 are not true.

Example 3.2. Let $L = \{1, 2, ..., 8\}$, Then $(L, \wedge, \vee)$ is a lattice, where $\forall a, b \in L$, $a \wedge b = min\{a, b\}$, $a \vee b = max\{a, b\}$. Let ρ be an equivalence relation on L with the following equivalence classes:
$[1]_\rho = \{1, 4, 8\}$,
$[2]_\rho = \{2, 5, 7\}$,
$[3]_\rho = \{3\}$,
$[6]_\rho = \{6\}$,
and $A = \{3, 5, 7\}$, $B = \{2, 6\}$. Then:

$\underline{\mathfrak{A}}_\rho(A) = \{3\}$,
$\underline{\mathfrak{A}}_\rho(B) = \{6\}$,
$\underline{\mathfrak{A}}_\rho(A \cup B) = \{2, 3, 5, 6, 7\}$,
$\overline{\mathfrak{A}}_\rho(A) = \{2, 3, 5, 7\}$,
$\overline{\mathfrak{A}}_\rho(B) = \{2, 5, 6, 7\}$,
$\overline{\mathfrak{A}}_\rho(A \cap B) = \emptyset$,
and so $\overline{\mathfrak{A}}_\rho(A) \cap \overline{\mathfrak{A}}_\rho(B) \nsubseteq \overline{\mathfrak{A}}_\rho(A \cap B)$, $\underline{\mathfrak{A}}_\rho(A \cup B) \nsubseteq \underline{\mathfrak{A}}_\rho(A) \cup \underline{\mathfrak{A}}_\rho(B)$.

Corollary 3.3. For every approximation space (L, ρ),
(i) For every $A \subseteq L$, $\underline{\mathfrak{A}}_\rho(A)$ and $\overline{\mathfrak{A}}_\rho(A)$ are definable sets,
(ii) For every $x \in L$, $[x]_\rho$ is definable set.

Proof. It is immediately by Proposition 3.1 (parts (5), (6), (7), (8) and (15)). □

If A and B are non-empty subsets of L, let $A \wedge B$ and $A \vee B$ denotes the following sets:

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\},\ A \vee B = \{a \vee b \mid a \in A, b \in B\}.$$

Proposition 3.4. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B) = \overline{\mathfrak{A}}_\rho(A \wedge B)$.

Proof. Suppose z be any element of $\overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B)$, then $z = a \wedge b$ for some $a \in \overline{\mathfrak{A}}_\rho(A)$, $b \in \overline{\mathfrak{A}}_\rho(B)$, hence $[a]_\rho \cap A \neq \emptyset$ and $[b]_\rho \cap B \neq \emptyset$ and so there exist $x \in [a]_\rho \cap A$ and $y \in [b]_\rho \cap B$. Therefore $x \wedge y \in A \wedge B$

and $x \wedge y \in [a]_\rho \wedge [b]_\rho = [a \wedge b]_\rho$ hence $[a \wedge b]_\rho \cap (A \wedge B) \neq \emptyset$ and so $\overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B) \subseteq \overline{\mathfrak{A}}_\rho(A \wedge B)$.

Conversely, let $x \in \overline{\mathfrak{A}}_\rho(A \wedge B)$ then $[x]_\rho \cap (A \wedge B) \neq \emptyset$ hence there exists $y \in [x]_\rho$ and $y \in A \wedge B$ and so $y = a \wedge b$ for some $a \in A$ and $b \in B$. Now we have $x \in [y]_\rho = [a \wedge b]_\rho = [a]_\rho \wedge [b]_\rho$. Then there exist $x' \in [a]_\rho$ and $y' \in [b]_\rho$ such that $x = x' \wedge y'$. Since $a \in [x']_\rho \cap A$ and $b \in [y']_\rho \cap B$, hence $x' \in \overline{\mathfrak{A}}_\rho(A)$ and $y' \in \overline{\mathfrak{A}}_\rho(B)$, which yields that $x = x' \wedge y' \in \overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B)$ and so $\overline{\mathfrak{A}}_\rho(A \wedge B) \subseteq \overline{\mathfrak{A}}_\rho(A) \wedge \overline{\mathfrak{A}}_\rho(B)$. $\square$

Proposition 3.5. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\overline{\mathfrak{A}}_\rho(A) \vee \overline{\mathfrak{A}}_\rho(B) = \overline{\mathfrak{A}}_\rho(A \vee B)$.

Proof. The proof is similar to the proof of Proposition 3.4, by considering the suitable modification by using the definition of $A \vee B$. $\square$

Proposition 3.6. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B) \subseteq \underline{\mathfrak{A}}_\rho(A \wedge B)$.

Proof. Suppose x be any element of $\underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B)$ then $x = a \wedge b$ for some $a \in \underline{\mathfrak{A}}_\rho(A)$ and $b \in \underline{\mathfrak{A}}_\rho(B)$. Hence $[a]_\rho \subseteq A$ and $[b]_\rho \subseteq B$. Since $[a \wedge b]_\rho = [a]_\rho \wedge [b]_\rho \subseteq A \wedge B$, we get $a \wedge b \in \underline{\mathfrak{A}}_\rho(A \wedge B)$ and so $x \in \underline{\mathfrak{A}}_\rho(A \wedge B)$. $\square$

The following example shows that the converse of Proposition 3.6 is not true.

Example 3.7. Let $L = \{0, 1, 2, ..., 11\}$, Then $(L, \wedge, \vee)$ is a lattice, where $\forall a, b \in L$, $a \wedge b = min\{a, b\}$, $a \vee b = max\{a, b\}$. Let ρ be a complete congruence relation on L with the following equivalence classes:
$[0]_\rho = \{0, 1, 2\}$,
$[3]_\rho = \{3, 4, 5\}$,
$[6]_\rho = \{6, 7, 8\}$,
$[9]_\rho = \{9, 10, 11\}$,
and $A = \{1, 3, 4, 5\}$, $B = \{0, 1, 2, 6, 8\}$. Then:

$\underline{\mathfrak{A}}_\rho(A) = \{3, 4, 5\}$,
$\underline{\mathfrak{A}}_\rho(B) = \{0, 1, 2\}$,
$A \wedge B = \{0, 1, 2, 3, 4, 5\}$
$\underline{\mathfrak{A}}_\rho(A \wedge B) = \{0, 1, 2, 3, 4, 5\}$,
$\underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B) = \{0, 1, 2\}$
and so $\underline{\mathfrak{A}}_\rho(A \wedge B) \nsubseteq \underline{\mathfrak{A}}_\rho(A) \wedge \underline{\mathfrak{A}}_\rho(B)$.

Proposition 3.8. Let ρ be a complete congruence relation on L, and A, B non-empty subsets of L, then $\underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B) \subseteq \underline{\mathfrak{A}}_\rho(A \vee B)$.

Proof. The proof is similar to the proof of Proposition 3.6, by considering the suitable modification by using the definition of $A \vee B$. $\square$

The following example shows that $\underline{\mathfrak{A}}_\rho(A \vee B) \subseteq \underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B)$ does not hold in general.

Example 3.9. Let $L = \{0, 1, 2, ..., 8\}$, Then $(L, \wedge, \vee)$ is a lattice, where $\forall a, b \in L$, $a \wedge b = min\{a, b\}$, $a \vee b = max\{a, b\}$. Let ρ be a complete congruence relation on L with the following equivalence classes:
$[0]_\rho = \{0, 1, 2\}$,
$[3]_\rho = \{3, 4\}$,
$[5]_\rho = \{5, 6, 7, 8\}$,
and $A = \{3, 4, 5, 7\}$, $B = \{0, 1, 2, 3, 6, 8\}$. Then:

$\underline{\mathfrak{A}}_\rho(A) = \{3, 4\}$,
$\underline{\mathfrak{A}}_\rho(B) = \{0, 1, 2\}$,
$A \vee B = \{3, 4, 5, 6, 7, 8\}$,
$\underline{\mathfrak{A}}_\rho(A \vee B) = \{3, 4, 5, 6, 7, 8\}$,
$\underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B) = \{3, 4\}$,
and so $\underline{\mathfrak{A}}_\rho(A \vee B) \nsubseteq \underline{\mathfrak{A}}_\rho(A) \vee \underline{\mathfrak{A}}_\rho(B)$

Lemma 3.10. Let ρ_1 and ρ_2 be two complete congruence relations on L such that $\rho_1 \subseteq \rho_2$ and let A be a non-empty subset of L, then:
(i) $\underline{\mathfrak{A}}_{\rho_2}(A) \subseteq \underline{\mathfrak{A}}_{\rho_1}(A)$,
(ii) $\overline{\mathfrak{A}}_{\rho_1}(A) \subseteq \overline{\mathfrak{A}}_{\rho_2}(A)$.

Proof. It is straightforward. $\square$

The following Corollary follows from Lemma 3.10.

Corollary 3.11. Let ρ_1 and ρ_2 be two complete congruence relations on L and A a non-empty subset of L, then:
(i) $\underline{\mathfrak{A}}_{\rho_1}(A) \cap \underline{\mathfrak{A}}_{\rho_2}(A) \subseteq \underline{\mathfrak{A}}_{(\rho_1 \cap \rho_2)}(A)$,
(ii) $\overline{\mathfrak{A}}_{(\rho_1 \cap \rho_2)}(A) \subseteq \overline{\mathfrak{A}}_{\rho_1}(A) \cap \overline{\mathfrak{A}}_{\rho_2}(A)$.

Proposition 3.12. Let ρ be a congruence relation on L, and J be an ideal of L, then $\overline{\mathfrak{A}}_\rho(J)$ is an ideal of L.

Proof. Suppose $a, b \in \overline{\mathfrak{A}}_\rho(J)$ and $r \in L$, then $[a]_\rho \cap J \neq \emptyset$ and $[b]_\rho \cap J \neq \emptyset$. So there exist $x \in [a]_\rho \cap J$ and $y \in [a]_\rho \cap J$. Since J is an ideal of L, we have $x \vee y \in J$ and $x \vee y \in [a]_\rho \vee [b]_\rho \subseteq [a \vee b]_\rho$. Hence $[a \vee b]_\rho \cap J \neq \emptyset$ which implies $a \vee b \in \overline{\mathfrak{A}}_\rho(J)$. Also, we have $r \wedge x \in J$ and $r \wedge x \in [r]_\rho \wedge [a]_\rho \subseteq [r \wedge a]_\rho$. So $[r \wedge a]_\rho \cap J \neq \emptyset$ which implies $r \wedge a \in \overline{\mathfrak{A}}_\rho(J)$. Therefore $\overline{\mathfrak{A}}_\rho(J)$ is an ideal of L. □

Similarly, if ρ is a congruence relation on L and J is a sublattice of L, then $\overline{\mathfrak{A}}_\rho(J)$ is a sublattice of L.

Proposition 3.13. Let ρ be a complete congruence relation on L, and J be an ideal of L, then $\underline{\mathfrak{A}}_\rho(J)$ is an ideal of L.

Proof. Suppose a, $b \in \underline{\mathfrak{A}}_\rho(J)$ and $r \in L$, then $[a]_\rho \subseteq J$ and $[b]_\rho \subseteq J$. So $[a \vee b]_\rho = [a]_\rho \vee [b]_\rho \subseteq J$, and $[r \wedge a]_\rho = [a]_\rho \wedge [b]_\rho \subseteq J$. Hence $a \vee b \in \underline{\mathfrak{A}}_\rho(J)$ and $r \wedge a \in \underline{\mathfrak{A}}_\rho(J)$. □

Similarly, if ρ is a complete congruence relation on L and J is a sublattice of L, then $\underline{\mathfrak{A}}_\rho(J)$ is a sublattice of L.

Definition 3.14. Let ρ be a congruence relation on L and $\mathfrak{A}_\rho(A) = (\underline{\mathfrak{A}}_\rho(A), \overline{\mathfrak{A}}_\rho(A))$ a *rough set* in the approximation space (L, ρ). If $\underline{\mathfrak{A}}_\rho(A)$ and $\overline{\mathfrak{A}}_\rho(A)$ are ideals (resp. sublattice) of L, then we call $\mathfrak{A}_\rho(A)$ a *rough ideal* (resp. *sublattice*). Note that a rough sublattice also is called a *rough lattice*.

Corollary 3.15. (i) Let ρ, be a congruence relation on L, and I an ideal of L then $\mathfrak{A}_\rho(I)$ is a rough ideals.

(ii) Let ρ be a complete congruence relation on L and J a sublattice of L, then $\mathfrak{A}_\rho(J)$ is a rough lattice.

Proof. It is obtained by 3.12 and 3.13. □

Let L and L' be two lattices, a map $f : L \to L'$ is said to be *homomor−phism* or (*lattice homomorphism*) if for all a, $b \in L$, $f(a \wedge b) = f(a) \wedge f(b)$, and $f(a \vee b) = f(a) \vee f(b)$.

Now, let L and L' be two lattices and $f : L \to L'$ a homomorphism. It is well known, $\theta = \{(a, b) \in L \times L \mid f(a) = f(b)\} \subseteq L \times L$ is a congruence relation on L. Because if $a\theta b$ then $f(a) = f(b)$ and for all $z \in L$, we have $f(a \wedge z) = f(a) \wedge f(z) = f(b) \wedge f(z) = f(b \wedge z)$. Therefor $(a \wedge z)\ \theta\ (b \wedge z)$, and similarly $(a \vee z)\ \theta\ (b \vee z)$.

Theorem 3.16. Let L and L' be two lattices and $f : L \to L'$ a homomorphism. If A is a non-empty subset of L, then $f(\overline{\mathfrak{A}}_\theta(A)) = f(A)$.

Proof. Since $A \subseteq \overline{\mathfrak{A}}_\theta(A)$ it follows that $f(A) \subseteq f(\overline{\mathfrak{A}}_\theta(A))$.

Conversely, let $y \in f(\overline{\mathfrak{A}}_\theta(A))$. Then there exists an element $x \in \overline{\mathfrak{A}}_\theta(A)$, such that $f(x) = y$, so we have $[x]_\theta \cap A \neq \emptyset$. Thus there exists an element $a \in [x]_\theta \cap A$. Then $a \in [x]_\theta$, hence $x\theta a$, and so $f(x) = f(a) \in f(A)$, therefore $f(\overline{\mathfrak{A}}_\theta(A)) \subseteq f(A)$. $\square$

Let $f : L \to L'$ be a homomorphism and A a subset of L, Since $\underline{\mathfrak{A}}_\theta(A) \subseteq A$ it follows that $f(\underline{\mathfrak{A}}_\theta(A)) \subseteq f(A)$. But the following example shows that, in general, $f(\underline{\mathfrak{A}}_\theta(A)) \neq f(A)$.

Example 3.17. Let $(L, \wedge, \vee)$ and $(L', \wedge, \vee)$ be two lattices where $L = \{1, 2, 3, 4\}$; and $L' = \{5, 6, 7\}$; and for all s, t in L or L', $s \wedge t= min\{s, t\}$ and $s \vee t = max\{s, t\}$. The map $f : L \to L'$ given by

$$f(4) = f(3)= 7,\ f(2) = 6,\ f(1) = 5,$$

is a homomorphism. We have $\theta = \{3, 4\}$. Suppose $A = \{1, 2\}$, then $f(A) = \{5, 6\}$, $\underline{\mathfrak{A}}_\theta(A) = \emptyset$ and $f(\underline{\mathfrak{A}}_\theta(A)) = \emptyset$, and so $f(\underline{\mathfrak{A}}_\theta(A)) \neq f(A)$.

The lower and upper approximations can be presented in an equivalent form as follows:

Let L be a lattice, ρ a congruence relation on L, and A a non-empty subset of L. Then we define $\overline{\vee}$ and $\overline{\wedge}$ on $L/\rho = \{[x]_\rho \mid x \in L\}$, by

$$[x]_\rho \overline{\vee} [y]_\rho = [x \vee y]_\rho,\ [x]_\rho \overline{\wedge} [y]_\rho = [x \wedge y]_\rho.$$

This relation is well-defined, since if $[x_1]_\rho = [x_2]_\rho$ and $[y_1]_\rho = [y_2]_\rho$, then $x_1 \rho x_2$ and $y_1 \rho y_2$. Since ρ is a congruence relation we have $(x_1 \vee y_1)\rho(x_2 \vee y_1)$ and $(x_2 \vee y_1)\rho(x_2 \vee y_2)$. Then $(x_1 \vee y_1)\rho(x_2 \vee y_2)$, so $[x_1 \vee y_1]_\rho = [x_2 \vee y_2]_\rho$. Therefore $[x_1]_\rho \overline{\vee} [y_1]_\rho = [x_2]_\rho \overline{\vee} [y_2]_\rho$.

It is easy to see that $(L/\rho, \overline{\vee}, \overline{\wedge})$, is a lattice. Also if $A \neq \emptyset$, and $A \subseteq L$ put $\underline{\underline{\mathfrak{A}}}_\rho(A) = \{[x]_\rho \in L/\rho \mid [x]_\rho \subseteq A\}$ and $\overline{\overline{\mathfrak{A}}}_\rho(A) = \{[x]_\rho \in L/\rho \mid [x]_\rho \cap A \neq \emptyset\}$.

Proposition 3.18. Let ρ be a congruence relation on L and J be an ideal of L, then $\overline{\overline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ.

Proof. Assume that $[a]_\rho$, $[b]_\rho \in \overline{\overline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \in L/\rho$. Then $[a]_\rho \cap J \neq \emptyset$ and $[b]_\rho \cap J \neq \emptyset$, so there exist $x \in [a]_\rho \cap J$ and $y \in [b]_\rho \cap J$. Since J is an ideal of L, we have $x \vee y \in J$ and $r \wedge x \in J$. Also, we have $x \vee y \in [a]_\rho \vee [b]_\rho \subseteq [a \vee b]_\rho$, and $r \wedge x \in [r]_\rho \wedge [a]_\rho \subseteq [r \wedge a]_\rho$. Therefore $[a \vee b]_\rho \cap J \neq \emptyset$ and $[r \wedge a]_\rho \cap J \neq \emptyset$, which imply $[a]_\rho \vee [b]_\rho \in \overline{\overline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \wedge [a]_\rho \in \overline{\overline{\mathfrak{A}}}_\rho(J)$. Therefore $\overline{\overline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ. $\square$

Proposition 3.19. Let ρ be a complete congruence relation on L and J be an ideal of L, then $\underline{\underline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ.

Proof. Assume that $[a]_\rho$, $[b]_\rho \in \underline{\underline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \in L/\rho$. Then $[a]_\rho \subseteq J$ and $[b]_\rho \subseteq J$. Since J is an ideal of L, we have $a \vee b \in J$ and $r \wedge a \in J$ Therefore $[a]_\rho \vee [b]_\rho = [a \vee b]_\rho \subseteq J \vee J = J$, and $[r]_\rho \wedge [a]_\rho = [r \wedge a]_\rho \subseteq J$, which imply $[a]_\rho \vee [b]_\rho \in \underline{\underline{\mathfrak{A}}}_\rho(J)$ and $[r]_\rho \wedge [a]_\rho \in \underline{\underline{\mathfrak{A}}}_\rho(J)$. Therefore $\underline{\underline{\mathfrak{A}}}_\rho(J)$ is an ideal of L/ρ. $\square$

Proposition 3.20. (i) Let ρ be a congruence relation on L and J a sublattice of L, then $\overline{\overline{\mathfrak{A}}}_\rho(J)$ is a sublattice of L/ρ.

(ii) Let ρ be a complete congruence relation on L and J a sublattice of L, then $\underline{\underline{\mathfrak{A}}}_\rho(J)$ is a sublattice of L/ρ.

Proof. Similar to the proof of propositions 3.13, 3.18 and 3.19. $\square$

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