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ON STATIONARY MODES OF NONLINEAR DIELECTRIC SLAB STRUCTURES: A NEW MATHEMATICAL APPROACH

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INTRODUCTION

The amplitude E(x) of stationary T.E. modes propagating in nonlinear dielectric slab structures obey the following nonlinear ordinary differential equation [1, 2, 3, 4]:

$$E''+(-\beta^2+\varepsilon(x,E))E=0 \qquad (1)$$

where x ranges in $(-\infty,+\infty)$, the prime denotes differentiation with respect to x, β is the wave number in a direction orthogonal to x, $\varepsilon(x,E)$ is the nonlinear permittivity, depending both on x and E. In (1) the lenghts have been non dimensionalized with respect to the free space wavelength.

Solutions to eq. (1) representing guided waves must also satisfy the asymptotic conditions at infinity

$$\lim_{|x|\to+\infty} E, E' = 0 \qquad (2)$$

For a single slab extending in the x-range (-a,a), the form typically taken by $\varepsilon(x,E)$ is [1].

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$$\varepsilon(x,E) = \varepsilon^0_{1,2} + g_{1,2}(|E|)$$

where the indices 1,2 refer to the ranges $|x|>a$ and $|x|<a$, respectively, and $\varepsilon^0_{1,2}$ are constants; $g_{1,2}(|E|)$ are non decreasing (non increasing) functions of $|E|$, such that $g_{1,2}(0)=0$, representing the effect of focussing (defocussing) nonlinear polarization of the media. At the interfaces between the media 1 and 2, that is at $x=\pm a$, E and E' must be continuous.

The linear case, as obtained by setting $g_{1,2}=0$ in eq. (1), reduces to a linear eigenvalue problem that admits solutions only for discrete values of β, when $\gamma_1=b^2-\varepsilon^0_1>0$ and $\gamma_2=\beta^2-\varepsilon^0_2<0$. On the contrary, the nonlinear problem exhibits a totally different class of solutions, for any β, which we shall study in the extreme case when the corresponding linear problem has no solution, that is when $\gamma_1>\gamma_2>0$; we shall also assume that $0<g_1<g_2$, that $g_{1,2}$ is monotone increasing (focussing nonlinearity), and $\gamma_1<\sup g_1$.

Qualitative properties of the solutions will be pointed out making use of phase plane analysis of the structure of the orbits of the dynamical system associated with (1), a technique which has been already applied to nonlinear eigenvalue problems in bounded domains arising in population dynamics and reaction diffusion equations [5], [6].

QUALITATIVE PROPERTIES OF THE SOLUTIONS

Let us consider the two plane dynamical systems

$$\begin{cases} u' = v \\ v' = (\gamma_1 - g_1(|u|)u \end{cases} \tag{1a}$$

and

$$\begin{cases} u' = v \\ v' = (\gamma_2 - g_2(|u|)u \end{cases} \tag{1b}$$

where u represents E. It is clear that any solution of (1) is composed of two pieces of orbits of (1a) issuing and entering (0,0) matched at $x=\pm a$ by a piece of orbit of (1b) in a "time" interval 2a.

We remark that by reflection symmetry of the slab structure around x=0, if u(x) is a solution, then u(-x) and -u(-x) are solutions, therefore when in the following we shall speak of uniqueness, it will be meant uniqueness apart from the change of sign of u of x or of both.

Assuming that $g_{1,2}$ are Lipschitz continuous, the qualitative structure of the orbits of system (1a) is represented in figure (1).

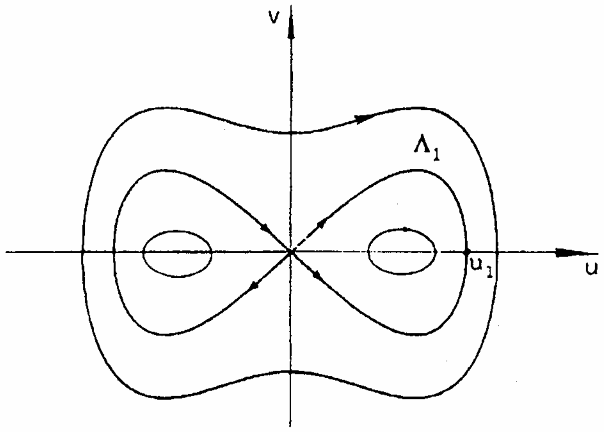

Fig. 1

Note the homoclinic loop Λ_1 whose branches issuing and entering (0,0) provide the pieces of solutions of eq. (1) for $|x|>a$ that satisfy the asymptotic conditions at infinity.

On the other hand, the qualitative structure of the orbits of system (1b) is represented in figure (2), where also the homoclinic loop Λ_1 is reported. The conditions $\gamma_1>\gamma_2>0$ and $0<g_1<g_2$ garantee that the second homoclinic loop Λ_2 is inside the first one Λ_1. It is clear from the above figure that the solutions of problem (1) (2) can be made to correspond in a one to one manner to the pieces of periodic orbits of (1b) connecting Λ_1 with itself in a 'time' interval 2a. This can only be achieved by means of periodic orbits of (1b) outside the homoclinic loop Λ_2 and crossing the u axis in the range (u_2,u_1), where $(u_2,0)$ and $(u_1,0)$ are the intersections of Λ_2 and Λ_1 respectively with the u axis.

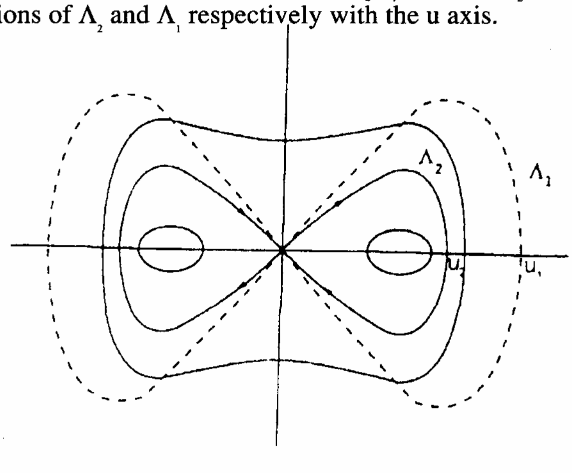

Fig. 2

We shall show that any periodic orbit, say $P_{\bar{u}}$, through $(\bar{u},0)$, with $u_2<\bar{u}<u_1$ intersects Λ_1 at a single point in the quadrant u>0, v>0. In fact we note first that the smooth functions

$$V_{1,2}(u) = \int_0^u u(g_{1,2} - \gamma_{1,2})du, \quad \text{by the conditions on } \gamma_{1,2} \text{ and } g_{1,2}$$

stated above, take the form qualitatively reproduced in fig. (3). In particular $V_2>V_1$ except at 0=(0,0).

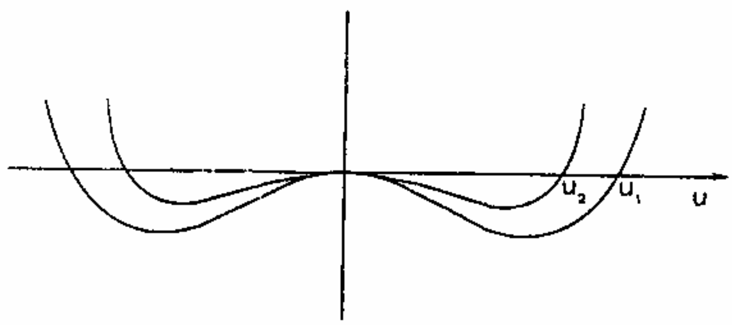

Fig. 3

The homoclinic loop Λ_1 satisfies the equation

$$\frac{v^2}{2} + V_1(u) = 0$$

while any periodic orbit $P_{\bar{u}}$, outside Λ_2 satisfies $\frac{v^2}{2} + V_2(u) = V_2(\bar{u}) > 0$, with $\bar{u} > u_2$. The intersections of $P_{\bar{u}}$ with Λ_1 are then given by

$$V_2(u) - V_1(u) = V_2(\bar{u}) \tag{3}$$

Now, as $V_2(u)$-$V_1(u)$, and $V_2(\bar{u})$ are monotone increasing for positive values of their arguments, it follows that for any $\bar{u}\in(u_2,u_1)$ equation (3) has one positive solution, say $h(\bar{u})$. Moreover it is easy to see that $h(\bar{u}) < \bar{u}, \lim_{\bar{u}\to u_2} h(\bar{u}) = 0$, $\lim_{\bar{u}\to u_1} h(\bar{u}) = u_1$ and $h'(\bar{u}) > 0$.

Let us Λ_1^{ij} the piece of Λ_1 belonging to the quadrant of the (u,v)-plane corresponding to the set of signs $i,j \in \{+,-\}$; moreover let us denote by $T(\bar{u})$ the period of $P_{\bar{u}}$ and let

$\tau(\bar{u}) = \int_{h(\bar{u})}^{\bar{u}} \frac{du}{\sqrt{2(V_2(\bar{u}) - V_2(u))}}$, be the time needed to connect Λ_1^{++} to the u axis along $P_{\bar{u}}$, being $u_2 < \bar{u} < u_1$. We can classify the types of matchings and of solutions of problem (1) (2), into three classes

$S^{(k)}$: $P_{\bar{u}}$ connects Λ_1^{++} to Λ_1^{+-}; that is

$$\sigma_k(\bar{u}) \overset{\text{def}}{=} \frac{k}{2} T(\bar{u}) + \tau(\bar{u}) = a, \qquad k = 0,1,2\ldots \tag{4}$$

The corresponding solutions of problem (1) (2) will then be given by the piecewise smooth curve $\widehat{OP^-} \cup P_{\bar{u}}^{(k)} \cup \widehat{P^-P^+} \cup \widehat{P^+0}$ where $P_{\bar{u}}^{(k)}$ denotes the k-fold iterate of the periodic orbit $P_{\bar{u}}$ (See Fig. 4 in the case k=0)

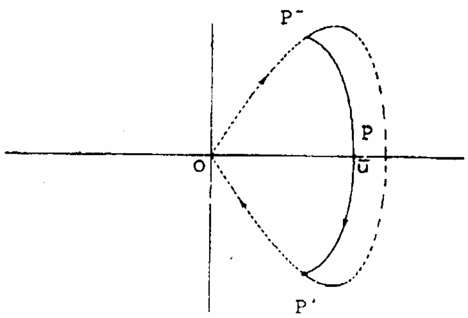

Fig. 4

Mode $S^{(k)}$ is symmetric and has 2k zeros in (-a,a).

$A^{(k)}$: $P_{\bar{u}}$ connects Λ_1^{++} to Λ_1^{-+}; that is

$$\alpha_k(\bar{u}) \overset{\text{def}}{=} ((2k+1)/4) T(\bar{u}) + \tau(\bar{u}) = a, \qquad k = 0,1,2\ldots \tag{5}$$

(Figure 5 in the case k=0). Mode $A^{(k)}$ is antisymmetric and has 2k+1 zeros in (-a,a).

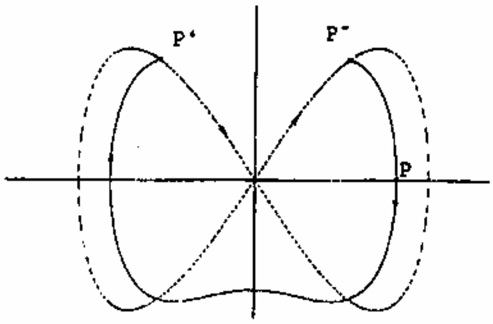

Fig. 5

$W^{(k)}$: $P_{\bar{u}}$ connects Λ_1^{++} to Λ_1^{--}; that is

$$W_k(\bar{u}) \stackrel{def}{=} ((2k+1)/4)T(\bar{u}) = a, \quad k = 0,1,2\ldots \tag{6}$$

Mode $W^{(0)}$ is neither symmetric nor antisymmetric (Figure 6 in the case k=0).

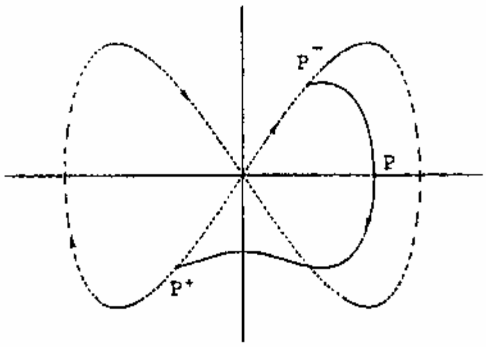

Fig. 6

In order to discuss the existence and uniqueness of the above solutions, we note that $\tau(\bar{u})$ is monotone increasing in the range (u_2,u_1), and $\lim_{\bar{u}\to u_2} \tau(\bar{u}) = +\infty$, $\lim_{\bar{u}\to u_1} \tau(\bar{u}) = 0$. As a consequences mode $S^{(0)}$ always exists and is unique.

On the other hand

$$T(\bar{u}) = 4\int_0^{\bar{u}} \frac{du}{\sqrt{2(V_2(\bar{u}) - V_2(u))}} \quad \text{is monotone}$$

increasing in (u_2,u_1), and $\lim_{\bar{u}\to u_2} T(\bar{u}) = +\infty$,

$$\lim_{\bar{u}\to u_1} T(\bar{u}) \underline{\underline{\text{def}}}\, T_1 = 4\int_0^{u_1} \frac{du}{\sqrt{2\left(V_2(\bar{u}) - V_2(u)\right)}}$$

The qualitative graphs of the functions α_k, σ_k and w_k are reported in figure (7)

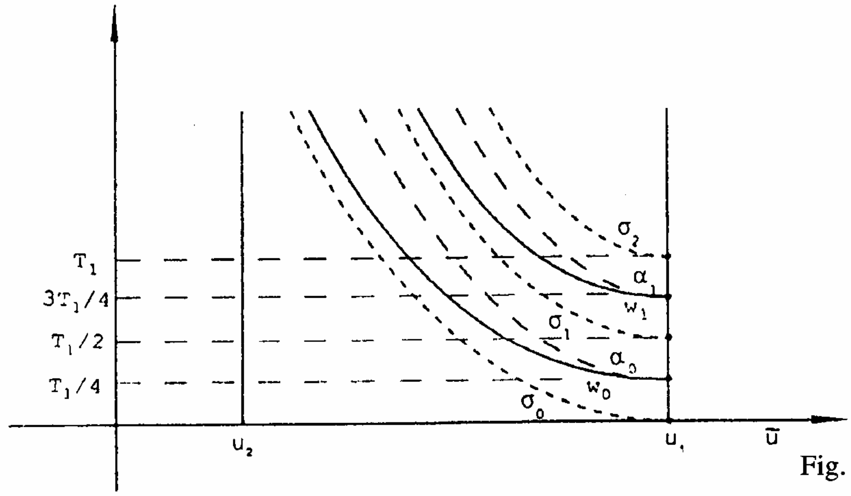

Fig. 7

If we denote by $\bar{\sigma}_k$, $\bar{\alpha}_k$ and $\bar{w}_k$ the $\bar{u}$ values solving equation (4) (5) and (6) respectively, it is clear that

$$\bar{\sigma}_0 < \bar{w}_0 < \bar{\alpha}_0 < \ldots \bar{\sigma}_{k-1} < \bar{w}_{k-1} < \bar{\alpha}_{k-1} < \bar{\sigma}_k < \bar{w}_k < \bar{\alpha}_k < \ldots$$

whenever such solutions exist. Moreover, if one of the above solutions exists then also the previous ones in the above list exist. In particular, for $((2k+1)/4)T_1<a<((k+1)/a)T_1$ all the modes less then $S^{(k+1)}$ exist, while if $(K/2)T_1<a<((2k-1)/4)T_1$, all the modes less then $W^{(k)}$ exist. Therefore $W^{(k)}$ and $A^{(k)}$ are generated by increase of a through $(k/2)T_1$, while $S^{(k)}$ is generated by increase of a through $(k/2)T_1$.

CONCLUSIONS

The qualitative effects of the non-linear polarization of the media on the propagation of TE modes in dielectric slab structures has been pointed out by use of phase-plane analysis of the associated second order, ordinary differential equation. In particular symmetric structures that do not allow for the existence of linear bounded modes have been shown to admit a class of non-linear modes of three types: symmetric, antisymmetric and neither symmetric nor antisymmetric. The analyses based on the direct numerical integration of the equation pubblished so far [9] [10] do not seem to have pointed out this and other important pieces of information which, on the other hand, can be obtained in an almost elementary way by use of the analysis presented in this paper.

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