THE INVERSE POTENTIAL PROBLEM OF ELETTROCARDIOLOGY IN TERMS OF SOURCES

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1. INTRODUCTION

The aim of this work is to solve the inverse problem of electrocardiology in terms of sources in a model of cardiac muscle i.e. to identify intracardiac sources from surface conductor potentials.

The solution of this problem is directly related to the localization of the foci of ectopic ventricular beats during surgery.

The oblique dipole layer model of the depolarization wavefront is assumed to describe correctly the potential field at a distance from cardiac sources [1].

A few instants after ectopic cardiac excitation the wavefront is limited to a small region surrounding the ectopic focus. In this case the potential field at a small distance from the wavefront can be approximated by a linear quadrupole centered at the ectopic focus. Without loss of generality we studied the identification of one dipolar source from potential data given on the cardiac muscle surface, because multipolar sources may be represented by superposition of single dipoles.

In our model we take into account the influence of cardiac muscle anisotropy due to fiber rotation on generated potential distributions.

The proposed method is being applied to the localization of ectopic intracardiac foci in animal experiments in view of future clinical use.

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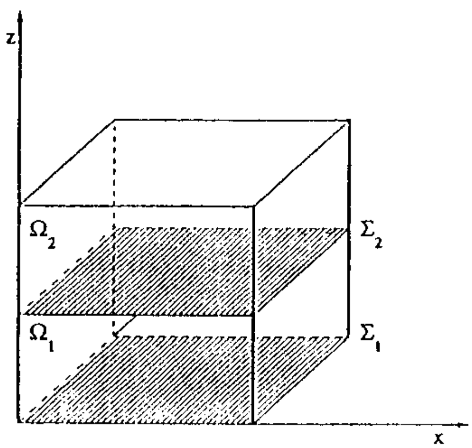
2. THE MATHEMATICAL FORMULATION OF THE INVERSE PROBLEM

Let Ω be an insulated conductor volume consisting of the regions Ω_2 homogeneous,representing blood and Ω_1, anisotropic, representing a portion of the myocardium as a set of superimposed layers of parallel fibers with fiber direction rotating from endocardium to epicardium.(fig.1).



$\Omega=\Omega_1 \cup \Omega_2$

$\Gamma=\partial\Omega$

Σ_1, Σ_2 epi- and endocardial surfaces with Σ_2 separating Ω_1 and Ω_2

T_K , k=1,2 conductivity tensor in Ω_K, with $T_2= \sigma I$

T conductivity tensor in Ω

$T=T_K$ in Ω_K

fig. 1

In order to evaluate a dipolar source g from the knowledge of the potential distribution u on the cardiac surface Σ (fig.1), we must solve the following <u>*inverse problem*</u> :

$$(1) \qquad \begin{cases} -\nabla \cdot (T\nabla u) = g & \text{in} \\ u = z & \text{on } \Sigma=\Sigma_1 \cup \Sigma_2 \\ (T\nabla u) \cdot \underline{n} = 0 & \text{on } \Gamma \\ u_1 = u_2 & \text{on } \Sigma_2 \\ (T_1 \nabla u_1) \cdot \underline{n} = (T_2 \nabla u_2) \cdot \underline{n} & \text{on } \Sigma_2 \end{cases}$$

$-\nabla \cdot T\nabla$ is unbounded, implying that the solution g does not depend continuously on the data z

The problem is *ill-posed*

We may formulate the problem (1) in the following way:

$$(2) \qquad \begin{cases} \mathcal{A}g = z \\ z = u(g)_{|\Sigma} \end{cases}$$

where $\mathcal{A}$ is called *transfer operator*, which turns out to be

linear, continuous, injective, with unbounded inverse.

Thanks to the properties of the transfer operator, it is possible to utilize the *Tikhonov's regularization method* (Tikhonov et al. 1976) to stabilize the problem, so that we can approximate the solution of the problem (2) with the solution of the following stable problem:

$$\boxed{\inf_{g\in U} \left\{ \|\mathcal{A}g-z\|_F^2 + \lambda \|g\|_U^2 \right\}}$$

where λ is called *regularization parameter*, $g\in U$, $z\in F$, U and F are normed spaces. $\forall\ \lambda>0$, we call g_λ *regularized solution* of the problem. So that we find a class of regularized solutions

$$\left\{ g_\lambda : \lambda>0 \right\}.$$

3. DISCRETIZATION AND REGULARIZATION OF THE PROBLEM

In order to find the solution, we assign some dipoles in Ω_I. Each dipole is obtained as a linear combination of three unitary dipoles, oriented along the direction of the orthogonal axes, so we search for a linear combination of them. Let n=3p the number of the unitary dipoles. In the applications z is measured on a finite set of locations P_K, (k=1,...,s) on the surface Σ. For the potential superposition principle, the following equations hold:

$$\sum_{i=1}^{n} \alpha_i U_i(P_k) = z_\delta(P_k) \qquad k=1,\dots,s$$

where $U_i(P_k)$ is the potential generated by the i-th unitary dipole in

P_k. We know this value by solving the forward problem for every unitary dipole;

$z_\delta(P_k)$ is the observed potential in P_k affected by error related to measurements

α_i is the unknown i-th unitary dipole moment.

In matrix form we have:

(3) $$A_h \underline{\alpha} = \underline{z}_\delta$$

where $A_h = \left(U_i(P_k) \right)_{i=1,\dots,n}^{k=1,\dots,s}$, is the computed transfer matrix with error h: $\|A_h - A\| = h < \infty$

$\underline{z}_\delta$ is an experimental determination of the exact potential $\underline{z}$ such that: $\|\underline{z} - \underline{z}_\delta\| < \delta$.

Since the problem (2) is ill-posed, the problem (4) is <u>*ill-conditioned*</u> ($C(A_h) \gg 1$). We stabilize it by means of Tikhonov's regularization method, so we look for the solution of the following stable problem:

$$\boxed{\inf_{\underline{\alpha} \in \mathbb{R}^n} \left\{ \frac{1}{s} \| A_h \underline{\alpha} - \underline{z}_\delta \|^2 + \lambda \| \underline{\alpha} \|^2 \right\}}$$

4. CRITERIA FOR THE SELECTION OF THE REGULARIZATION PARAMETER

In order to estimate the best value of λ we need some information on the solution.

Gabor and Nelson [3] gave some results on the estimation of the "heart vector" based on the integration of the potential over the bounding surface Γ of the isotropic and homogeneous conductor volume Ω. We have extended these results to the case of inhomogeneous anisotropic media. Let **i** be the vector of current density and $s = \nabla \cdot \mathbf{i}$ the source strength, whose sum over the whole body is nil; then the resultant vector dipole moment of such a source system is defined as

$$M = \int_{\Omega} rs \, dv$$

where **r** is the radius vector from an arbitrary origin and dv is the volume element. By Ohm's law, **i** = -A grad u, and from Green's lemma we obtain

$$M_x = \int_{\Gamma} F_x(\underline{\sigma},\vartheta) u d\Gamma \quad M_y = \int_{\Gamma} F_y(\underline{\sigma},\vartheta) u d\Gamma \quad M_z = \int_{\Gamma} F_z(\underline{\sigma}) u d\Gamma$$

where $F_x(\underline{\sigma},\vartheta)$, $F_y(\underline{\sigma},\vartheta)$ and $F_z(\underline{\sigma})$ are related to the Ω anisotropy.

Then we have information on the moment of the heart vector which is equal to $\|g\|^2$ or in discrete terms equal to $\|\underline{\alpha}\|^2$.

If $\tilde{u}$ is a realization of the potential u on the surface Γ, it is possible to give an estimate of $\|\alpha\|^2$ by means of $\tilde{c}_1^2 = M_x^2 + M_y^2 + M_z^2$.

In order to select the regularization parameter λ (Morozov 1968) we use a criterion for the following auxiliary function

$$\gamma(\lambda) = \|\underline{\alpha}_\lambda\|^2$$

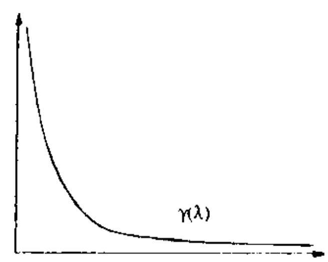


fig. 2

and consider the equation

$$\gamma(\lambda) = c^2$$

If c^2 is a value assumed by the function γ, the solution λ_c of the equation is unique. This method is equivalent to minimize $\|A_h\underline{\alpha}-z_\delta\|^2$ on the compact $\|\underline{\alpha}\|=c$.

5. MATHEMATICAL FORMULATION OF THE FORWARD PROBLEM

The *forward problem* consists in determining the potential u(x) in Ω given the dipolar source. Every dipole is simulated by two opposite electric charges placed at a small distance and the potential must satisfy the following conditions: continuity of the potential and flux across internal boundaries, zero flux on the boundary Γ.

The problem is described by the solution of the following Neumann problem for the Poisson's equation (G. Di Cola et al 1990) and can be formulated weakly as follows:

find $u \in V$ so that:

$$a(u,v) = \langle g,v \rangle \quad \forall v \in V$$

where:

$$a(u,v) = \int_{\Omega} (\nabla v)^T T \nabla u \; d\Omega \quad , \qquad \langle g,v \rangle = \int_{\Omega} gv \; d\Omega$$

$$H^1(\Omega) \subset V \subset L^2(\Omega)$$

$u = u_k$ in Ω_k $k=1,2$	
q_i	electric charges
x_i', x_i''	charge positions
$g = \sum_{i=1}^{n} q_i \left\{\delta(x_i') - \delta(x_i'')\right\}$	dipolar source with $\int_{\Omega} g \; d\Omega = 0$
$S = \mathrm{diag}(\sigma_l, \sigma_t, \sigma_t)$	σ_l, σ_t are the conductivity coefficients along a direction respectively parallel and perpendicular to the fiber
$R = \begin{pmatrix} \cos(\vartheta) & -\mathrm{sen}(\vartheta) & 0 \\ \mathrm{sen}(\vartheta) & \cos(\vartheta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	fiber rotation matrix
$T_1 = RSR^T$	symmetric positive definite matrix related to the conductivity tensor in Ω_1
σ	conductivity in Ω_2
$T_2 = \sigma I$	conductivity matrix of region Ω_2
$T = T_k$ in Ω_k $k=1,2$	conductivity tensor.

Thanks to the Fredholm alternative theorem, the solution $u(x)$ of the problem (1) exists and it is unique if:

$$\int_{\Omega} u(x) \; d\Omega = 0;$$

the solution $u(x) \in H(\Omega)$ with $H^1(\Omega) \subset H(\Omega) \subset L^2(\Omega)$ and depends continuously on g.

6. NUMERICAL SOLUTION OF THE FORWARD PROBLEM: THE FINITE ELEMENT METHOD

- $\Omega = \bigcup_{K\in\tau_h} K = K_1 \cup K_2 \cup \ldots \cup K_N$
- K parallelepipeds of τ_h
- $Q(k)=\left\{ v \mid v \text{ trilinear on } K \right\}$
- $V_h=\left\{ v_h \in C^0(\Omega): v_h \in Q(K), \forall K \in \tau_h \right\}$
- $v_h(x)= \sum_{j=1}^{M} \eta_i \phi_j(x) \qquad \eta_j(x)=v_h(N_j) \qquad x\in \Omega\cup\Gamma$
- N_j interior nodes of the grid
- $\phi_j \in V_h \qquad \phi_j(N_j)=\delta_{ij}=\begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \quad i,j=1,\ldots,N$

The discrete problem consists in finding $u_h \in V_h$ such that:

$$a(u_h,v_h)=(f,v_h) \qquad \forall\ v_h \in V_h$$

7. THE STUDY OF A SAMPLE PROBLEM

We made the following simulation: we divided the cubic region Ω into the equal regions Ω_1 and Ω_2 with anisotropy coefficients:

$$\sigma=\sigma_x=3\sigma_y=3\sigma_z \qquad \text{and} \qquad \vartheta(z_2)=90^\circ$$

z_2 corresponding to Σ_2

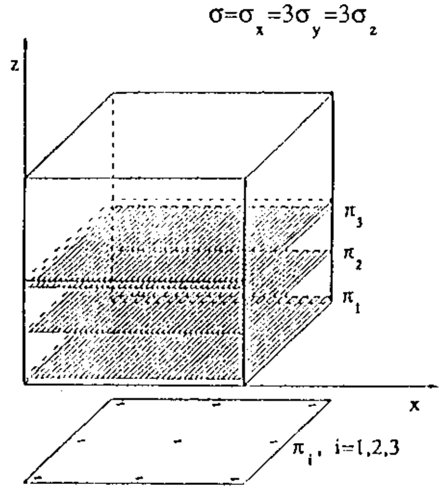


fig. 3

We fixed the possible active dipoles on three different planes in a uniform way as shown in fig.3 and solved the forward problem for every unitary dipole considered, to construct the transfer matrix.

In order to test the method, we studied a sample problem with data obtained by numerical simulation of the potentials generated by a single dipole. A noise of variance $\sigma^2 I$ was added to the surface data. Figures 4a and 5a show some results obtained without regularizing the problem, figures 4b and 5b show the correspondent regularized solutions when we assume $\lambda=\lambda_c$, which identify correctly the assigned dipole. The number of the observations is determined by the realistically available heart surface measurements (a few hundred).

We found the solution by means of a *singular value decomposition* of the matrix $A_h=UWV^T$ (U and V orthogonal matrix, $W=diag(\sigma_1 ... \sigma_s)$) minimizing:

$$F(\underline{\alpha},\lambda)=\frac{1}{s}\ \|UWV^T\underline{\alpha}-UU^T\underline{z}_\delta\|^2+\lambda\ \|\underline{V}^T\alpha\|^2$$

By introducing $\underline{y}=V^T\underline{\alpha}$ and $\tilde{\underline{b}}=U^T\underline{z}_\delta$: $F(\underline{\alpha},\lambda)=\frac{1}{s}\ \|W\underline{y}-\tilde{\underline{b}}\|^2+\lambda\ \|\underline{y}\|^2$,

hence: $$y_\lambda^k=\tilde{b}_k\frac{\sigma_k}{\sigma_k^2+s\lambda} \qquad \text{and} \qquad \underline{\alpha}_\lambda=V\underline{y}_\lambda$$

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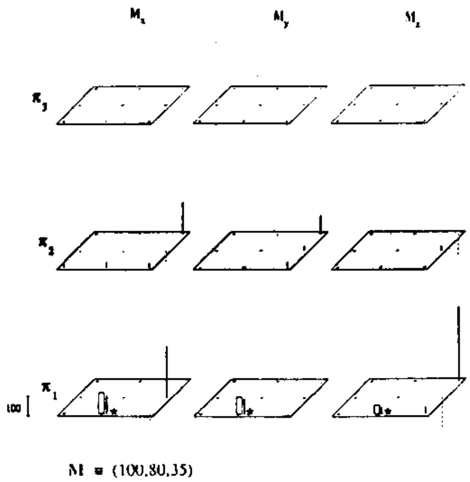

fig.4a

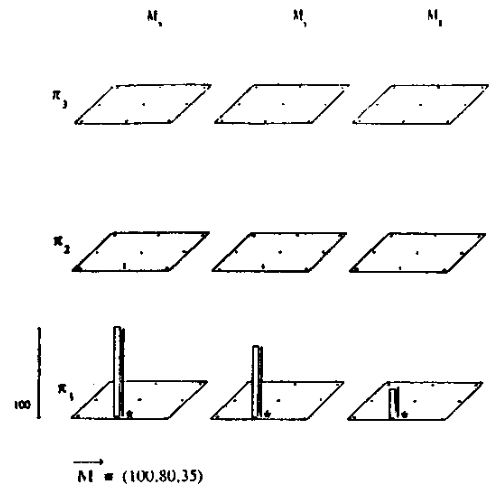

fig.4b

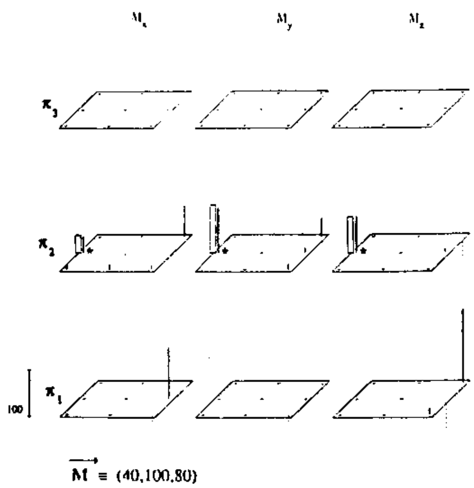

fig.5a

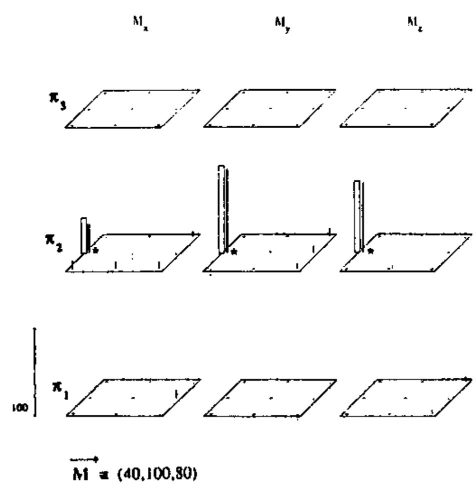

fig.5b

exact solution   computed solution

REFERENCES

[1] P.Colli-Franzone et al. (1982) Potential Fields Generated by Oblique Dipole Layers Modeling Excitation Wavefront in the Anisotropic Myocardium. Comparison with Potential Fields Elicited by Paced Dog Hearts in a Volume Conductor, *Circulation Research*, 51, 330-346

[2] G. Di Cola et al. (1990) Numerical simulation of electric fields generated by dipolar sources in tridimensional anisotropic non homogeneous media 17th International Congress on Electrocardiology Florence 26-29 Set. 1990

[3] D. Gabor, C.V. Nelson (1954). Determination of the resultant dipole of the heart from measurements on the body surface, *Journal of applied physics*,25, 4

[4] V.A. Morozov (1968). The error principle in the solution of operational equations by the regularization method. *Zh. vychisl. Mat. mat. Fiz.*, 8, 2, 295-309.

[5] A. Tikhonov, V. Arsenine (1976). *Méthodes de résolution de problemes mal posés*, Editions MIR-Moscow.

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