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Solvable groups derived from fuzzy hypergroups

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Abstract

In this paper we introduce the smallest equivalence relation ξ^* on a finite fuzzy hypergroup S such that the quotient group S/ξ^*, the set of all equivalence classes, is a solvable group. The characterization of solvable groups via strongly regular relation is investigated and several results on the topic are presented.

Key words: Fuzzy hypergroups, strongly regular relation, solvable groups, fundamental relation.

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1 Introduction

In mathematics, more specifically in the field of group theory, a solvable group or soluble group is a group that can be constructed from Abelian groups using extensions. Equivalently, a solvable group is a group whose derived series terminates in the trivial subgroup. All Abelian groups are trivially solvable a subnormal series being given by just the group itself and the trivial group. But non-Abelian groups may or may not be solvable. A small example of a solvable, non-nilpotent group is the symmetric group S3. In fact, as the smallest simple non-Abelian group is A5, (the alternating group of degree 5) it follows that every group with order less than 60 is solvable. The study of fuzzy hyperstructures is an interesting research topic for fuzzy sets. There are many works on the connections between fuzzy sets and hyperstructures. This can be considered into three groups. A *first* group of papers studies *crisp* hyperoperations defined through fuzzy sets. This study was initiated by Corsini in [3, 4] and then continued by other researchers. A *second* group of papers concerns the *fuzzy hyperalgebras*. This is a direct extension of the concept of fuzzy algebras. This was initiated by Zahedi in [12]. A *third* group was introduced by Corsini and Tofan in [5]. The basic idea in this group of papers is the following: a multioperation assigns to every pair of elements of S a non-empty subset of S, while a fuzzy multioperation assigns to every pair of elements of S a nonzero fuzzy set on S. This idea was continuated by Sen, Ameri and Chowdhury in [10] where fuzzy semihypergroups are introduced. The fundamental relations are one of the most important and interesting concepts in fuzzy hyperstructures that ordinary algebraic structures are derived from fuzzy hyperstructures by them. Fundamental relation α^* on fuzzy hypersemigroups is studied in [1].Also in [8], the smallest strongly regular equivalence relation γ^* on a fuzzy hypersemigroup S such that S/γ^* is a commutative semigroup is studied. In this paper, we introduce and study the fundamental relation ξ^* of a finite fuzzy hypergroup S such that S/ξ^* is a solvable group. Finally, we introduce the concept of ξ-part of a fuzzy hypergroup and we determines necessary and sufficient conditions such that the relation ξ to be transitive.

2 Preliminary

Recall that for a non-empty set S, a fuzzy subset μ of S is a function from S into the real unite interval $[0, 1]$. We denote the set of all nonzero fuzzy subsets of S by $F^*(S)$. Also for fuzzy subsets μ_1 and μ_2 of S, then μ_1 is *smaller* than μ_2 and write $\mu_1 \leq \mu_2$ iff for all $x \in S$, we have $\mu_1(x) \leq \mu_2(x)$. Define $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ as follows: $\forall x \in S, \quad (\mu_1 \vee \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$.

A *fuzzy hyperoperation* on S is a mapping $\circ : S \times S \mapsto F^*(S)$ written as $(a, b) \mapsto a \circ b = ab$. The couple $(S, \circ)$ is called a *fuzzy hypergropoid*.

Definition 2.1. *A fuzzy hypergropoid $(S, \circ)$ is called a fuzzy hypersemigroup if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset μ of S*

$$(a \circ \mu)(r) = \begin{cases} \bigvee_{t \in S}((a \circ t)(r) \wedge \mu(t)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

$$(\mu \circ a)(r) = \begin{cases} \bigvee_{t \in S} (\mu(t) \wedge (t \circ a)(r)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

for all $r \in S$.

Definition 2.2. *Let* μ, ν *be two fuzzy subsets of a fuzzy hypergropoid* $(S, \circ)$. *Then we define* $\mu \circ \nu$ *by* $(\mu \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, *for all* $t \in S$.

Definition 2.3. *A fuzzy hypersemigroup* $(S, \circ)$ *is called fuzzy hypergroup if* $x \circ S = S \circ x = \chi_S$, *for all* $x \in S$, *where* χ_S *is characteristic function of* S.

Example 2.1. *Consider a fuzzy hyperoperation* $\circ$ *on a non-empty set* S *by* $a \circ b = \chi_{\{a,b\}}$, *for all* $a, b \in S$. *Then* $(S, \circ)$ *is a fuzzy hypersemigroup and fuzzy hypergroup as well.*

Theorem 2.1. *Let* $(S, \circ)$ *be a fuzzy hypersemigroup. Then* $\chi_a \circ \chi_b = a \circ b$, *for all* $a, b \in S$.

Definition 2.4. *Let* ρ *be an equivalence relation on a fuzzy hypersemigroup* $(S, \circ)$, *we define two relations* $\overline{\rho}$ *and* $\overline{\overline{\rho}}$ *on* $F^*(S)$ *as follows: for* $\mu, \nu \in F^*(S)$; $\mu \overline{\rho} \nu$ *if* $\mu(a) > 0$ *then there exists* $b \in S$ *such that* $\nu(b) > 0$ *and* $a \rho b$, *also if* $\nu(x) > 0$ *then there exists* $y \in S$, *such that* $\mu(y) > 0$ *and* $x \rho y$. $\mu \overline{\overline{\rho}} \nu$ *if for all* $x \in S$ *such that* $\mu(x) > 0$ *and for all* $y \in S$ *such that* $\nu(y) > 0$, $x \rho y$.

Definition 2.5. *An equivalence relation* ρ *on a fuzzy hypersemigroup* $(S, \circ)$ *is said to be (strongly) fuzzy regular if* $a \rho b, a' \rho b'$ *implies* $a \circ a' \ \overline{\rho} \ b \circ b' (a \circ a' \ \overline{\overline{\rho}} \ b \circ b')$.

If ρ is a equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, then we consider the following hyperoperation on the quotient set S/ρ as follows:

for every $a\rho, b\rho \in S/\rho$

$$a\rho \oplus b\rho = \{c\rho : (a' \circ b')(c) > 0, a\rho a', b\rho b'\}$$

Theorem 2.2. *[2] Let* $(S, \circ)$ *be a fuzzy hypersemigroup and* ρ *be an equivalence relation on* S. *Then*
(i) the relation ρ *is fuzzy regular on* $(S, \circ)$ *iff* $(S/\rho, \oplus)$ *is a hypersemigroup.*
(ii) the relation ρ *is strongly fuzzy regular on* $(S, \circ)$ *iff* $(S/\rho, \oplus)$ *is a semigroup.*

3 New strongly regular relation ξ_n^*

Now in this paper we introduce and analyze a new strongly regular relation ξ_n^* on a fuzzy hypergroup S such that the quotient group S/ξ_n^* is solvable.

Definition 3.1. *Let (S, o) be a fuzzy hypergroup. We define*
1) $L_0(S) = S$
2) $L_{k+1}(S) = \{t \in S \mid (xy)(r) > 0, (tyx)(r) > 0, \quad in \quad which \quad x, y \in L_k(S), \quad for \quad some \quad r \in S\}$.
for all $k \geq 0$. Suppose that $n \in \mathbb{N}$ and $\xi_n = \cup_{m \geq 1} \xi_{m,n}$, where $\xi_{1,n}$ is the diagonal relation and for every integer $m > 1, \xi_{m,n}$ is the relation defined as follows:
$a\xi_{m,n}b \Longleftrightarrow \exists x_1, ..., x_m \in H(m \in \mathbb{N}), \exists \sigma \in \mathbb{S}_m : \sigma(i) = i, \quad if \quad z_i \notin L_n(H)$: $(x_1 o ... o x_m)(a) > 0$ and $(x_{\sigma_1} o ... o x_{\sigma_m})(b) > 0$.

It is clear that ξ_n is symmetric. Define for any $a \in S$, $a(a) = (\chi_a)(a) = 1$, thus ξ_n is reflexive. We take ξ_n^* to be transitive closure of ξ_n. Then it is an equivalence relation on H.

Corolary 3.1. *For every $n \in \mathbb{N}$, we have $\alpha^* \subseteq \xi_n^* \subseteq \gamma^*$.*

Theorem 3.1. *For every $n \in \mathbb{N}$, the relation ξ_n^* is a strongly regular relation.*

Proof. Suppose $n \in \mathbb{N}$. Clearly, $\xi_{m,n}$ is an equivalence relation. First we show that for each $x, y, z \in S$

$$x\xi_n y \Rightarrow xz\overline{\overline{\xi_n}}yz, \quad zx\overline{\overline{\xi_n}}zy \qquad (*).$$

If $x\xi_n y$, then there exists $m \in \mathbb{N}$ such that $x\xi_{m,n}y$, and so there exist $(z_1, \ldots, z_m) \in S^m$ and $\sigma \in S_m$ such that if $z_i \notin L_n(S)$ then $\prod_{i=1}^{m} z_i(x) > 0, \prod_{i=1}^{m} z_{\sigma(i)}(y) > 0$. Let $z \in S$, for any r, s such that $(xz)(r) > 0$ and $(yz)(s) > 0$. We have $((\prod_{i=1}^m z_i)z)(r) = \bigvee_p \{(\prod_{i=1}^{m} z_i)(p) \wedge (pz)(r)\}$. Let $p = x$, then $((\prod_{i=1}^{m} z_i)(z))r > 0, \sigma(i) = i, \quad if \quad z_i \notin L_n(S)$, $((\prod_{i=1}^{m} z_\sigma(i))(z))(s) = \bigvee_q \{(\prod_{i=1}^{m} z_\sigma(i))(q) \wedge (qz)(s)\}$. Let $q = y$, then $((\prod_{i=1}^{m} z_\sigma(i))(z))(s) > 0$, and $\sigma(i) = i, \quad if \quad z_i \notin L_n(S)$. Now suppose that $z_{m+1} = z$ and we define

$\sigma' \in S^m + 1$: $\sigma'(i) = \begin{cases} \sigma(i), & \forall i \in \{1, 2, \ldots, m\} \\ m+1, & i = m+1. \end{cases}$ Thus for all $r, s \in S$; $(\prod_{i=1}^{m} z_i)(r) > 0$, $(\prod_{i=1}^{m} z'_{\sigma})(s) > 0$; $\sigma'(i) = i$ if $z_i \notin L_n(S)$. Therefore $xz\overline{\overline{\xi_n}}yz$. Now if $x\xi_n^* y$, then there exists $k \in \mathbb{N}$ and $u_0 = x, u_1, \ldots, u_k = y \in S$ such that $u_0 = x\xi_n u_1 \xi_n u_2 \xi_n \ldots \xi_n u_m = y$, by the above result we have $u_0 z = xz\overline{\overline{\xi_n}}u_1 z\overline{\overline{\xi_n}}u_2 z\overline{\overline{\xi_n}} \ldots \overline{\overline{\xi_n}}u_k z = yz$ and so $xz\overline{\overline{\xi_n}}yz$. Similarly we can show that $zx\overline{\overline{\xi_n}}zy$. Therefore ξ_n^* is a strongly regular relation on S. □

Proposition 3.1. *For every $n \in \mathbb{N}$, we have $\xi_{n+1}^* \subseteq \xi_n^*$.*

Proof. Let $x\xi_{n+1}y$ so $\exists(z_1, ..., z_m) \in S^m; \exists\delta \in S_m : \delta(i) = i$ if $z_i \notin L_{n+1}(S)$, such that $(\prod_{i=1}^{m} z_i)(x) > 0$, $(\prod_{i=1}^{m} z_{\delta(i)})(y) > 0$. Now let $\delta_1 = \delta$, since $L_{n+1}(S) \subseteq L_n(S)$ so $x\xi_n y$.□

The next result immediately follows from previous theorem.

Corolary 3.2. *If S is a commutative hypergroup, then $\beta^* = \xi_n^*$.*

A group G is solvable if and only if $G^{(n)} = \{e\}$ for some $n \geq 1$ in which, $G^{(0)} = G$, $G^{(1)} = G'$, commutator subgroup of G, and inductively $G^{(i)} = (G^{(i-1)})'$.

Theorem 3.2. *If S is a fuzzy hypergroup and φ is a strongly regular relation on S, then*

$$L_{k+1}(S/\varphi)) = \langle \bar{t} \mid t \in L_k(S) \rangle$$

for $k \in \mathbb{N}$.

Proof. Suppose that $G = S/\varphi$ and $\overline{x} = \varphi(x)$ for all $x \in S$. We prove the theorem by induction on k. For $k = 0$ we have $L_1(G) = \langle \bar{t} \mid t \in L_0(S) \rangle$. Now suppose that $\overline{a} = \bar{t}$ where $t \in L_{k+1}(S)$ so there exist $r_1 \in S$; $(xy)(r_1) > 0$, $(tyx)(r_1) > 0$ in which $x, y \in L_k(S)$. Then $\overline{xy} = \overline{z_1}$; $(xy)(z_1) > 0$ and so $\overline{xy} = \overline{r_1}$. Also $\overline{tyx} = \overline{z_2}$; $(tyx)(z_2) > 0$ and $\overline{tyx} = \overline{r_1} = \overline{xy}$ which implies that $\bar{t} = [\overline{x}, \overline{y}]$. By hypotheses of induction we conclude that $\bar{t} \in L_{k+1}(G)$. Hence $\overline{a} = [\bar{t}, \bar{s}] \in L_{k+2}(G)$. Conversely, let $\overline{a} \in L_{k+2}(G)$. Then $\overline{a} = [\overline{x}, \overline{y}]$, where $\overline{x}, \overline{y} \in L_{k+1}(G)$, so by hypotheses of induction we have $\overline{x} = \overline{u}$ and $\overline{y} = \overline{v}$, where $u, v \in L_k(S)$. Let $c \in S$; $(uv)(c) > 0$ we show that there exists $t \in S$ such that $(tvu)(c) > 0$. Since $S \circ u = \chi_S$ and $c \in S$ then there exists $r \in S$ such that $(ru)(c) > 0$ and so by $r \in S = S \circ v$ there exist $t \in S$; $(tv)(r) > 0$. Therefore $(tvu)(c) = \bigvee_n((tv)(n) \wedge (nu)(c)) \geq (tv)(r) \wedge (ru)(c) > 0$. Thus $(uv)(c) > 0$, $(tvu)(c) > 0$ which implies that $t \in L_{k+1}(S)$. Now since

$\overline{uv} = \overline{c} = \overline{tvu}$, then $\overline{t} = [\overline{u}, \overline{v}] = [\overline{x}, \overline{y}] = \overline{a}$ and $t \in L_{k+1}(S)$. Therefore, $\overline{a} = \overline{t} \in \langle \overline{t}; t \in L_{k+1}(S) \rangle$.□

Theorem 3.3. *S/ξ_n^* is a solvable group of class at most $n+1$.*

Proof. Using Theorem 3.2, $L_k(S/\xi_n^*)$is an Abelian group and $L_{k+1}(S/\xi_n^*) = \{e\}$. □

4 On solvable groups derived from finite fuzzy hypergroups

In this section we introduce the smallest strongly relation ξ^* on a finite fuzzy hypergroup S such that H/ξ^* is a solvable group.

Definition 4.1. *Let S be a finite fuzzy hypergroup. Then we define the relation ξ^* on S by*

$$\xi^* = \bigcap_{n \geq 1} \xi_n^*.$$

Theorem 4.1. *The relation ξ^* is a strongly regular relation on a finite fuzzy hypergroup S such that S/ξ^* is a solvable group.*

Proof. Since $\xi^* = \bigcap_{n \geq 1} \xi_n^*$, it is easy to see that ξ^* is a strongly regular relation on S. By using Proposition 3.1, we conclude that there exists $k \in \mathbb{N}$ such that $\xi_{k+1}^* = \xi_k^*$. Thus $\xi* = \xi_k^*$ for some $k \in \mathbb{N}$. □

Theorem 4.2. *The relation ξ^* is the smallest strongly regular relation on a finite fuzzy hypergroup S such that S/ξ^* is a solvable group.*

Proof. Suppose ρ is a strongly regular relation on S such that $K = S/\rho$ is a solvable group of class c. Suppose that $x\xi y$. Then $x\xi_n y$, for some $n \in \mathbb{N}$ and so there exists $m \in \mathbb{N}$ such that

$x\xi_{mn}y \Longleftrightarrow \exists(z_1, ..z_m) \in S^m, \exists\delta \in S_m : \delta(i) = i$ if $z_i \notin L_n(S)$ such that $(\prod_{i=1}^m z_i)(x) > 0$, $(\prod_{i=1}^m z_{\delta(i)})(y) > 0$,

$$L_{c+1}(S/\rho) = \langle \rho(t); t \in L_c(S) \rangle = \{\rho(e)\},$$

and so $\rho(z_i) = \rho(e)$, for every $z_i \in L_c(S)$. Therefore $\rho(x) = \rho(y)$, which implies that $x\rho y$.□

5 Transitivity of ξ^*

In this section we introduce the concept of ξ-part of a fuzzy hypergroup and we determine necessary and sufficient condition such that the relation ξ to be transitive.

Definition 5.1. *Let X be a non-empty subset of S. Then we say that X is a ξ-part of S if the following condition holds: for every $k \in \mathbb{N}$ and $(z_1, ..., z_m) \in H^m$ and for every $\sigma \in S_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n(S)$, and there exists $x \in X$ such that $(\prod_{i=1}^{m} z_i)(x) > 0$, then for all $y \in S \backslash X$, $(\prod_{i=1}^{m} z_{\sigma(i)})(y) =$* 0.

Theorem 5.1. *Let X be a non-empty subset of a fuzzy hypergroup S. Then the following conditions are equivalent:*
1) X is a ξ-part of S,
2) $x \in X$, $x\xi y \Longrightarrow y \in X$,
3) $x \in X$, $x\xi^* y \Longrightarrow y \in X$.

Proof. (1) $\Longrightarrow$ (2) if $(x, y) \in S^2$ is a pair such that $x \in X$ and $x\xi y$, then there exist $(z_1, ..., z_i) \in S^k$; $(\prod_{i=1}^{m} z_i)(x) > 0$, $(\prod_{i=1}^{m} z_{\sigma(i)})(y) > 0$ and $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n(S)$. Since X is a ξ-part of S, we have $y \in X$.
(2) $\Longrightarrow$ (3) Suppose that $(x, y) \in S^2$ is a part such that $x \in X$ and $x\xi^* y$. Then there is $(z_1, ..., z_i) \in S^k$ such that $x = z_0 \xi z_1 \xi ... \xi z_k = y$. Now by using (2) k-times we obtain $y \in X$.
(3) $\Longrightarrow$ (1) For every $k \in \mathbb{N}$ and $(z_1, ..., z_i) \in S^k$ and for every $\sigma \in S_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n(S)$, then there exists $x \in X$; $(\prod_{i=1}^{m} z_i)(x) > 0$ and there exist $y \in S \backslash X$; $(\prod_{i=1} z_{\sigma(i)})(y) > 0$, then $x\xi_n y$ and so $x\xi y$. Therefore by (3) we have $y \in X$ which is a contradiction.□

Theorem 5.2. *The following conditions are equivalent:*
1) for every $a \in H$, $\xi(a)$ is a ξ-part of S,
2) ξ is transitive.

Proof. (1) $\Longrightarrow$ (2) Suppose that $x\xi^* y$. Then there is $(z_1, ..., z_i) \in S^k$ such that $x = z_0 \xi z_1 \xi ... \xi z_k = y$, since $\xi(z_i)$ for all $0 \leq i \leq k$, is a ξ-part, we have $z_i \in \xi(z_{i-1})$, for all $1 \leq i \leq k$. Thus $y \in \xi(x)$, which means that $x\xi y$.
(2) $\Longrightarrow$ (1) Suppose that $x \in S$, $z \in \xi(x)$ and $z\xi y$. By transitivity of ξ, we have $y \in \xi(x)$. Now according to the last theorem, $\xi(x)$ is a ξ-part of S.□

Definition 5.2. *The intersection of all ξ-parts which contain A is called ξ-closure of A in S and it will be denoted by $K(A)$.*

In what follows, we determine the set $W(A)$, where A is a non-empty subset of S. We set
1) $W_1(A) = A$ and
2) $W_{n+1}(A) = \{x \in S \mid \exists(z_1, ..., z_i) \in S^k : \ (\prod_{i=1}^{m} z_{(i)})(x) > 0, \exists\sigma \in S_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{n\geq 1} L_n(S)$ and there exists $a \in W_n(A)$; $(\prod_{i=1}^{m} z_{\sigma(i)})(a) > 0\}$.
We denote $W(A) = \bigcup_{n\geq 1} W_n(A)$.

Theorem 5.3. *For any non-empty subset of S, the following statements hold:*
1) $W(A) = K(A)$,
2) $K(A) = \cup_{a\in A} K(a)$.

Proof. 1) It is enough to prove:
a) $W(A)$ i a ξ-part,
b) if $A \subseteq B$ and B is a ξ-part, then $W(A) \subseteq B$.
In order to prove (a), suppose that $a \in W(A)$ such that $(\prod_{i=1} z_i)(a) > 0$ and $\sigma \in S_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{n\geq 1} L_n(S)$. Therefore, there exists $n \in \mathbb{N}$ such that $(\prod_{i=1}^{m} z_i)(a) > 0$ $a \in W_n(A)$. Now if there exists $t \in S$ such that $(\prod_{i=1} z_{\sigma(i)})(t) > 0$ we obtain $t \in W_{n+1}(A)$. Therefore, $t \in W(A)$ which is a contradiction. Thus $(\prod_{i=1}^{m} z_{\sigma(i)})(t) = 0$ and so $W(A)$ is a ξ-part. Now we prove (b) by induction on n. We have $W_1(A) = A \subseteq B$. Suppose that $W_n(A) \subseteq B$. We prove that $W_{n+1}(A) \subseteq B$. If $z \in W_{n+1}(A)$, then there exists $k \in \mathbb{N}$; $(z_1, ..., z_k) \in S^k$; $(\prod_{i=1}^{m} z_i)(z) > 0$ and there exists $\sigma \in S_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{t\geq 1} L_t(S)$ and there exists $t \in W_n(A)$; $(\prod_{i=1}^{m} z_{\sigma_i})(t) > 0$, since $W_n(A) \subseteq B$ we have $t \in B$ and $(\prod_{i=1}^{m} z_{\sigma_i})(t) > 0$. Now since B is ξ-part , $(\prod_{i=1}^{m} z_i)(z) > 0$ then $z \in B$.

2) It is clear that for all $a \in A$, $K(a) \subseteq K(A)$. By part 1), we have $K(A) = \cup_{n\geq 1} W_n(A)$ and $W_1(A) = A = \cup_{a\in A}\{a\}$. It is enough to prove that $W_n(A) = \cup_{a\in A} W_n(a)$, for all $n \in \mathbb{N}$. We follow by induction on n. Suppose it is true for n. We prove that $W_{n+1}(A) = \cup_{a\in A} W_{n+1}(a)$. If $z \in W_{n+1}(A)$, then there exists $k \in \mathbb{N}$, $(z_1, ..., z_k) \in S^k$; $(\prod_{i=1}^{m} z_i)z > 0$ and there exists $\sigma \in S_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t\geq 1} L_t(S)$ and there exist $a \in W_n(A)$; $(\prod_{i=1}^{m} z_{\sigma(i)})(a) > 0$. By the hypotheses of induction there exists $a \in W_n(A) = \cup_{b\in A} W_n(b)$; $(\prod_{i=1}^{m} z_{\sigma(i)})(a^{'}) > 0$ for some $a^{'} \in W_n(b)$ in which $b \in A$. Therefore, $z \in W_{n+1}(b)$, and so $W_{n+1}(A) \subseteq \cup_{b\in A} W_{n+1}(b)$. Hence $K(A) = \cup_{a\in A} K(a)$.□

Theorem 5.4. *The following relation is equivalence relation on H.*

$$xWy \Longleftrightarrow x \in W(y),$$

for every $(x, y) \in S^2$, where $W(y) = W(\{y\})$.

Proof. It is easy to see that W is reflexive and transitive. We prove that W is symmetric. To this, we check that:
1) for all $n \geq 2$ and $x \in S$, $W_n(W_2(x)) = W_{n+1}(x)$,
2) $x \in W_n(y)$ if and only if $y \in W_n(x)$.
We prove (1) by induction on n.

$W_2(W_2(x)) = \{z \mid \exists q \in \mathbb{N}, (a_1, ..., a_q) \in S^q; (\prod_{i=1} a_i)(z) > 0$ *and* $\exists \sigma \in S_k$ *such that* $\sigma(i) = i, if\ z_i \notin \cup_{s\geq 1} L_s(S)$ *and* $\exists y \in W_2(x)$; $(\prod_{i=1}^{m} a_{\sigma(i)})(y) > 0\} = W_3(x)$. Now we proceed by induction on n. Suppose $W_n(W_2(x)) = W_{n+1}(x)$ then
$W_{n+1}(W_2(x)) = \{z \mid \exists q \in \mathbb{N}, (a_1, ..., a_q) \in S^q; (\prod_{i=1}^{m} a_i)(z) > 0$ *and* $\exists \sigma \in S_k$ *such that* $\sigma(i) = i$, *if* $z_i \notin \cup_{s\geq 1} L_s(S)$ *and* $\exists t \in W_n(W_2(x)); (\prod_{i=1}^{m} a_{\sigma(i)})(t) > 0\} = W_{n+2}(x)$. Now we prove (2) by induction on n, too. It is clear that $x \in W_2(y)$ if and only if $y \in W_2(x)$. Suppose $x \in W_n(y)$ if and only if $y \in W_n(x)$. Let $x \in W_{n+1}(y)$, then there exists $q \in \mathbb{N}, (a_1, ..., a_q) \in S^q; (\prod_{i=1}^{m} a_i)(x) > 0$ *and* $\exists \sigma \in S_k$ *such that* $\sigma(i) = i$, *if* $a_i \notin$

$\cup_{s\geq 1}L_s(S)$ *and* $\exists t\in W_n(y); (\prod_{i=1}^{m} a_{\sigma(i)})t>0$. Now, $(\prod_{i=1}^{m} a_i)(x)>0, x\in W_1(x)$ and $(\prod_{i=1}^{m} a_{\sigma(i)})(t)>0$ implies that $t\in W_2(x)$. Since $t\in W_n(y)$, then by hypotheses of induction $y\in W_n(t)$ and we see that $t\in W_2(x)$, therefore $y\in W_n(W_2(x))=W_{n+1}(x)$. □

Remark 5.1. *If S is a fuzzy hypergroup, then S/ξ^* is a group. We define $\omega_S=\phi^{-1}(1_{S/\xi^*})$, in which $\phi: S\rightarrow S/\xi^*$ is the canonical projection.*

Lemma 5.1. *If S is a fuzzy hypergroup and M is a non-empty subset of S, then*
(i) $\phi^{-1}(\phi(M))=\{x\in S:(\omega_S M)(x)>0\}=\{x\in S:(M\omega_S)(x)>0\}$
(ii) If M is a ξ part of S, then $\phi^{-1}(\phi(M))=M$.

Proof. (i) Let $x\in S$ and $(t,y)\in\omega_S\times M$ such that $(ty)(x)>0$, so $\phi(x)=\phi(t)\oplus\phi(y)=1_{S/\xi^*}\oplus\phi(y)=\phi(y)$, therefore $x\in\phi^{-1}(\phi(y))\subset\phi^{-1}(\phi(M))$. Conversely, for every $x\in\phi^{-1}(\phi(M))$, there exists $b\in M$ such that $\phi(x)=\phi(b)$. By reproducibility, $a\in S$ exists such that $(ab)(x)>0$, so $\phi(b)=\phi(x)=\phi(a)\oplus\phi(b)$. This implies $\phi(a)=1_{S/\xi^*}$ and $a\in\phi^{-1}(1_{S/\xi^*})=\omega_S$. Therefore $(\omega_S M)(x)>0$.

In the same way, we can prove that $\phi^{-1}(\phi(M))=\{x\in S:(M\omega_S)(x)>0\}$.
(ii) We know $M\subseteq\phi^{-1}(\phi(M))$. If $x\in\phi^{-1}(\phi(M))$, then there exists $b\in M$ such that $\phi(x)=\phi(b)$. Therefore $x\in\xi^*(x)=\xi^*(b)$. Since M is a ξ part of S and $b\in M$, by Lemma 5.1, we conclude $\xi^*(b)\subseteq M$ and $x\in M$. □

Definition 5.3. *Let $(S,\cdot)$ be a fuzzy hypergroup. $K\subseteq S$ is called a fuzzy subhypergroup of S if*
i) $(a\cdot b)\cdot c=a\cdot(b\cdot c)$, for all $a,b,c\in S$
ii) $a\cdot K=\chi_K$, for all $a\in K$.

Theorem 5.5. *ω_S is a fuzzy subhypergroup of S, which is also a ξ-part of S.*

Proof. Clearly, $\omega_S\subseteq S$ and so $(a\cdot b)\cdot c=a\cdot(b\cdot c)$, for all $a,b,c\in\omega_S$. Now we show that $\omega_S y=\chi_{\omega_S}$ for all $y\in\omega_S$. Let $x,y\in\omega_S$, then there exists $u\in S$ such that $(uy)(x)>0$. Therefore, $\overline{uy}=\overline{x}$, which implies that $\overline{u}=1$. Thus $u\in\omega_S$. Consequently, $\omega_S y=\chi_{\omega_S}$. Hence, ω_S is a fuzzy subhypergroup of S. Now we prove that $K(y)=\phi^{-1}(\phi(\{y\}))=\{x\in S:(\omega_S y)(x)>0\}=\omega_S$.

$$
\begin{aligned}
z \in \phi^{-1}(\phi(\{y\})) &\iff \varphi(z) = \varphi(y) \\
&\iff \xi^*(z) = \xi^*(y) \\
&\iff z\xi^* y \\
&\iff z \in \xi^*(z) = \omega(\{y\}) = K(y).
\end{aligned}
$$

Also since $y \in \omega_S$, then $\{x \in S : (\omega_S y)(x) > 0\} = \{x \in S : (\chi_{\omega_S})(x) > 0\} = \omega_S$. Therefore $K(y) = \omega_S$ and so ω_S is ξ part. $\square$

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