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Sums of generalized convergent harmonic series with eight periodically repeated numerators

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Abstract

This contribution deals with the generalized convergent harmonic series with eight periodically repeated numerators and it is a follow-up to author's papers dealing with the generalized alternating harmonic series with two up to seven periodically repeated numerators. It is derived the only expression of the last numerator depending on preceding numerators for which this series converges. Then the formula for the sum of this series is analytically derived. This analytical result is numerically verified by using the CAS Maple 16.

Keywords: alternating harmonic series, geometric series, sum of the series, CAS Maple.

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1 Introduction and basic notions

Let us recall the basic terms and notions. The *harmonic series* is the sum of reciprocals of all natural numbers (except zero), so this is the series

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots.$$

The divergence of this series can be proved e.g. by using the integral test or the comparison test of convergence. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is known as the *alternating harmonic series*. This series converges by the alternating series test. In particular, the sum is equal to the natural logarithm of 2:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2\,.$$

2 Sum of generalized alternating harmonic series with two up to seven periodically repeated numerators

This paper is a continuation of the author's contributions [1], [2], [3], [4], [5] and [6]. The paper [1] deals, among others, with the generalized alternating harmonic series with two periodically repeated numerators $(1, a)$, i.e with the series of the form

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} + \frac{a}{2n} \right) = \frac{1}{1} + \frac{a}{2} + \frac{1}{3} + \frac{a}{4} + \frac{1}{5} + \frac{a}{6} + \frac{1}{7} + \frac{a}{8} + \frac{1}{9} + \frac{a}{10} + \cdots\,,$$

where $a \in \mathbb{R}$. In entire agreement with the well-known fact it was derived that the only one value of the coefficient a, for which this series converges, is $a = -1$ and that the sum of this series is $s = \ln 2$.

The paper [2] deals with the generalized alternating harmonic series with three periodically repeated numerators $(1, a, b)$, i.e. with the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} + \frac{a}{3n-1} + \frac{b}{3n} \right) = \frac{1}{1} + \frac{a}{2} + \frac{b}{3} + \frac{1}{4} + \frac{a}{5} + \frac{b}{6} + \frac{1}{7} + \frac{a}{8} + \frac{b}{9} + \cdots\,.$$

It was derived that the only value of the coefficient $b \in \mathbb{R}$, for which this series converges, is $b = -a - 1$, and that the sum of this series is given by the formula

$$s(a) = \frac{a+1}{2} \ln 3 - \frac{a-1}{6\sqrt{3}}\, \pi\,.$$

The contribution [3] deals with the generalized alternating harmonic series with four periodically repeated numerators $(1, a, b, c)$, i.e. with the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{4n-3} + \frac{a}{4n-2} + \frac{b}{4n-1} + \frac{c}{4n} \right) = \frac{1}{1} + \frac{a}{2} + \frac{b}{3} + \frac{c}{4} + \frac{1}{5} + \frac{a}{6} + \frac{b}{7} + \frac{c}{8} + \cdots\,.$$

It was derived that the only value of the coefficient $c \in \mathbb{R}$, for which this series converges, is $c = -a - b - 1$ and it was also derived that the sum of this series is

$$s(a, b) = \frac{2a + 3b + 3}{4} \ln 2 - \frac{b - 1}{8} \pi .$$

The paper [4] is about the generalized alternating harmonic series with five periodically repeated numerators $(1, a, b, c, d)$, i.e. with the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{5n-4} + \frac{a}{5n-3} + \frac{b}{5n-2} + \frac{c}{5n-1} + \frac{d}{5n} \right) = \\ = \frac{1}{1} + \frac{a}{2} + \frac{b}{3} + \frac{c}{4} + \frac{d}{5} + \frac{1}{6} + \frac{a}{7} + \frac{b}{8} + \frac{c}{9} + \frac{d}{10} + \cdots .$$

It was derived that the only value of the coefficient $d \in \mathbb{R}$, for which this series converges, is $d = -a-b-c-1$. It was also derived that the sum of this series is

$$s(a, b, c) = \frac{1 + a + b + c}{4} \ln 5 + \frac{\sqrt{5}(1 - a - b + c)}{20} \ln \frac{3 + \sqrt{5}}{2} + \\ + \frac{\sqrt{5}(1 - c) + 1 + 2a - 2b - c}{\sqrt{10}\sqrt{5 + \sqrt{5}}} \arctan \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}}{5 - \sqrt{5}} + \\ + \frac{\sqrt{5}(1 - c) - (1 + 2a - 2b - c)}{\sqrt{10}\sqrt{5 - \sqrt{5}}} \arctan \frac{\sqrt{2}\sqrt{5 - \sqrt{5}}}{5 + \sqrt{5}} .$$

The contribution [5] is about the generalized alternating harmonic series with six periodically repeated numerators $(1, a, b, c, d, e)$, i.e. with the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{6n-5} + \frac{a}{6n-4} + \frac{b}{6n-3} + \frac{c}{6n-2} + \frac{d}{6n-1} + \frac{e}{6n} \right) = \\ = \frac{1}{1} + \frac{a}{2} + \frac{b}{3} + \frac{c}{4} + \frac{d}{5} + \frac{e}{6} + \cdots .$$

It was derived that the only value of the coefficient $e \in \mathbb{R}$, for which this series converges, is $e = -a-b-c-d-1$. It was also derived that the sum of this series is

$$s(a, b, c, d) = \frac{1 + b + d}{3} \ln 2 + \frac{1 + a + c + d}{4} \ln 3 + \frac{3 + a - c - 3d}{12\sqrt{3}} \pi .$$

The paper [6] deals with the generalized alternating harmonic series with seven periodically repeated numerators $(1, a, b, c, d, e, f)$, i.e. with the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{7n-6} + \frac{a}{7n-5} + \frac{b}{7n-4} + \frac{c}{7n-3} + \frac{d}{7n-2} + \frac{e}{7n-1} + \frac{f}{7n} \right) .$$

It was derived that the only value of the coefficient $f \in \mathbb{R}$, for which this series converges, is $f = -a-b-c-d-e-1$. It was also derived that the sum of this series is

$$\begin{aligned} s(a,b,c,d,e) = {} & 0.4440431881a + 0.2555147273b + 0.1530792957c + \\ & + 0.0861379681d + 0.0375971731e + 0.9695377967\,. \end{aligned}$$

3 Sum of generalized alternating harmonic series with eight periodically repeated numerators

Now, we deal with the numerical series of the form

$$\begin{aligned} \sum_{n=1}^{\infty} \bigg(& \frac{1}{8n-7} + \frac{a}{8n-6} + \frac{b}{8n-5} + \frac{c}{8n-4} + \frac{d}{8n-3} + \frac{e}{8n-2} + \\ & + \frac{f}{8n-1} + \frac{g}{8n} \bigg) = \frac{1}{1} + \frac{a}{2} + \frac{b}{3} + \frac{c}{4} + \frac{d}{5} + \frac{e}{6} + \frac{f}{7} + \frac{g}{8} + \\ & + \frac{1}{9} + \frac{a}{10} + \frac{b}{11} + \frac{c}{12} + \frac{d}{13} + \frac{e}{14} + \frac{f}{15} + \frac{g}{16} + \cdots\,, \end{aligned} \tag{1}$$

where $a, b, c, d, e, f, g \in \mathbb{R}$. This series we shall call *generalized alternating harmonic series with eight periodically repeated numerators* $(1, a, b, c, d, e, f, g)$. We express the numerator g, for which the series (1) converges, as a function of the numerators a, b, c, d, e, f, and determine the sum of this series.

The power series corresponding to the series (1) has evidently the form

$$\begin{aligned} \sum_{n=1}^{\infty} \bigg(& \frac{x^{8n-7}}{8n-7} + \frac{ax^{8n-6}}{8n-6} + \frac{bx^{8n-5}}{8n-5} + \frac{cx^{8n-4}}{8-4} + \frac{dx^{8n-3}}{8n-3} + \frac{ex^{8n-2}}{8n-2} + \\ & + \frac{fx^{8n-1}}{8n-1} + \frac{gx^{8n}}{8n} \bigg)\,. \end{aligned} \tag{2}$$

We denote its sum by $s(x)$. The series (2) is for $x \in (-1, 1)$ absolutely convergent, so we can rearrange it and rewrite it in the form

$$\begin{aligned} s(x) = {} & \sum_{n=1}^{\infty} \frac{x^{8n-7}}{8n-7} + a\sum_{n=1}^{\infty} \frac{x^{8n-6}}{8n-6} + b\sum_{n=1}^{\infty} \frac{x^{8n-5}}{8n-5} + c\sum_{n=1}^{\infty} \frac{x^{8n-4}}{8n-4} + \\ & + d\sum_{n=1}^{\infty} \frac{x^{8n-3}}{8n-3} + e\sum_{n=1}^{\infty} \frac{x^{8n-2}}{8n-2} + f\sum_{n=1}^{\infty} \frac{x^{8n-1}}{8n-1} + g\sum_{n=1}^{\infty} \frac{x^{8n}}{8n}\,. \end{aligned} \tag{3}$$

If we differentiate the series (3) term-by-term, where $x \in (-1, 1)$, we get

$$\begin{aligned} s'(x) = & \sum_{n=1}^{\infty} x^{8n-8} + a \sum_{n=1}^{\infty} x^{8n-7} + b \sum_{n=1}^{\infty} x^{8n-6} + c \sum_{n=1}^{\infty} x^{8n-5} + \\ & + d \sum_{n=1}^{\infty} x^{8n-4} + e \sum_{n=1}^{\infty} x^{8n-3} + f \sum_{n=1}^{\infty} x^{8n-2} + g \sum_{n=1}^{\infty} x^{8n-1}. \end{aligned} \tag{4}$$

After reindexing and fine arrangement the series (4) we obtain

$$\begin{aligned} s'(x) = & \sum_{n=0}^{\infty} x^{8n} + ax \sum_{n=0}^{\infty} x^{8n} + bx^2 \sum_{n=0}^{\infty} x^{8n} + cx^3 \sum_{n=0}^{\infty} x^{8n} + \\ & + dx^4 \sum_{n=0}^{\infty} x^{8n} + ex^5 \sum_{n=0}^{\infty} x^{8n} + fx^6 \sum_{n=0}^{\infty} x^{8n} + gx^7 \sum_{n=0}^{\infty} x^{8n}, \end{aligned}$$

that is

$$s'(x) = \left(1 + ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7\right) \sum_{n=0}^{\infty} \left(x^8\right)^n. \tag{5}$$

When we summate the convergent geometric series on the right-hand side of (5) with the first term 1 and the ratio x^8, where $\left|x^8\right| < 1$, i.e. for $x \in (-1, 1)$, we get

$$s'(x) = \frac{1 + ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7}{1 - x^8}.$$

We convert this fraction using the CAS Maple 16 to partial fractions and get

$$s'(x) = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1} + \frac{Ex + F}{x^2 + \sqrt{2}x + 1} + \frac{G}{x + 1} + \frac{H}{x - 1},$$

where $x \in (-1, 1)$ and

$$\begin{aligned} A &= \frac{a - c + e - g}{4}, & B &= \frac{1 - b + d - f}{4}, \\ C &= \frac{-1 + b + \sqrt{2}c + d - f - \sqrt{2}g}{4\sqrt{2}}, & D &= \frac{\sqrt{2} + a - c - \sqrt{2}d - e + g}{4\sqrt{2}}, \\ E &= \frac{1 - b + \sqrt{2}c - d + f - \sqrt{2}g}{4\sqrt{2}}, & F &= \frac{\sqrt{2} - a + c - \sqrt{2}d + e - g}{4\sqrt{2}}, \\ G &= \frac{1 - a + b - c + d - e + f - g}{8}, & H &= \frac{-1 - a - b - c - d - e - f - g}{8}. \end{aligned} \tag{6}$$

The sum $s(x)$ of the series (2) we obtain by integration in the form

$$s(x) = \int \left(\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} + \frac{Ex+F}{x^2+\sqrt{2}x+1} + \frac{G}{x+1} + \frac{H}{x-1} \right) \mathrm{d}x =$$

$$= \frac{A}{2} \int \frac{2x}{x^2+1}\,\mathrm{d}x + B \int \frac{1}{x^2+1}\,\mathrm{d}x + \int \frac{C(2x-\sqrt{2})/2 + D + C\sqrt{2}/2}{x^2-\sqrt{2}x+1}\,\mathrm{d}x +$$

$$+ \int \frac{E(2x+\sqrt{2})/2 + F - E\sqrt{2}/2}{x^2+\sqrt{2}x+1}\,\mathrm{d}x + G\ln|x+1| + H\ln|x-1| + K,$$

so

$$s(x) = \frac{A}{2}\ln(x^2+1) + B\arctan x + \frac{C}{2}\ln(x^2-\sqrt{2}x+1) +$$

$$+ \frac{2D+C\sqrt{2}}{2} \int \frac{\mathrm{d}x}{(x-\sqrt{2}/2)^2 + (\sqrt{2}/2)^2} + \frac{E}{2}\ln(x^2+\sqrt{2}x+1) +$$

$$+ \frac{2F-E\sqrt{2}}{2} \int \frac{\mathrm{d}x}{(x+\sqrt{2}/2)^2 + (\sqrt{2}/2)^2} + G\ln|x+1| + H\ln|x-1| + K =$$

$$= \frac{A}{2}\ln(x^2+1) + B\arctan x + \frac{C}{2}\ln(x^2-\sqrt{2}x+1) + \frac{2D+C\sqrt{2}}{\sqrt{2}} \times$$

$$\times \arctan\frac{2x-\sqrt{2}}{\sqrt{2}} + \frac{E}{2}\ln(x^2+\sqrt{2}x+1) + \frac{2F-E\sqrt{2}}{\sqrt{2}}\arctan\frac{2x+\sqrt{2}}{\sqrt{2}} +$$

$$+ G\ln|x+1| + H\ln|x-1| + K,$$

where K is the constant of integration and where we used the formulas

$$\int \frac{f'(t)}{f(t)}\,\mathrm{d}t = \ln|f(t)| + K \qquad \text{and} \qquad \int \frac{\mathrm{d}t}{t^2+\alpha^2} = \frac{1}{\alpha}\arctan\frac{t}{\alpha} + K\,.$$

From the condition $s(0) = 0$, and because we have $\ln 1 = 0$, $\arctan 0 = 0$, $\arctan(\pm 1) = \pm\frac{\pi}{4}$, we obtain

$$\frac{2D+C\sqrt{2}}{\sqrt{2}} \cdot \frac{-\pi}{4} + \frac{2F-E\sqrt{2}}{\sqrt{2}} \cdot \frac{\pi}{4} + K = 0\,,$$

hence

$$K = \frac{\pi}{4\sqrt{2}}\left(2D + C\sqrt{2} - 2F + E\sqrt{2}\right).$$

Because $2(D-F)+(C+E)\sqrt{2} = \dfrac{a-e}{\sqrt{2}}$, we get $K = \dfrac{\pi}{4\sqrt{2}} \cdot \dfrac{a-e}{\sqrt{2}} = \dfrac{(a-e)\pi}{8}$.

After application the relations (6), where

$$\sqrt{2}D + C = \frac{1+\sqrt{2}a + b - d - \sqrt{2}e - f}{4\sqrt{2}}\,,$$

$$\sqrt{2}F - E = \frac{1 - \sqrt{2}a + b - d + \sqrt{2}e - f}{4\sqrt{2}},$$

we get

$$\begin{aligned} s(x) = {} & \frac{a - c + e - g}{8} \ln(x^2 + 1) + \frac{1 - b + d - f}{4} \arctan x + \\ & + \frac{-1 + b + \sqrt{2}c + d - f - \sqrt{2}g}{8\sqrt{2}} \ln(x^2 - \sqrt{2}x + 1) + \\ & + \frac{1 + \sqrt{2}a + b - d - \sqrt{2}e - f}{4\sqrt{2}} \arctan(\sqrt{2}x - 1) + \\ & + \frac{1 - b + \sqrt{2}c - d + f - \sqrt{2}g}{8\sqrt{2}} \ln(x^2 + \sqrt{2}x + 1) + \\ & + \frac{1 - \sqrt{2}a + b - d + \sqrt{2}e - f}{4\sqrt{2}} \arctan(\sqrt{2}x + 1) + \\ & + \frac{1 - a + b - c + d - e + f - g}{8} \ln|x + 1| - \\ & - \frac{1 + a + b + c + d + e + f + g}{8} \ln|x - 1| + \frac{(a - e)\pi}{8}. \end{aligned}$$

Now, we will deal with the convergence of the power series (2) in the point $x = 1$. Substituting $x = 1$ to the series (2) – it can be done by the extended version of Abel's theorem (see [7], p. 23) – we get the numerical series (1). By the integral test we can prove that the series (1) converges if and only if $H = 0$, i.e. for $g = -a - b - c - d - e - f - 1$. Simplifying the formula for $s(x)$ above, where $g = -a - b - c - d - e - f - 1$, and for $x = 1$ we get

$$\begin{aligned} s(1) = {} & \frac{1 + 2a + b + d + 2e + f}{8} \ln 2 + \frac{1 - b + d - f}{4} \arctan 1 + \\ & + \frac{\sqrt{2} - 1 + \sqrt{2}a + (\sqrt{2} + 1)b + 2\sqrt{2}c + (\sqrt{2} + 1)d + \sqrt{2}e + (\sqrt{2} - 1)f}{8\sqrt{2}} \times \\ & \times \ln(2 - \sqrt{2}) + \frac{1 + \sqrt{2}a + b - d - \sqrt{2}e - f}{4\sqrt{2}} \arctan(\sqrt{2} - 1) + \\ & + \frac{\sqrt{2} + 1 + \sqrt{2}a + (\sqrt{2} - 1)b + 2\sqrt{2}c + (\sqrt{2} - 1)d + \sqrt{2}e + (\sqrt{2} + 1)f}{8\sqrt{2}} \times \\ & \times \ln(2 + \sqrt{2}) + \frac{1 - \sqrt{2}a + b - d + \sqrt{2}e - f}{4\sqrt{2}} \arctan(\sqrt{2} + 1) + \\ & + \frac{1 + b + d + f}{4} \ln 2 - 0 + \frac{(a - e)\pi}{8}. \end{aligned}$$

Because $\ln 1 = 0$, $\arctan 1 = \dfrac{\pi}{4}$, $\arctan(\sqrt{2}-1) = \dfrac{\pi}{8}$, $\arctan(\sqrt{2}+1) = \dfrac{3\pi}{8}$, we have

$$s(1) = \frac{1-b+d-f}{16}\pi + \frac{-1+b+d-f+\sqrt{2}(1+a+b+2c+d+e+f)}{8\sqrt{2}} \times$$
$$\times \ln(2-\sqrt{2}) + \frac{1+b-d-f+\sqrt{2}(a-e)}{32\sqrt{2}}\pi +$$
$$+ \frac{1-b-d+f+\sqrt{2}(1+a+b+2c+d+e+f)}{8\sqrt{2}}\ln(2+\sqrt{2}) +$$
$$+ \frac{3(1+b-d-f)-3\sqrt{2}(a-e)}{32\sqrt{2}}\pi +$$
$$+ \frac{3(1+b+d+f)+2(a+e)}{8}\ln 2 + \frac{a-e}{8}\pi\,.$$

After simplification and after re-mark $s(1)$ as $s(a,b,c,d,e,f)$ we obtain

$$s(a,b,c,d,e,f) = \frac{1+2a-b+d-2e-f}{16}\pi +$$
$$+ \frac{\sqrt{2}-a+\sqrt{2}b-\sqrt{2}d+e-\sqrt{2}f}{16}\pi + \frac{1-b-d+f}{8\sqrt{2}}\ln(3+2\sqrt{2}) +$$
$$+ \frac{1+a+b+2c+d+e+f+3+2a+3b+3d+2e+3f}{8}\ln 2\,.$$

Finally, we get the required formula

$$s(a,b,c,d,e,f) = \frac{\sqrt{2}(1+b-d-f)+1+a-b+d-e-f}{16}\pi +$$
$$+ \frac{1-b-d+f}{8\sqrt{2}}\ln(3+2\sqrt{2}) + \frac{4(1+b+d+f)+3a+2c+3e}{8}\ln 2\,. \quad (7)$$

4 Numerical verification

We have solved the problem to determine the sum $s(a,b,c,d,e,f)$ above for several values of a, b, c, d, e, f by using the basic programming language of the computer algebra system Maple 16. It was used the following simple procedure `sumgenhar1abcdefg`. As a sample of the hexads (a,b,c,d,e,f) we took 12 hexads

$$(1,0,0,0,0,0)\,,\; (0,1,0,0,0,0)\,,\; (0,0,1,0,0,0)\,,\; (0,0,0,1,0,0)\,,$$

$$(0,0,0,0,1,0),\ (0,0,0,0,0,1),\ (0,0,0,0,0,0),\ (-1000,0,0,0,0,0),$$
$$(6,-1,0,1,-6,-1),\ (-1,1,-1,1,-1,1),\ (-2,1,-2,1,-2,1),$$
$$\text{and}\ (1/2,1/4,1/8,1/16,1/32,1/64).$$

It was chosen $t = 10^6$ summands with 8 terms

$$\frac{1}{8n-7}+\frac{a}{8n-6}+\frac{b}{8n-5}+\frac{c}{8n-4}+\frac{d}{8n-3}+ \\ +\frac{e}{8n-2}+\frac{f}{8n-1}-\frac{a+b+c+d+e+f+1}{8n}$$

for the computations whose results will be compared with the results obtained by the formula (7). The procedure `sumgenhar1abcdefg` consists of the following commands:

```
sumgenhar1abcdefg:=proc(t,a,b,c,d,e,f)
   local g,r,k,s,w;
   s:=0; r:=0;
   g:=-a-b-c-d-e-f-1;
   for k from 1 to t do
       r:=1/(8*k-7)+a/(8*k-6)+b/(8*k-5)+c/(8*k-4)+d/(8*k-3)
          +e/(8*k-2)+f/(8*k-1)+g/(8*k);
       s:=s+r;
   end do;
   print("t=",k-1,"s(",a,b,c,d,e,f,")=",evalf[20](s));
   w:=Pi*(sqrt(2)*(1+b-d-f)+1+a-b+d-e-f)/16
        +ln(3+2*sqrt(2))*(1-b-d+f)/(8*sqrt(2))
        +ln(2)*(4*(1+b+d+f)+3*a+2*c+3*e)/8
   print("s(",a,b,c,d,e,f")=",evalf[20](w));
end proc:
```

Computation of the twelve sums $s(10^6,a,b,c,d,e,f)$ took about 47 hours and 39 minutes. The relative quantification accuracies of the twelve sums $s(10^6,a,b,c,d,e,f)$, that is the ratio

$$\left|\frac{s(10^6,a,b,c,d,e,f)-s(a,b,c,d,e,f)}{s(10^6,a,b,c,d,e,f)}\right|,$$

have here place value about 10^{-7}.

The results of the procedure above are presented in the Table 1, where the computed sums are denoted briefly $s(10^6)$ instead of $s(10^6,a,b,c,d,e,f)$ and the sums $s(a,b,c,d,e,f)$ are denoted as $s(abcdef)$ and are evaluated by means of the formula (7):

a	b	c	d	e	f	$s(10^6)$	$s(abcdef)$
1	0	0	0	0	0	0.9591518	0.9591520
0	1	0	0	0	0	1.4326892	1.4326894
0	0	1	0	0	0	1.1496962	1.1496964
0	0	0	1	0	0	1.0858461	1.0858463
0	0	0	0	1	0	1.0399901	1.0399903
0	0	0	0	0	1	1.0047597	1.0047598
0	0	0	0	0	0	0.9764095	0.9764096
-10^3	0	0	0	0	0	-455.30323	-455.30332
6	-1	0	1	-6	-1	3.1415922	3.1415927
-1	1	-1	1	-1	1	0.6931471	0.6931472
-2	1	-2	1	-2	1	0.0000001	0
1/2	1/4	1/8	1/16	1/32	1/64	1.3035043	1.3035045

Table 1: The approximate values of the sums of the generalized harmonic series with periodically repeating numerators $(1, a, b, c, d, e, f, -a-b-c-d-e-f-1)$ for 12 hexads (a, b, c, d, e, f)

5 Conclusion

We dealt with the generalized convergent harmonic series with eight periodically repeated numerators $(1, a, b, c, d, e, f, g)$, where $a, b, c, d, e, f, g \in \mathbb{R}$, i.e. with the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}+\frac{a}{8n-6}+\frac{b}{8n-5}+\frac{c}{8n-4}+\frac{d}{8n-3}+\frac{e}{8n-2}+\frac{f}{8n-1}+\frac{g}{8n}\right).$$

We derived that the only value of the numerator g, for which this series converges, is $g = -a - b - c - d - e - f - 1$, and we derived that the sum of this series is given by the formula

$$s(a, b, c, d, e, f) = \frac{\sqrt{2}(1+b-d-f)+1+a-b+d-e-f}{16}\pi + \\ + \frac{1-b-d+f}{8\sqrt{2}}\ln(3+2\sqrt{2}) + \frac{4(1+b+d+f)+3a+2c+3e}{8}\ln 2 .$$

This formula allows to determine another sums whose periodically repeated numerators need not be $(1, a, b, c, d, e, f, -a-b-c-d-e-f-1)$, but also $(k, \ell, m, n, p, q, r, -k-\ell-m-n-p-q-r)$, for $k, \ell, m, n, p, q, r \in \mathbb{R}$, at least one nonzero. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{64}{8n-7}+\frac{32}{8n-6}+\frac{16}{8n-5}+\frac{8}{8n-4}+\frac{4}{8n-3}+\frac{2}{8n-2}+\frac{1}{8n-1}-\frac{127}{8n}\right)$$

has the sum $S(64,32,16,8,4,2,1,-127) =$

$$= 64\cdot s\left(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64}\right) \doteq 64\cdot 1.303\,504 \doteq 83.424\,.$$

There are special series with sums expressed only by one summand or with the special or the null sum. It can be easily derived that

$$s(a,b,0,-b,-a,-1) = \frac{1+a-b+\sqrt{2}(1+b)}{8}\,\pi\,,$$

$$\text{so e.g.} \qquad s(6,-1,0,1,-6,-1) = \pi\,,$$

$$s(a,1,c,1,a,1) = \frac{8+3a+c}{4}\ln 2\,, \qquad \text{so e.g.} \qquad s(-1,1,-1,1,-1,1) = \ln 2\,,$$

$$\text{and} \qquad s(-2,-1,0,1,2,-1) = 0\,, \qquad s(-2,1,-2,1,-2,1) = 0\,.$$

We verified the main result (7) by computing 12 sums by using the CAS Maple 16. The generalized convergent harmonic series with eight periodically repeated numerators so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a relatively simple formula.

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