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# The sum of the series of reciprocals of the cubic polynomials with triple non-positive integer root

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**Abstract**

This contribution, which is a follow-up to author's paper [1] dealing with the sums of the series of reciprocals of some quadratic polynomials, deals with the series of reciprocals of the cubic polynomials with triple non-positive integer root. Three formulas for the sum of this kind of series expressed by means of harmonic numbers are derived and presented, together with one approximate formula, and verified by several examples evaluated using the basic programming language of the computer algebra system Maple 16. This contribution can be an inspiration for teachers who are teaching the topic Infinite series or as a subject matter for work with talented students.

**Key words**: telescoping series, harmonic numbers, CAS Maple, Riemann zeta function.

**2000 AMS**: 40A05, 65B10.

## 1 Introduction

Let us recall some basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ . The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each $n$

as the sum $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$. The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit $s$ if and only if the sequence of partial sums $\{s_n\}$ converges to $s$, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a *sum* $s$ and write $\sum_{k=1}^{\infty} a_k = s$.

The *nth harmonic number* is the sum of the reciprocals of the first $n$ natural numbers: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$. The *generalized harmonic numbers* of order $n$ in power $r$ is the sum

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r}\,, \tag{1}$$

where $H_{n,1} = H_n$ are harmonic numbers. Every generalized harmonic number of order $n$ in power $m$ can be written as a function of generalized harmonic number of order $n$ in power $m-1$ using formula (see [2]):

$$H_{n,m} = \sum_{k=1}^{n-1} \frac{H_{k,m-1}}{k(k+1)} + \frac{H_{n,m-1}}{n}\,, \tag{2}$$

whence

$$H_{n,2} = \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n}\,, \qquad H_{n,3} = \sum_{k=1}^{n-1} \frac{H_{k,2}}{k(k+1)} + \frac{H_{n,2}}{n}\,.$$

Therefore

$$H_{n,3} = \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \left( \sum_{i=1}^{k-1} \frac{H_i}{i(i+1)} + \frac{H_k}{k} \right) + \frac{1}{n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n^2}\,,$$

thus

$$H_{n,3} = \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \left( \sum_{i=1}^{k-1} \frac{H_i}{i(i+1)} + \frac{H_k}{k} + \frac{H_k}{n} \right) + \frac{H_n}{n^2}\,. \tag{3}$$

From formula (1), where $r = 1, 2, 3$ and $n = 1, 2, \ldots, 8$, we get this table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $H_n$ | 1 | $\frac{3}{2}$ | $\frac{11}{6}$ | $\frac{25}{12}$ | $\frac{137}{60}$ | $\frac{49}{20}$ | $\frac{363}{140}$ | $\frac{761}{280}$ |
| $H_{n,2}$ | 1 | $\frac{5}{4}$ | $\frac{49}{36}$ | $\frac{205}{144}$ | $\frac{5269}{3600}$ | $\frac{5369}{3600}$ | $\frac{266681}{176400}$ | $\frac{1077749}{705600}$ |
| $H_{n,3}$ | 1 | $\frac{9}{8}$ | $\frac{251}{216}$ | $\frac{2035}{1728}$ | $\frac{256103}{216000}$ | $\frac{28567}{24000}$ | $\frac{9822481}{8232000}$ | $\frac{78708473}{65856000}$ |

# 2 The sum of the series of reciprocals of the cubic polynomials with triple non-positive integer root

We deal with the problem to determine the sum $s(a,a,a)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^3}$$

for non-positive integers $a$, i.e. to determine the sum $s(0,0,0)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots , \tag{4}$$

the sum $s(-1,-1,-1)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^3} = \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots = s(0,0,0) - s_1(0,0,0) = s(0,0,0) - 1 \,,$$

the sum $s(-2,-2,-2)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)^3} = \frac{1}{3^3} + \frac{1}{4^3} + \cdots = s(0,0,0) - s_2(0,0,0) = s(0,0,0) - \frac{9}{8}$$

etc. Clearly, we get the formula

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^3} = s(0,0,0) - s_{-a}(0,0,0) \,, \tag{5}$$

where $s_{-a}(0,0,0)$ is the $(-a)$th partial sum of the series (4). Several values of the $n$th partial sums $s_n(0,0,0)$, briefly denoted by $s_n$, are: $s_{100} \doteq 1.2020074$, $s_{1000} \doteq 1.2020564$, $s_{10000} \doteq 1.2020569$, $s_{100000} \doteq 1.2020569$. Let us note that the series $s(0,0,0)$ converges to the *Apéry's constant* $1.202056903159\ldots$, which represents the value $\zeta(3)$ of the *Riemann zeta function*

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \,.$$

The partial sums $s_n(0,0,0)$ so present the generalized harmonic numbers $H_{n,3}$. According to formula (5) is

$$s(a,a,a) = \zeta(3) - H_{-a,3} \,, \tag{6}$$

then using formula (3) we get

**Theorem 2.1.** *The series* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k-a)^3}$, *where* $a$ *is a negative integer, has the sum*

$$s(a,a,a)=\zeta(3)-\sum_{k=1}^{-a-1}\frac{1}{k(k+1)}\left(\sum_{i=1}^{k-1}\frac{H_i}{i(i+1)}+\frac{H_k}{k}-\frac{H_k}{a}\right)-\frac{H_{-a}}{a^2}. \quad (7)$$

Now, we express formula (3) in another form. We have

$$\begin{aligned}H_{n,3}=&\frac{1}{2}\left(\frac{H_1}{1}+\frac{H_1}{n}\right)+\frac{1}{6}\left(\frac{H_1}{2}+\frac{H_2}{2}+\frac{H_2}{n}\right)+\\&+\frac{1}{12}\left(\frac{H_1}{2}+\frac{H_2}{6}+\frac{H_3}{3}+\frac{H_3}{n}\right)+\frac{1}{20}\left(\frac{H_1}{2}+\frac{H_2}{6}+\frac{H_3}{12}+\frac{H_4}{4}+\frac{H_4}{n}\right)+\\&+\frac{1}{30}\left(\frac{H_1}{2}+\frac{H_2}{6}+\frac{H_3}{12}+\frac{H_4}{20}+\frac{H_5}{5}+\frac{H_5}{n}\right)+\cdots\\\cdots&+\frac{1}{(n-3)(n-2)}\left(\frac{H_1}{2}+\frac{H_2}{6}+\cdots+\frac{H_{n-4}}{(n-4)(n-3)}+\frac{H_{n-3}}{n-3}+\frac{H_{n-3}}{n}\right)+\\&+\frac{1}{(n-2)(n-1)}\left(\frac{H_1}{2}+\frac{H_2}{6}+\cdots+\frac{H_{n-3}}{(n-3)(n-2)}+\frac{H_{n-2}}{n-2}+\frac{H_{n-2}}{n}\right)+\\&+\frac{1}{(n-1)n}\left(\frac{H_1}{2}+\frac{H_2}{6}+\cdots+\frac{H_{n-2}}{(n-2)(n-1)}+\frac{H_{n-1}}{n-1}+\frac{H_{n-1}}{n}\right)+\frac{H_n}{n^2},\end{aligned}$$

i.e.

$$\begin{aligned}H_{n,3}=&\frac{H_1}{1\cdot 2}\left(\frac{1}{1}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\\&+\frac{H_2}{2\cdot 3}\left(\frac{1}{2}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\\&+\frac{H_3}{3\cdot 4}\left(\frac{1}{3}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\cdots\\\cdots&+\frac{H_{n-3}}{(n-3)(n-2)}\left(\frac{1}{n-3}+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\\&+\frac{H_{n-2}}{(n-2)(n-1)}\left(\frac{1}{n-2}+\frac{1}{(n-1)n}+\frac{1}{n}\right)+\frac{H_{n-1}}{(n-1)n}\left(\frac{1}{n-1}+\frac{1}{n}\right)+\frac{H_n}{n^2}.\end{aligned} \quad (8)$$

Because $\dfrac{1}{k(k+1)}=\dfrac{1}{k}-\dfrac{1}{k+1}$, then the $n$th partial sum $t_n$ of the telescoping series $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k(k+1)}=\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\cdots$ is

$$t_n=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{2}-\frac{1}{n+2}\,,$$

for the expressions in the first three parentheses of formula (8) we get

$$\begin{aligned}
\frac{1}{1}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n}&=1+t_{n-2}+\frac{1}{n}=\\
&=1+\left(\frac{1}{2}-\frac{1}{n}\right)+\frac{1}{n}=\frac{3}{1\cdot 2}\,,\\
\frac{1}{2}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n}&=\frac{1}{2}+t_{n-2}-t_1+\frac{1}{n}=\\
&=\frac{1}{2}+\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{5}{2\cdot 3}\,,\\
\frac{1}{3}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n}&=\frac{1}{3}+t_{n-2}-t_2+\frac{1}{n}=\\
&=\frac{1}{3}+\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{7}{3\cdot 4}
\end{aligned}$$

and analogously for the expressions in the last three parentheses of formula (8) we get

$$\begin{aligned}
\frac{1}{n-3}+\frac{1}{(n-2)(n-1)}+\frac{1}{(n-1)n}+\frac{1}{n}&=\frac{1}{n-3}+t_{n-2}-t_{n-4}+\frac{1}{n}=\\
&=\frac{1}{n-3}+\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{n-2}\right)=\frac{2n-5}{(n-3)(n-2)}\,,\\
\frac{1}{n-2}+\frac{1}{(n-1)n}+\frac{1}{n}&=\frac{1}{n-2}+t_{n-2}-t_{n-3}+\frac{1}{n}=\\
&=\frac{1}{n-2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{n-1}=\frac{2n-3}{(n-2)(n-1)}\,,\\
\frac{1}{n-1}+\frac{1}{n}&=\frac{2n-1}{(n-1)n}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
H_{n,3}=\frac{H_1}{1\cdot 2}\cdot\frac{3}{1\cdot 2}+\frac{H_2}{2\cdot 3}\cdot\frac{5}{2\cdot 3}+\frac{H_3}{3\cdot 4}\cdot\frac{7}{3\cdot 4}+\cdots\\
\cdots+\frac{H_{n-2}}{(n-2)(n-1)}\cdot\frac{2n-3}{(n-2)(n-1)}+\frac{H_{n-1}}{(n-1)n}\cdot\frac{2n-1}{(n-1)n}+\frac{H_n}{n^2}\,,
\end{aligned}$$

hence

$$H_{n,3}=\frac{3H_1}{(1\cdot 2)^2}+\frac{5H_2}{(2\cdot 3)^2}+\cdots+\frac{(2n-3)H_{n-2}}{[(n-2)(n-1)]^2}+\frac{(2n-1)H_{n-1}}{[(n-1)n]^2}+\frac{H_n}{n^2}\,,$$

thus

$$H_{n,3}=\sum_{k=1}^{n-1}\frac{(2k+1)H_k}{[k(k+1)]^2}+\frac{H_n}{n^2}\,. \tag{9}$$

From formulas (6) and (9) we obtain

**Theorem 2.2.** *The series* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k-a)^3}$, *where $a$ is a negative integer, has the sum*

$$s'(a,a,a)=\zeta(3)-\sum_{k=1}^{-a-1}\frac{(2k+1)H_k}{[k(k+1)]^2}-\frac{H_{-a}}{a^2}\,. \tag{10}$$

**Remark 2.1.** In [4] it is derived that a good approximation for the partial sum $s_n(0,0,0)$ is the expression

$$\sum_{k=1}^{n}\frac{1}{k^3}\approx\zeta(3)-\frac{1}{4}\left(\frac{2}{n^2}-\frac{2}{n^3}+\frac{1}{n^4}\right). \tag{11}$$

It is stated that for small $n$, say, $n=5$, the relative error in the above approximation is vanishingly small, i.e. about 0.03%, and that for larger $n\sim 1000$, the error is swamped by machine precision.

If we use formulas (11) and (5), where $a$ is a negative integer, we get

$$\begin{aligned}s(a,a,a)&=s(0,0,0)-s_{-a}(0,0,0)\ =\\ &=\zeta(3)-\sum_{k=1}^{-a}\frac{1}{k^3}\approx\zeta(3)-\left[\zeta(3)-\frac{1}{4}\left(\frac{2}{(-a)^2}-\frac{2}{(-a)^3}+\frac{1}{(-a)^4}\right)\right],\end{aligned}$$

so we have an approximate formula $s(a,a,a)\approx\dfrac{1}{4}\left(\dfrac{2}{a^2}+\dfrac{2}{a^3}+\dfrac{1}{a^4}\right)$ and an approximate sum

$$\overline{s}(a,a,a)=\frac{2a^2+2a+1}{4a^4}\,. \tag{12}$$

**Example 2.1.** Evaluate the sum of the series

$$\sum_{k=1}^{\infty}\frac{1}{(k+5)^3}$$

by means of formula: **i)** (7), **ii)** (10), **iii)** (5), **iv)** (12) and compare obtained results.

**Solution:**

**i)** The series has by Theorem 2.1, where $a=-5$, the sum

$$s(-5,-5,-5)=\zeta(3)-\sum_{k=1}^{4}\frac{1}{k(k+1)}\left(\sum_{i=1}^{k-1}\frac{H_i}{i(i+1)}+\frac{H_k}{k}+\frac{H_k}{5}\right)-\frac{H_5}{25}\,.$$

The last summand $\dfrac{H_5}{25}=\dfrac{137/60}{25}=\dfrac{137}{1500}$. Now, we evaluate the middle summand:

$$\begin{aligned}\sum_{k=1}^{4}\frac{1}{k(k+1)}\left(\sum_{i=1}^{k-1}\frac{H_i}{i(i+1)}+\frac{H_k}{k}+\frac{H_k}{5}\right)&=\frac{1}{1\cdot 2}\left(\frac{H_1}{1}+\frac{H_1}{5}\right)+\\ &+\frac{1}{2\cdot 3}\left(\frac{H_1}{1\cdot 2}+\frac{H_2}{2}+\frac{H_2}{5}\right)+\frac{1}{3\cdot 4}\left(\frac{H_1}{1\cdot 2}+\frac{H_2}{2\cdot 3}+\frac{H_3}{3}+\frac{H_3}{5}\right)+\\ &+\frac{1}{4\cdot 5}\left(\frac{H_1}{1\cdot 2}+\frac{H_2}{2\cdot 3}+\frac{H_3}{3\cdot 4}+\frac{H_4}{4}+\frac{H_4}{5}\right).\end{aligned}$$

If we denote this summand as $S$ and use the values of the first five harmonic numbers from the table above, we get

$$\begin{aligned}S&=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{5}\right)+\frac{1}{6}\left(\frac{1}{2}+\frac{3/2}{2}+\frac{3/2}{5}\right)+\frac{1}{12}\left(\frac{1}{2}+\frac{3/2}{6}+\frac{11/6}{3}+\frac{11/6}{5}\right)+\\ &+\frac{1}{20}\left(\frac{1}{2}+\frac{3/2}{6}+\frac{11/6}{12}+\frac{25/12}{4}+\frac{25/12}{5}\right)+\\ &=\frac{1}{2}\cdot\frac{6}{5}+\frac{1}{6}\cdot\frac{31}{20}+\frac{1}{12}\cdot\frac{311}{180}+\frac{1}{20}\cdot\frac{265}{144}=\frac{1891}{1728}\,.\end{aligned}$$

Altogether we have

$$s(-5,-5,-5)=\zeta(3)-\frac{1891}{1728}-\frac{137}{1500}=\zeta(3)-\frac{256103}{216000}\doteq 0.016394866122\,.$$

**ii)** By Theorem 2.2 we get an easy and effective way how to obtain the required sum:

$$\begin{aligned}s'(-5,-5,-5)&=\zeta(3)-\sum_{k=1}^{4}\frac{(2k+1)H_k}{[k(k+1)]^2}-\frac{H_5}{25}=\\ &=\zeta(3)-\frac{3H_1}{(1\cdot 2)^2}-\frac{5H_2}{(2\cdot 3)^2}-\frac{7H_3}{(3\cdot 4)^2}-\frac{9H_4}{(4\cdot 5)^2}-\frac{H_5}{25}\,.\end{aligned}$$

By means of the first five values of the harmonic numbers we have

$$s'(-5,-5,-5) = \zeta(3) - \frac{3}{4}\cdot 1 - \frac{5}{36}\cdot\frac{3}{2} - \frac{7}{144}\cdot\frac{11}{6} - \frac{9}{400}\cdot\frac{25}{12} - \frac{1}{25}\cdot\frac{137}{60} =$$
$$= \zeta(3) - \frac{256103}{216000} \doteq 0.016394866122\,.$$

**iii)** The third and in this case much more easily way, how to determine the sum $s(-5,-5,-5)$, is to use formula (5) and the value of $s_5(0,0,0) = H_{5,3}$ from the table above. So we immediately obtain the required result:

$$s(-5,-5,-5) = s(0,0,0) - s_5(0,0,0) = \zeta(3) - \frac{256103}{216000} \doteq 0.016394866122\,.$$

**iv)** If we use formula (12), we get the approximate sum

$$\overline{s}(-5,-5,-5) = \frac{2(-5)^2+2(-5)+1}{4(-5)^4} = \frac{2\cdot 5^2 - 2\cdot 5+1}{4\cdot 5^4} = \frac{41}{2500} = 0.0164\,.$$

Formulas (7), (10), and (5) give identical result $0.016394866122$, while formula (12) gives approximate result $0.0164$. The relative error of the fourth approximate result is $3.13\cdot 10^{-4} \sim 0.03\,\%$.

# 3 Numerical verification

We solve the problem to determine the values of the sum $s(a,a,a)$ of the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k-a)^3}$ for $a = -1,-2,\ldots,-10,-99,-100,-500,-999,-1000$. We use on the one hand an approximate evaluation of the sum

$$s(a,a,a,t) = \sum_{k=1}^{t}\frac{1}{(k-a)^3},$$

where $t = 10^6$, and formula (12) for approximate evaluation sum $\overline{s}(a,a,a)$, and on the other hand formulas (7) and (10) for evaluation the sum $s(a,a,a)$ We compare 15 quadruplets of the sums $s(a,a,a)$, $s'(a,a,a)$, $s(a,a,a,10^6)$, and $\overline{s}(a,a,a)$ to verify formulas (7) and (10) and to determine the relative error of two approximate sums $s(a,a,a,10^6)$ and $\overline{s}(a,a,a)$. We use procedures `hnum` and `rp3aaaneg` written in the basic programming language of the CAS Maple 16 and one **for** statement:

```
hnum:=proc(n)
    local m,h; h:=0;
    for m from 1 to n do
        h:=h+1/m;
    end do;
end proc;

rp3aaaneg:=proc(a,t)
    local i,k,A,A2,s,s1,s2,s3,saaa,s2aaa,sumaaa,sumaaaline,z3;
    A:=-a; A2:=A*A; s:=0; saaa:=0; s2aaa:=0; sumaaa:=0;
    z3:=1.2020569031595942854O;
    for k from 1 to A-1 do
        s1:=0; s2:=0; s3:=0;
        if k-1=0 then s2:=0 else
        for i from 1 to k-1 do
            s2:=s2+hnum(i)/(i*(i+1));
        end do;
        end if;
        s2:=s2+hnum(k)/k+hnum(k)/A;
        s1:=s1+s2/(k*(k+1));
        s:=s+s1;
        s3:=s3+((2*k+1)*hnum(k))/(k*k*(k+1)*(k+1));
    end do;
    saaa:=z3-s-hnum(A)/A2; s2aaa:=z3-s3-hnum(A)/A2;
    print("a=",a,":saaa=",evalf[20](saaa),s2aaa=",evalf[20](s2aaa));
    for k from 1 to t do
        sumaaa:=sumaaa+1/((k-a)*(k-a)*(k-a));
    end do;
    print("sumaaa(",t,")=",evalf[20](sumaaa));
    sumaaaline:=(2*A2+2*a+1)/(4*A2*A2);
    print("sumaaaline=",evalf[20](sumaaaline));
    print("rerrsumaaa=",evalf[20]((abs(sumaaa-saaa))/saaa));
    print("rerrsumaaaline=",evalf[20]((abs(sumaaaline-saaa))/saaa));
end proc:

A:=[-1,-2,-3,-4,-5,-6,-7,-8,-9,-10,-99,-100,-500,-999,-1000];
for a in A do
    rp3aaaneg(a,1000000);
end do;
```

The approximate values of the sums $s(a,a,a)$ and $s(a,a,a,10^6)$, denoted briefly $s$ and $s(10^6)$, and the sum $\overline{s}(a,a,a)$, denoted $\overline{s}$, obtained by the procedures above and rounded to 9 decimals, are written into the following table (the sums $s'(a,a,a)$ give identically values as the sums $s(a,a,a)$):

| $a$ | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ |
|---|---|---|---|---|---|
| $s$ | 0.202056903 | 0.077056903 | 0.040019866 | 0.024394866 | 0.016394866 |
| $s(10^6)$ | 0.202056903 | 0.077056903 | 0.040019866 | 0.024394866 | 0.016394866 |
| $\overline{s}$ | 0.250000000 | 0.078125000 | 0.040123457 | 0.024414063 | 0.016400000 |
| $a$ | $-6$ | $-7$ | $-8$ | $-9$ | $-10$ |
| $s$ | 0.011765236 | 0.008849784 | 0.006896659 | 0.005524917 | 0.004524917 |
| $s(10^6)$ | 0.011765236 | 0.008849785 | 0.006896660 | 0.005524917 | 0.004524917 |
| $\overline{s}$ | 0.011766975 | 0.008850479 | 0.006896973 | 0.005525072 | 0.004525000 |
| $a$ | $-99$ | $-100$ | $-500$ | $-999$ | $-1000$ |
| $s$ | 0.000050502 | 0.000049502 | 0.000001996 | 0.000000501 | 0.000000500 |
| $s(10^6)$ | 0.000050502 | 0.000049502 | 0.000001996 | 0.000000500 | 0.000000499 |
| $\overline{s}$ | 0.000050503 | 0.000049503 | 0.000001996 | 0.000000501 | 0.000000500 |

Computation of 15 quadruplets of the sums $s(a,a,a)$, $s'(a,a,a)$, $s(a,a,a,10^6)$ and $\overline{s}(a,a,a)$ took about 21 hours and 30 minutes. The relative errors of the approximate sums $s(a,a,a,10^6)$, i.e. the ratios

$$\big|[s(a,a,a,10^6) - s(a,a,a)]/s(a,a,a)\big|,$$

range from $10^{-10}$ (for $a=-1$) to $10^{-5}$ (for $a=-1000$), and the relative errors of the approximate sums $\overline{s}(a,a,a)$, i.e. the ratios

$$\big|[\overline{s}(a,a,a) - s(a,a,a)]/s(a,a,a)\big|,$$

range from $10^{-1}$ (for $a=-1$) to $10^{-5}$ (for $a=-1000$).

# 4 Conclusion

We dealt with the sum of the series of reciprocals of the cubic polynomials with triple non-positive integer root $a$, i.e. with the series $\sum\limits_{k=1}^{\infty}\frac{1}{(k-a)^3}$. We stated that its sum clearly can be for great number of members $t$ (we used $t=10^6$) approximately computed by formula

$$s(a,a,a,t) = \sum_{k=1}^{t}\frac{1}{(k-a)^3},$$

we derived that the approximate value of its sum is for a negative $a$ given by simple formula

$$\overline{s}(a,a,a) = \frac{2a^2+2a+1}{4a^4},$$

and we derived that the precise value of the sum is for a negative $a$ given by formula $s(a,a,a) = \zeta(3) - H_{-a,3}$ , i.e. by formula

$$s(a,a,a) = \zeta(3) - \sum_{k=1}^{-a-1} \frac{1}{k(k+1)} \left( \sum_{i=1}^{k-1} \frac{H_i}{i(i+1)} + \frac{H_k}{k} - \frac{H_k}{a} \right) - \frac{H_{-a}}{a^2} \,,$$

and also by easier formula

$$s'(a,a,a) = \zeta(3) - \sum_{k=1}^{-a-1} \frac{(2k+1)H_k}{[k(k+1)]^2} - \frac{H_{-a}}{a^2} \,.$$

We verified these results by computing 15 quadruplets of the four sums above for $a = -1, -2, \dots, -10,\ -99, -100, -500, -999, -1000$ by using the CAS Maple 16 and compared their values. The series of reciprocals of the cubic polynomials with triple non-positive integer root so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a formula which can be expressed in closed form.

# References

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