

# Finite $H_v$ -Fields with Strong-Inverses

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## Abstract

The largest class of hyperstructures is the class of  $H_v$ -structures. This is the class of hyperstructures where the equality is replaced by the non-empty intersection. This extremely large class can be used to define several objects that are not possible to be defined in the classical hypergroup theory. It is convenient, in applications, to use more axioms and conditions to restrict the research in smaller classes. In this direction, in the present paper we continue our study on  $H_v$ -structures which have strong-inverse elements. More precisely we study the small finite cases.

**Keywords:** hyperstructure;  $H_v$ -structure; hope; strong-inverse elements.

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## 1 Introduction

First we present some basic definitions on hyperstructures, mainly on the weak hyperstructures introduced in 1990 [7].

**Definition 1.1.** *Hyperstructures are called the algebraic structures equipped with, at least, one hyperoperation. Abbreviate: **hyperoperation=hope**. The **weak hyperstructures** are called  **$H_v$ -structures** and they are defined as follows:*

*In a set  $H$  equipped with a hope  $\cdot : H \times H \rightarrow \wp(H) - \{\emptyset\}$ , we abbreviate by*

**WASS** the **weak associativity**:  $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$  and by

**COW** the **weak commutativity**:  $xy \cap yx \neq \emptyset, \forall x, y \in H$ .

*The hyperstructure  $(H, \cdot)$  is called an  **$H_v$ -semigroup** if it is WASS, is called  **$H_v$ -group** if it is reproductive  $H_v$ -semigroup:*

$$xH = Hx = H, \forall x \in H.$$

*$(R, +, \cdot)$  is called  **$H_v$ -ring** if the hopes  $(+)$  and  $(\cdot)$  are WASS, the reproduction axiom is valid for  $(+)$ , and  $(\cdot)$  is **weak distributive** with respect to  $(+)$ :*

$$x(y + z) \cap (xy + xz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

An  $H_v$ -group is called *cyclic* [6], [8], if there is an element, called *generator*, which the powers have union the underline set. The minimal power with this property is called the period of the generator. If there is an element and a special power, the minimum one, is the underline set, then the  $H_v$ -group is called *single-power cyclic*.

For more definitions, results and applications on hyperstructures and mainly on the  $H_v$ -structures, see books as [1], [2], [8] and papers as [6], [9], [8], [10], [11], [12], to mention but a few of them. An extreme class of hyperstructures is the following: An  $H_v$ -structure is called **very thin** if and only if, all hopes are operations except one, with all hyperproducts to be singletons except only one, which is a subset with cardinality more than one.

The **fundamental relations**  $\beta^*$  and  $\gamma^*$  are defined, in  $H_v$ -groups and  $H_v$ -rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively. Normally to find the fundamental classes is very hard job. The basic theorems on the fundamental classes are analogous to the following:

**Theorem 1.1.** [8] *Let  $(H, \cdot)$  be an  $H_v$ -group and let us denote by  $U$  the set of all finite products of elements of  $H$ . We define the relation  $\beta$  in  $H$  as follows:  $x\beta y$  iff  $\{x, y\} \subset u$  where  $u \in U$ . Then the fundamental relation  $\beta^*$  is the transitive closure of the relation  $\beta$ .*

**Proof.** See [7], [8].

An element of a hyperstructure is called **single** if its fundamental class is a singleton.

**Motivation** for  $H_v$ -structures:

*The quotient of a group with respect to an invariant subgroup is a group.*

*F. Marty states that, the quotient of a group by any subgroup is a hypergroup.*

*Now, the quotient of a group with respect to any partition is an  $H_v$ -group.*

We remark that in  $H_v$ -groups (or even in hypergroups in the sense of F. Marty) we do not have necessarily any 'unit' element, consequently neither 'inverses'. However, we may have more than one unit elements and for each element of an  $H_v$ -group we may have one inverse element or more than one inverse element.  $\square$

**Definition 1.2.** *Let  $(H, \cdot)$  be an  $H_v$ -semigroup. An element  $e$ , is called left unit if  $ex \ni x, \forall x \in H$ , it is called right unit if  $xe \ni x, \forall x \in H$  and it is called **unit** element if it is both left and right unit element. For given unit  $e$ , an element  $x \in H$ , has a left inverse with respect to  $e$ , any element  $x_{le}$  if  $x_{le} \cdot x \ni e$ , it has a right inverse element  $x_{re}$  if  $x \cdot x_{re} \ni e$ , and it has an inverse  $x_e$  with respect to  $e$ , if  $e \in x_e \cdot x \cap x \cdot x_e$ . Denote by  $E_l$  the set of all left unit elements, by  $E_r$  the set of all right unit elements, and by  $E$  the set of unit elements.*

**Definition 1.3.** [16], [5] *Let  $(H, \cdot)$  be an  $H_v$ -semigroup. An element is called **strong-inverse** if it is an inverse to  $x$  with respect to all unit elements.*

**Remark 1.1.** *We remark that an element  $x_s$  is a strong-inverse to  $x$ , if  $E \subset x_s \cdot x \cap x \cdot x_s$ . Therefore the strong-inverse property it is not exists in the classical structures.*

**Definition 1.4.** *Let  $(H, \cdot), (H, \otimes)$  be  $H_v$ -semigroups defined on the same  $H$ .  $(\cdot)$  is **smaller** than  $(\otimes)$ , and  $(\otimes)$  **greater** than  $(\cdot)$ , if and only if, there exists an automorphism  $f \in \text{Aut}(H, \otimes)$  such that  $xy \subset f(x \otimes y), \forall x, y \in H$ . Then  $(H, \otimes)$  contains  $(H, \cdot)$  and write  $\cdot \leq \otimes$ . If  $(H, \cdot)$  is a structure, then it is basic and  $(H, \otimes)$  is an  $H_b$ -structure.*

**The Little Theorem.** In a set, greater hopes of the ones which are WASS or COW, are also WASS or COW, respectively.

The fundamental relations are used for general definitions, thus, for example, in order to define the general  $H_v$ -field one uses the fundamental relation  $\gamma^*$ :

**Definition 1.5.** [7], [8], [9] *The  $H_v$ -ring  $(R, +, \cdot)$  is an  **$H_v$ -field** if the quotient  $R/\gamma^*$  is a field. The definition of the  $H_v$ -field introduced a new class of hyperstructures [12]: The  $H_v$ -semigroup  $(H, \cdot)$  is  $h/v$ -group if the quotient  $H/\beta^*$  is a group.*

More complicated hyperstructures can be defined as well:

**Definition 1.6.** Let [8]  $(R, +, \cdot)$  be an  $H_v$ -ring,  $(\mathcal{M}, +)$  be a COW  $H_v$ -group and there exists an external hope

$$\cdot : \mathbf{R} \times \mathcal{M} \rightarrow \wp(\mathcal{M}) : (a, x) \rightarrow ax$$

such that  $\forall a, b \in R$  and  $\forall x, y \in \mathcal{M}$  we have

$$a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

then  $\mathcal{M}$  is called an  $H_v$ -module over  $F$ . In the case of an  $H_v$ -field  $F$  instead of an  $H_v$ -ring  $R$ , then the  **$H_v$ -vector space** is defined.

In the above cases the fundamental relation  $\epsilon^*$  is defined to be the smallest equivalence relation such that the quotient  $\mathcal{M}/\epsilon^*$  is a module (resp. vector space) over the fundamental ring  $R/\gamma^*$  (resp. fundamental field  $F/\gamma^*$ ).

The general definition of an  $H_v$ -Lie algebra was given in [14] as follows:

**Definition 1.7.** Let  $(\mathbf{L}, +)$  be an  $H_v$ -vector space over the  $H_v$ -field  $(\mathbf{F}, +, \cdot)$ ,  $\phi : \mathbf{F} \rightarrow \mathbf{F}/\gamma^*$  the canonical map and  $\omega_F = \{x \in \mathbf{F} : \phi(x) = 0\}$ , where 0 is the zero of the fundamental field  $\mathbf{F}/\gamma^*$ . Similarly, let  $\omega_L$  be the core of the canonical map  $\phi' : \mathbf{L} \rightarrow \mathbf{L}/\epsilon^*$  and denote by the same symbol 0 the zero of  $L/\epsilon^*$ . Consider the bracket (commutator) hope:

$$[,] : \mathbf{L} \times \mathbf{L} \rightarrow \wp(L) : (x, y) \rightarrow [x, y]$$

then  $\mathbf{L}$  is an  $H_v$ -Lie algebra over  $F$  if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$\begin{aligned} & [\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset \\ & [x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset, \\ & \forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in \mathbf{F} \end{aligned}$$

(L2)  $[x, x] \cap \omega_L \neq \emptyset, \forall x \in \mathbf{L}$

(L3)  $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \forall x, y \in \mathbf{L}$

**Definition 1.8.** [10] Let  $(H, \cdot)$  be a hypergroupoid. We say that we remove the element  $h \in H$ , if we simply consider the restriction of  $(\cdot)$  on  $H - \{h\}$ . We say that the element  $\underline{h} \in H$  absorbs the element  $h \in H$  if we replace  $h$ , whenever it appears, by  $\underline{h}$ . We say that the element  $\underline{h} \in H$  merges with the element  $h \in H$ , if we take as product of  $x \in H$  by  $\underline{h}$ , the union of the results of  $x$  with both  $h$  and  $\underline{h}$ , and consider  $h$  and  $\underline{h}$  as one class, with representative the element  $\underline{h}$ .

## 2 Large classes of $H_v$ -structures and applications

The large class of, so called, P-hyperstructures was appeared in 80's to represent hopes of constant length [6]. Since then several classes of P-hopes were introduced and studied [8], [4], [11].

**Definition 2.1.** Let  $(G, \cdot)$  be a groupoid, then for all  $P$  such that  $\emptyset \neq P \subset G$ , we define the following hopes called **P-hopes**:  $\forall x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP),$$

$$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The  $(G, \underline{P}), (G, \underline{P}_r), (G, \underline{P}_l)$  are called *P-hyperstructures*. The most usual case is when  $(G, \cdot)$  is semigroup, then we have

$$x\underline{P}y = (xP)y \cup x(Py) = xPy$$

and  $(G, \underline{P})$  is a semihypergroup.

It is immediate the following: Let  $(G, \cdot)$  be a group, then for all subsets  $P$  such that  $\emptyset \neq P \subset G$ , the hyperstructure  $(G, \underline{P})$ , where the P-hope is  $x\underline{P}y = xPy$ , becomes a hypergroup in the sense of Marty, i.e. the strong associativity is valid. The P-hope is of constant length, i.e. we have  $|x\underline{P}y| = |P|$ . We call the hyperstructure  $(G, \underline{P})$ , *P-hypergroup*.

In [4], [15] a modified P-hope was introduced which is appropriate for the e-hyperstructures:

**Construction 2.1.** Let  $(G, \cdot)$  be abelian group and  $P$  any subset of  $G$  with more than one elements. We define the hyperoperation  $\times_P$  as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope  $P_e$ -hope. The hyperstructure  $(G, \times_P)$  is an abelian  $H_v$ -group.

Another large class is the one on which a new hope ( $\partial$ ) in a groupoid is defined.

**Definition 2.2.** [13]. Let  $(G, \cdot)$  be groupoid (resp. hypergroupoid) and  $f : G \rightarrow G$  be a map. We define a hope ( $\partial$ ), called *theta-hope* or simply  $\partial$ -hope, on  $G$  as follows

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G. \text{ (resp. } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G)$$

If  $(\cdot)$  is commutative then  $(\partial)$  is commutative. If  $(\cdot)$  is COW, then  $(\partial)$  is COW.

Let  $(G, \cdot)$  be groupoid (resp. hypergroupoid) and  $f : G \rightarrow \mathbf{P}(G) - \{\emptyset\}$  be multivalued map. We define the hope  $(\partial)$ , on  $G$  as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \quad \forall x, y \in G$$

**Properties.** If  $(G, \cdot)$  is a semigroup then:

- (a) For every  $f$ , the hope  $(\partial)$  is WASS.
- (b) If  $f$  is homomorphism and projection, or idempotent:  $f^2 = f$ , then  $(\partial)$  is associative.

Let  $(G, \cdot)$  be a groupoid and  $f_i : G \rightarrow G, i \in I$ , be a set of maps on  $G$ . We consider the map  $f_{\cup} : G \rightarrow \mathbf{P}(G)$  such that  $f_{\cup}(x) = \{f_i(x) | i \in I, \}$  called the *the union* of the  $f_i(x)$ . We define the *union  $\partial$ -hope*, on  $G$  if we consider as map the  $f_{\cup}(x)$ . A special case for a given map  $f$ , is to take the union of this with the identity map. We consider the map  $\underline{f} \equiv f \cup (id)$ , so  $\underline{f}(x) = \{x, f(x)\}, \forall x \in G$ , which we call  *$b-\partial$ -hope*. Then we have

$$x\partial y = \{xy, f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G$$

Motivation for the definition of the  $\partial$ -hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for given functions  $s(x), t(x)$ , we have  $s\partial t = \{s't, st'\}$  where  $(')$  denotes the derivative.

**Proposition 2.1.** Let  $(G, \cdot)$  be group and  $f(x) = a$ , a constant map. Then  $(G, \partial)/\beta^*$  is a singleton.

**Proof.** For all  $x$  in  $G$  we can take the hyperproduct of the elements,  $a^{-1}$  and  $a^{-1}x$

$$a^{-1}\partial(a^{-1} \cdot x) = \{f(a^{-1}) \cdot a^{-1} \cdot x, a^{-1} \cdot f(a^{-1} \cdot x)\} = \{x, a\}.$$

thus  $x\beta a, \forall x \in G$ , so  $\beta^*(x) = \beta^*(a)$  and  $(G, \partial)/\beta^*$  is singleton.  $\square$

Special case if  $(G, \cdot)$  be a group and  $f(x) = e$ , then  $x\partial y = \{x, y\}$ , is the incidence hope.

Taking the application on the derivative, consider all polynomials of the first degree  $g_i(x) = a_i x + b_i$ . We have  $g_1\partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\}$ , so it is a hope on the set of first degree polynomials. Moreover all polynomials  $x + c$ , where  $c$  be a constant, are units.

The Lie-Santilli *isotopies* born to solve Hadronic Mechanics problems. Santilli [4], [15], proposed a 'lifting' of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is

reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed correspond to  $H_v$ -structures called *e-hyperfields* which are used in physics or biology. Therefore, in this theory the units and the inverses are playing very important role. The Construction 2.1, is used, last years, in this theory.

**Example 2.1.** Consider the 'small' ring  $(\mathbf{Z}_4, +, \cdot)$ , suppose that we want to construct non-degenerate  $H_v$ -field where 0 and 1 are scalars with respect for both addition and multiplication, and moreover every element of  $\mathbf{Z}_4$  has a unique opposite and every non-zero element has a unique inverse. Then on the multiplication tables of these operations the lines and columns of the elements 0 and 1 remain the same. The sum  $2+2=0$  and the product  $3 \cdot 3 = 1$ , remain the same. In the results of the sums  $2+3=1$ ,  $3+2=1$  and  $3+3=2$  one can put respectively, the elements 3, 3 and 0. In the results of the products  $2 \cdot 2 = 0$ ,  $2 \cdot 3 = 2$  and  $3 \cdot 2 = 2$  one can put respectively, the elements 2, 0 and 0. Then in all those enlargements, even only one enlargement is used, we obtain

$$(\mathbf{Z}_4, +, \cdot) / \beta^* \cong \mathbf{Z}_2$$

Therefore this construction gives 49  $H_v$ -fields.

### 3 Strong-inverse elements.

We present now some hyperstructures, results and examples of hyperstructures with strong-inverse elements.

**Properties 3.1.** Let  $(G, \cdot)$  be a group, take  $P$  such that  $\emptyset \neq P \subset G$  and the  $P$ -hypergroup  $(G, \underline{P})$ , where  $x\underline{P}y = xPy$ . We have the following

*Units:* In order an element  $u$  to be right unit of the  $P$ -hypergroup  $(G, \underline{P})$ , we must have  $x\underline{P}u = xPu \ni x, \forall x \in G$ . In fact the set  $Pu$  must contain the unit element  $e$  of the group  $(G, \cdot)$ . Thus, all the elements of the set  $P^{-1}$ , are right units. The same is valid for the left units, therefore, the set of all units is the  $P^{-1}$ .

*Inverses:* Let  $u$  be a unit in  $(G, \underline{P})$ , then, for given  $x$  in order to have an inverse element  $x'$  with respect to  $u$ , we must have  $x\underline{P}x' = xPx' \ni u$ , so taking  $xpx' = u$ , we obtain that all the elements of the form  $x' = p^{-1}x^{-1}u$  are inverses to  $x$  with respect to the unit  $u$ .

**Theorem 3.1.** [16] Let  $(G, \cdot)$  be a group, then for all normal subgroups  $P$  of  $G$ , the hyperstructure  $(G, \underline{P})$ , where  $x\underline{P}y = xPy, \forall x, y \in G$ , is a hypergroup with strong inverses. Moreover, for any inverse  $x'$  of  $x \in G$ , with respect to any unit, we have  $x\underline{P}x' = P$ .

**Proof.** Let  $x \in G$ , take an inverse  $x' = p^{-1}x^{-1}u$  with respect to the unit  $u = p_k^{-1}$ , for any  $p$ . Then we have  $x\underline{P}x' = xPx'$ . But, since  $P$  is normal subgroup, we have

$$x\underline{P}x' = xp^{-1}x^{-1}p_k^{-1}P = xp^{-1}x^{-1}P = xp^{-1}Px^{-1} = xPx^{-1} = P$$

Remark that in this case,  $P^{-1} = P$ , is the set of all units, thus all inverses are strong.  $\square$

**Properties 3.2.** Let  $(G, \cdot)$  be groupoid and  $f : G \rightarrow G$  be a map and  $(G, \partial)$  the corresponding  $\partial$ -structure, then we have the following:

*Units:* In order an element  $u$  to be right unit, we must have

$$x\partial u = \{f(x) \cdot u, x \cdot f(u)\} \ni x.$$

But, the unit must not depend on the  $f(x)$ , so  $f(u) = e$ , where  $e$  be unit in  $(G, \cdot)$  which must be a monoid. The same it is obtained for the left units. So the elements of  $\text{kern}f = \{u : f(u) = e\}$ , are the units of  $(G, \partial)$ .

*Inverses:* Let  $u$  be a unit in  $(G, \partial)$ , then  $(G, \cdot)$  is a monoid with unit  $e$  and  $f(u) = e$ . For given  $x$  in order to have an inverse element  $x'$  with respect to  $u$ , we must have

$$x\partial x' = \{f(x) \cdot x', x \cdot f(x')\} \ni u \text{ and } x'\partial x = \{f(x') \cdot x, x' \cdot f(x)\} \ni u.$$

So the only cases, which do not depend on the image  $f(x')$ , are

$$x' = (f(x))^{-1}u \text{ and } x' = u(f(x))^{-1}$$

the right and left inverses, respectively. We have two-sided inverses iff  $f(x)u = uf(x)$ .

*Remark [16]:* Since the inverses are depending on the units, therefore they are not strong.

The following constructions, originated from the properties the strong-inverse elements have, gives a minimal hyperstructure which have strong-inverse elements. This is a necessary enlargement in order all the elements to be strong-inverses.

**Construction 3.1.** Let  $(G, \cdot)$  be a group with unit  $e$ . Consider a finite set  $E = \{e_i | i \in I\}$ . On the set  $\underline{G} = (G - \{e\}) \cup E$  we define a hope  $(\times)$  as follows:

$$\begin{cases} e_i \times e_j = \{e_i, e_j\}, \forall e_i, e_j \in E \\ e_i \times x = x \times e_j = x, \forall e_i \in E, x \in G - \{e\} \\ x \times y = x \cdot y \text{ if } x \cdot y \in G - \{e\} \text{ and } x \times y = E \text{ if } x \cdot y = e \end{cases}$$

Then the hyperstructure  $(\underline{G}, \times)$  is a hypergroup. The set of unit elements is  $E$  and all the elements are strong-inverse. Moreover we have  $(\underline{G}, \times)/\beta^* \cong (G, \cdot)$ .



### Finite $H_v$ -Fields with Strong-Inverses

**Proof.** For the associativity we have the cases

$$(e_i \times e_j) \times e_k = e_i \times (e_j \times e_k) = \{e_i, e_j, e_k\}, \forall e_i, e_j, e_k \in E$$

$$(x \times y) \times z = x \times (y \times z) = x \cdot y \cdot z \text{ or } E, \forall x, y, z \in \underline{G} \text{ and not all of them belong to } E.$$

In the second case there is no matter if the product of two inverse elements appears. The only difference is that the result is singleton and in some cases the result is equal to the set  $E$ .

Therefore the strong associativity is valid. Moreover the reproductivity is valid and the set  $E$  is the set of units in  $(\underline{G}, \times)$ .

Two elements of  $\underline{G}$  are  $\beta^*$  equivalent if they belong to any finite  $\times$ -product of elements of  $\underline{G}$ . Thus all fundamental classes are singletons except the set of units  $E$ . That means that we have  $(\underline{G}, \times)/\beta^* \cong (G, \cdot)$ .  $\square$

**Construction 3.2.** Let  $(G, \cdot)$  be an  $H_v$ -group with only one unit element  $e$  and every element has a unique inverse. Consider a finite set  $E = \{e_i | i \in I\}$ . On the set  $\underline{G} = (G - \{e\}) \cup E$  we define a hope  $(\times)$  as follows:

$$\begin{cases} e_i \times e_j = \{e_i, e_j\}, \forall e_i, e_j \in E \\ e_i \times x = x \times e_j = x, \forall e_i \in E, x \in G - \{e\} \\ x \times y = x \cdot y \text{ if } x \cdot y \in G - \{e\} \text{ and } x \times y = E \text{ if } x \cdot y = e \end{cases}$$

Then the hyperstructure  $(\underline{G}, \times)$  is an  $H_v$ -group. The set of unit elements is  $E$  and all the elements are strong-inverse. Moreover we have

$$(\underline{G}, \times)/\beta^* \cong (G, \cdot)/\beta^*.$$

**Proof.** For the associativity we have the cases

$$(e_i \times e_j) \times e_k = e_i \times (e_j \times e_k) = \{e_i, e_j, e_k\}, \forall e_i, e_j, e_k \in E$$

$$(x \times y) \times z = x \times (y \times z) = x \cdot y \cdot z \text{ or } E, \forall x, y, z \in \underline{G} \text{ and not all of them belong to } E.$$

Therefore the WASS is valid. Moreover the reproductivity is valid and the set  $E$  is the set of units in  $(\underline{G}, \times)$ .

Two elements of  $\underline{G}$  are  $\beta^*$  equivalent if they belong to any finite  $\times$ -product of elements of  $\underline{G}$ . So, all fundamental classes correspond to the fundamental classes of  $(G, \cdot)$ , with an enlargement of the class of  $e$  into  $E$ . Thus, we have  $(\underline{G}, \times)/\beta^* \cong (G, \cdot)/\beta^*$ .  $\square$

We remark that the above constructions give a great number of hyperstructures with strong-inverses because we can enlarge then in any result except if the result is  $E$ .

Now we present a result on strong-inverses on a general finite case.

**Theorem 3.2.** *The minimum non-degenerate, i.e. have non-degenerate fundamental field, h/v-fields with strong-inverses with respect to both sum-hope and product-hope, obtained by enlarging the ring  $(\mathbf{Z}_{2p}, +, \cdot)$ , where  $p > 2$  is prime number, and which has fundamental field isomorphic to  $(\mathbf{Z}_p, +, \cdot)$ , is defined as follows:*

*The sum-hope  $(\oplus)$  is enlarged from  $(+)$  by setting*

- (1).  $p(\oplus)\kappa = \kappa(\oplus)p = \kappa + E, \forall \kappa \in \mathbf{Z}_{2p}$ , where  $E = \{0, p\}$  be the set of zeros
- (2). *whenever the result is 0 and p we enlarge it by setting p and 0, respectively.*

*The product-hope  $(\otimes)$  is enlarged from  $(\cdot)$  by setting*

- (3).  $(p + 1) \otimes \kappa = \kappa \otimes (p + 1) = \kappa U, \forall \kappa \in \mathbf{Z}_{2p}$ , where  $U = \{1, p + 1\}$  be the set of units
- (4). *whenever the result is 1 and p+1 we enlarge it by setting p+1 and 1, respectively.*

*The fundamental classes are of the form  $\underline{\kappa} = \{\kappa, \kappa + p\}, \forall \kappa \in \mathbf{Z}_{2p}$*

**Proof.** In order to have non degenerate case, since we have  $2p$  elements, in both, sum-hope and product-hope, is to take the zero-set  $E = \{0, p\}$  and unit-set  $U = \{1, p + 1\}$ . In order to have strong-opposites, we have to enlarge, according to Remark 1.1, as in (2). Moreover, in order to have strong-inverses, we have to enlarge, again according to Remark 1.1, as in (4).

From the above definition of the sum-hope it is to see that the fundamental classes are of the form  $\underline{\kappa} = \{\kappa, \kappa + p\}, \forall \kappa \in \mathbf{Z}_{2p}$ .

For the above classes for the product-hope mod(2p), we have  $\forall \kappa, \lambda \in \mathbf{Z}_{2p}$ ,

$$\begin{aligned} \underline{\kappa} \otimes \underline{\lambda} &= \{\kappa, \kappa + p\} \cdot \{\lambda, \lambda + p\} = \{\kappa\lambda, \kappa(\lambda + p), (\kappa + p)\lambda, (\kappa + p)(\lambda + p)\} = \\ &= \{\kappa\lambda, \kappa\lambda + \kappa p, \kappa\lambda + p\lambda, \kappa\lambda + \kappa p + p\lambda + pp\} = \{\kappa\lambda, \kappa\lambda + p\} \end{aligned}$$

Because, if  $\kappa$  or  $\lambda$  are odd numbers then  $\kappa\lambda + \kappa p$  or  $\kappa\lambda + p\lambda$ , respectively, are equal mod $2p$  to  $\kappa\lambda + p$ . Moreover, if both  $\kappa$  and  $\lambda$  are even numbers then we have,  $\kappa\lambda + \kappa p + p\lambda + pp = \kappa\lambda + p$ .

From the above we remark that the fundamental classes  $\underline{\kappa} = \{\kappa, \kappa + p\}, \forall \kappa \in \mathbf{Z}_{2p}$ , are formed from sum-hope and they are remain the same in the product-hope. Finally the fundamental field is isomorphic to  $(\mathbf{Z}_p, +, \cdot)$ .  $\square$

As example of the above Theorem we present the case for  $p=5$ .

**Example 3.1.** *In the case of the h/v-field  $(\mathbf{Z}_{10}, \oplus, \otimes)$ , i.e.  $p=5$ , we have the following multiplicative tables:*

*Finite  $H_v$ -Fields with Strong-Inverses*

| $\oplus$ | <b>0</b> | <b>1</b> | <b>2</b> | <b>3</b> | <b>4</b> | <b>5</b> | <b>6</b> | <b>7</b> | <b>8</b> | <b>9</b> |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| <b>0</b> | 0        | 1        | 2        | 3        | 4        | 5,0      | 6        | 7        | 8        | 9        |
| <b>1</b> | 1        | 2        | 3        | 4        | 5,0      | 6,1      | 7        | 8        | 9        | 0,5      |
| <b>2</b> | 2        | 3        | 4        | 5,0      | 6        | 7,2      | 8        | 9        | 0,5      | 1        |
| <b>3</b> | 3        | 4        | 5,0      | 6        | 7        | 8,3      | 9        | 0,5      | 1        | 2        |
| <b>4</b> | 4        | 5,0      | 6        | 7        | 8        | 9,4      | 0,5      | 1        | 2        | 3        |
| <b>5</b> | 5,0      | 6,1      | 7,2      | 8,3      | 9,4      | 0,5      | 1,6      | 2,7      | 3,8      | 4,9      |
| <b>6</b> | 6        | 7        | 8        | 9        | 0,5      | 1,6      | 2        | 3        | 4        | 5,0      |
| <b>7</b> | 7        | 8        | 9        | 0,5      | 1        | 2,7      | 3        | 4        | 5,0      | 6        |
| <b>8</b> | 8        | 9        | 0,5      | 1        | 2        | 3,8      | 4        | 5,0      | 6        | 7        |
| <b>9</b> | 9        | 0,5      | 1        | 2        | 3        | 4,9      | 5,0      | 6        | 7        | 8        |

and

| $\otimes$ | <b>0</b> | <b>1</b> | <b>2</b> | <b>3</b> | <b>4</b> | <b>5</b> | <b>6</b> | <b>7</b> | <b>8</b> | <b>9</b> |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| <b>0</b>  | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        | 0        |
| <b>1</b>  | 0        | 1,6      | 2        | 3        | 4        | 5        | 6,1      | 7        | 8        | 9        |
| <b>2</b>  | 0        | 2        | 4        | 6,1      | 8        | 0        | 2        | 4        | 6,1      | 8        |
| <b>3</b>  | 0        | 3        | 6,1      | 9        | 2        | 5        | 8,3      | 1,6      | 4        | 7        |
| <b>4</b>  | 0        | 4        | 8        | 2        | 6,1      | 0        | 4        | 8        | 2        | 6,1      |
| <b>5</b>  | 0        | 5        | 0        | 5        | 0        | 5        | 0,5      | 5        | 0        | 5        |
| <b>6</b>  | 0        | 6,1      | 2        | 8,3      | 4        | 0,5      | 6,1      | 2,7      | 8        | 4,9      |
| <b>7</b>  | 0        | 7        | 4        | 1,6      | 8        | 5        | 2,7      | 9        | 6,1      | 3        |
| <b>8</b>  | 0        | 8        | 6,1      | 4        | 2        | 0        | 8        | 6,1      | 4        | 2        |
| <b>9</b>  | 0        | 9        | 8        | 7        | 6,1      | 5        | 4,9      | 3        | 2        | 1,6      |

Moreover, it is easy to see that the fundamental field is isomorphic to  $(\mathbf{Z}_5, +, \cdot)$ .

## References

- [1] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, 1993.
- [2] P. Corsini and V. Leoreanu, *Application of Hyperstructure Theory*, Klower Academic Publishers, 2003.
- [3] B. Davvaz and V. Leoreanu, *Hyperring Theory and Applications* Int. Acad. Press, 2007
- [4] B. Davvaz, R.M Santilli, and T. Vougiouklis, *Studies of multi-valued hyperstructures for the characterization of matter-antimatter systems and their extension*, Algebras, Groups and Geometries 28(1), (2011), 105-116.

- [5] R. Mahjoob, T. Kaplani and T. Vougiouklis,  *$H_v$ -groups with strong-inverses*, Submitted.
- [6] T. Vougiouklis, *Generalization of  $P$ -hypergroups*, Rend. Circolo Mat. Palermo, Ser. II, 36, (198), (1987), 114-121.
- [7] T. Vougiouklis, (1991). *The fundamental relation in hyperrings. The general hyperfield*, 4<sup>th</sup> AHA, Xanthi 1990, World Scientific, (1991), 203-211.
- [8] T. Vougiouklis, *Hyperstructures and their Representations*, Monographs in Math., Hadronic, 1994.
- [9] T. Vougiouklis, *Some remarks on hyperstructures*, Contemporary Math., Amer. Math. Society, 184, (1995), 427-431.
- [10] T. Vougiouklis, *Enlarging  $H_v$ -structures*, Algebras and Combinatorics, ICAC'97, Hong Kong, Springer-Verlag, (1999), 455-463.
- [11] T. Vougiouklis, *On  $H_v$ -rings and  $H_v$ -representations*, Discrete Math., Elsevier, 208/209, (1999), 615-620.
- [12] T. Vougiouklis, *The  $h/v$ -structures*, J. Discrete Math. Sciences and Cryptography, V.6, N.2-3, (2003), 235-243.
- [13] T. Vougiouklis,  *$\partial$ -operations and  $H_v$ -fields*, Acta Math. Sinica, (Engl. Ser.), V.24, N.7, (2008), 1067-1078.
- [14] T. Vougiouklis, *The Lie-hyperalgebras and their fundamental relations*, Southeast Asian Bull. Math., V. 37(4), (2013), 601-614.
- [15] T. Vougiouklis, *Hypermathematics,  $H_v$ -structures, hypernumbers, hypermatrices and Lie-Santilli admissibility*, American J. Modern Physics, 4(5), (2015), 34-46.
- [16] T. Vougiouklis, and T. Kaplani, *Special Elements on  $P$ -hopes and  $\partial$ -hopes*, Southeast Asian Bulletin Mathematics, Vol. 40(3), (2016), 451-460.