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Vougiouklis Contributions in the Field of Algebraic Hyperstructures

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Abstract

Thomas Vougiouklis was born in 1948, Greece. He has many contributions to algebraic hyperstructures. H_v-structures are some of his main contributions. In this article, we study some of Vougiouklis ideas in the field of algebraic hyperstructures as follows: (1) Semi-direct hyperproduct of two hypergroups; (2) Representation of hypergroups; (3) Fundamental relation in hyperrings; (4) Commutative rings obtained from hyperrings; (5) H_v-structures; (6) The uniting elements method; (7) The e-hyperstructures; (8) Helix-hyperoperations.
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1 Semi-direct hyperproduct of two hypergroups

For all natural numbers $n > 1$, define the relation β_n on a semihypergroup H, as follows: $a\beta_n b$ if and only if there exist $x_1, \ldots, x_n \in H$ such that $\{a, b\} \subseteq \prod_{i=1}^{n} x_i$, and take $\beta = \bigcup_{n\geq 1} \beta_n$, where $\beta_1 = \{(x,x) \mid x \in H\}$ is the diagonal relation on H. Denote by β^* the transitive closure of β. The relation β^* is a strongly regular relation. This relation was introduced by Koskas [10] and studied mainly by Freni [9], proving the following basic result: If H is hypergroup, then $\beta = \beta^*$. Note that, in general, for a semihypergroup may be $\beta \neq \beta^*$. Moreover, the relation β^* is the smallest equivalence relation on a hypergroup H, such that the quotient H/β^* is a group. The *heart* ω_H of a hypergroup H is defined like the set of all elements x of H, for which the equivalence class $\beta^*(x)$ is the identity of the quotient group H/β^*. Vougiouklis in [13] studied the fundamental relation introduced by Koskas. He used the quotient set in order to define a semi-direct hyperproduct of two hypergroups. He obtained an extension of hypergroups by hypergroups. Let A, B be two hypergroups and consider the group $AutA$. Let $\widehat{\ } : B/\beta^* \to AutA$ be an arbitrary homomorphism, where we denote $\widehat{\beta^*(b)}$ by $\widehat{b}$. Then, in $A \times B$ a hyperproduct can be defined as follows: $(a,b) \odot (c,d) = \{(x,y) \mid x \in a\widehat{b}(c),\ y \in bd\}$ Then, $A \times B$ becomes a hypergroup called semi-direct hyperproduct of A and B corresponding to $\widehat{\ }$ and it is denoted by $A \widehat{\times} B$. Vougiouklis proved that $A\widehat{\times}B/\beta^*_{A\widehat{\times}B} \cong A/\beta^*_A \widehat{\times} B/\beta^*_B$ [13].

2 Representation of hypergroups

Vougiouklis in a sequence of papers studied the representations of hypergroups. For instance, in [15], a class of hypermatrices to represent hypergroups is introduced and application on class of P-hypergroups is given. Hypermatrices are matrices with entries of a semihyperring. The product of two hypermatrices (a_{ij}) and (b_{ij}) is the hyperoperation given in the usual manner $(a_{ij}) \cdot (b_{ij}) = \{(c_{ij}) \mid c_{ij} \in \sum_{k=1}^{n} a_{ik}b_{kj}\}$. Vougiouklis problem is the following one: For a given hypergroup H, find a semihyperring R such that to have a representation of H by hypermatrices with entries from R. Recall that if $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then a map $T : H \to M_R$ is called a representation if $T(x) \cdot T(y) = \{T(z) \mid z \in xy\} = T(xy)$, for all $x, y \in H$. He obtained an induced representation T^* for the hypergroup algebra of H, see [14].

3 Fundamental relation in hyperrings

Vougiouklis introduced the notion of fundamental relation in the context of general hyperrings [16, 17]. A multivalued system $(R, +, \cdot)$ is a (general) hyperring if (1) $(R, +)$ is a hypergroup; (2) $(R, \cdot)$ is a semihypergroup; (3) $(\cdot)$ is (strong) distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. In this paragraph, we use the term of a hyperring, instead of the term of a general hyperring, intending the above definition. A hyperring may be commutative with respect to $(+)$ or $(\cdot)$. If R is commutative with respect to both $(+)$ and $(\cdot)$, then it is a commutative hyperring. The above definition contains the class of multiplicative hyperrings and additive hyperrings as well. In the above hyperstructures, Vougiouklis introduced the equivalence relation γ^*, which is similar to the relation β^*, defined in every hypergroup. Let $(R, +, \cdot)$ be a hyperring. He defined the relation γ as follows: $a\gamma b$ if and only if $\{a, b\} \subseteq u$, where u is a finite sum of finite products of elements of R. Denote the transitive closure of γ by γ^*. The equivalence relation γ^* is called the fundamental equivalence relation in R. According to the distributive law, every set which is the value of a polynomial in elements of R is a subset of a sum of products in R. Let $\mathcal{U}$ be the set of all finite sums of products of elements of R. We can rewrite the definition of γ^* on R as follows: $a\gamma^* b$ if and only if there exist $z_1, ..., z_{n+1} \in R$ with $z_1{=}a$, $z_{n+1}{=}b$ and $u_1, ..., u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ for $i \in \{1, ..., n\}$. Let $(R, +, \cdot)$ be a hyperring. Then the relation γ^* is the smallest equivalence relation in R such that the quotient R/γ^* is a ring [16]. The both $\oplus$ and $\odot$ on R/γ^* are defined as follows: $\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c)$, for all $c \in \gamma^*(a) + \gamma^*(b)$ and $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$, for all $d \in \gamma^*(a) \cdot \gamma^*(b)$. If $u = \sum_{j\in J}(\prod_{i\in I_j} x_i) \in \mathcal{U}$, then for all $z \in u$, we have $\gamma^*(u) = \oplus \sum_{j\in J}(\odot \prod_{i\in I_j} \gamma^*(x_i)) = \gamma^*(z)$, where $\oplus \sum$ and $\odot \prod$ denote the sum and the product of classes.

4 Commutative rings obtained from hyperrings

The commutativity in addition in rings can be related with the existence of the unit in multiplication. If e is the unit in a ring then for all elements a, b we have $(a + b)(e + e) = (a + b)e + (a + b)e = a + b + a + b$ and $(a + b)(e + e) = a(e + e) + b(e + e) = a + a + b + b$. So $a + b + a + b = a + a + b + b$ gives $b + a = a + b$. Therefore, when we say $(R, +, \cdot)$ is a hyperring, $(+)$ is not commutative and there is not unit in the multiplication. So the commutativity, as well as the existence of the unit, it is not assumed in the fundamental ring. Of course, we know there exist many rings (+ is commutative) while don't have unit. Davvaz and Vougiouklis were interested in the fundamental ring to be commutative with respect to both sum and product, that is, the fundamental ring be an ordinary commutative ring. Therefore they introduced the following definition. Let R be a hyperring. Define the relation α as follows: $x\alpha y$ if and only if $\exists n \in \mathbb{N}$, $\exists(k_1, \ldots, k_n) \in \mathbb{N}^n$, $\exists \sigma \in S_n$ and $[\exists(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$, $\exists \sigma_i \in S_{k_i}$, $(i = 1, \ldots, n)]$ such that $x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in \sum_{i=1}^{n} A_{\sigma(i)}$, where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$. The relation α is reflexive and symmetric. Let α^* be the transitive closure of α, then α^* is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$ [4]. The quotient R/α^* is a commutative ring [4]. Notice that they used the Greek letter α for the relation because of the 'A'belian. The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring [4].

5 H_v-structures

During the 4^{th} Congress of Algebraic Hyperstructures and Applications (Xanthi, 1990), Vougiouklis introduced the concept of the weak hyperstructures which are now named H_v-structures. Over the last 27 years this class of hyperstructure, which is the largest, has been studied from several aspects as well as in connection with many other topics of mathematics. The hyperstructure $(H, \cdot)$ is called an H_v-group if (1) $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$, for all $x, y, z \in H$; (2) $a \cdot H = H \cdot a = H$, for all $a \in H$. A motivation to obtain the above structures is the following. Let $(G, \cdot)$ be a group and R an equivalence relation on G. In G/R consider the hyperoperation $\odot$ such that $\overline{x} \odot \overline{y} = \{\overline{z} | \ z \in \overline{x} \cdot \overline{y}\}$, where $\overline{x}$ denotes the class of the element x. Then $(G, \odot)$ is an H_v-group which is not always a hypergroup [20]. Let $(H_1, \cdot)$, $(H_2, \star)$ be two H_v-groups. A map $f : H_1 \to H_2$ is called an H_v-homomorphism or weak homomorphism if $f(x \cdot y) \cap f(x) \star f(y) \neq \emptyset$, for all $x, y \in H_1$. The map f is called an inclusion homomorphism if $f(x \cdot y) \subseteq f(x) \star f(y)$, for all $x, y \in H_1$. Finally, f is called a strong homomorphism if $f(x \cdot y) = f(x) * f(y)$, for all $x, y \in H_1$. If f is onto, one to one and strong homomorphism, then it is called isomorphism, if moreover f is defined on the same H_v-group then it is called automorphism. It is an easy verification that the set of all automorphisms in H, written $AutH$, is a group. On a set H several H_v-structures can be defined. A partial order on those hyperstructures is introduced as follows. Let $(H, \cdot)$, $(H, \star)$ be two H_v-groups defined on the same set H. We call $\cdot$ less than or equal to $\star$, and write $\cdot \leq \star$, if there is $f \in Aut(H, *)$ such that $x \cdot y \subseteq f(x \star y)$, for all $x, y \in H$ [20]. A quasi-hypergroup is called a hypergroupoid $(H, \cdot)$ if the reproduction axiom is valid. In [20], it is proved that all the quasi-hypergroups with two elements are H_v-groups. It is also proved that up to the isomorphism there are exactly 18 different H_v-groups. If a hyperoperation is weak associative then every greater hyperoperation, defined on the same set is also weak associative. In [21], using this property, the set of all H_v-groups with a scalar unit defined on a set with three elements is determined, also, see [22]. Let $(H, \cdot)$ be an H_v-group. The relation β^* is the smallest equivalence relation on H such that the quotient H/β^*, the set of all equivalence classes, is a group. β^* is called the fundamental equivalence relation on H. According to [19] if $\mathcal{U}$ denotes the set of all the finite products of elements of H, then a relation β can be defined on H whose transitive closure is the fundamental relation β^*. The relation β is as follows: for x and y in H we write $x\beta y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$. We can rewrite the definition of β^* on H as follows: $a\beta^* b$ if and only if there exist $z_1, \ldots, z_{n+1} \in H$ with $z_1 = a,\ z_{n+1} = b$ and $u_1, \ldots, u_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i \ \ (i = 1, \ldots, n)$. The product $\odot$ on H/β^* is defined as follows: $\beta^*(a) \odot \beta^*(b) = \{\beta^*(c) |\ c \in \beta^*(a) \cdot \beta^*(b)\}$, for all $a, b \in H$. It is proved in [19] that $\beta^*(a) \odot \beta^*(b)$ is the singleton $\{\beta^*(c)\}$ for all $c \in \beta^*(a) \cdot \beta^*(b)$. In this way H/β^* becomes a hypergroup. If we put $\beta^*(a) \odot \beta^*(b) = \beta^*(c)$, then H/β^* becomes a group. A multi-valued system $(R, +, \cdot)$ is an H_v-ring if (1) $(R, +)$ is an H_v-group; (2) $(R, \cdot)$ is an H_v-semigroup; (3) $(\cdot)$ is weak distributive with respect to $(+)$, i.e., for all x, y, z in R we have $(x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \emptyset$ and $((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \emptyset$. Let $(R, +, \cdot)$ be an H_v-ring. Define γ^* as the smallest equivalence relation such that the quotient R/γ^* is a ring. Let us denote the set of all finite polynomials of elements of R over $\mathbb{N}$ by $\mathcal{U}$. Define the relation γ as follows: $x\gamma y$ if and only if $\{x, y\} \subseteq u$, where $u \in \mathcal{U}$. The fundamental equivalence relation γ^* is the transitive closure of the relation γ [12]. Vougiouklis also introduced H_v-vector spaces in [18].

6 The uniting elements method

In 1989, Corsini and Vougiouklis [1], introduced a method, the uniting elements method, to obtain stricter algebraic structures, from given ones, through hyperstructure theory. This method was introduced before the introduction of the H_v-structures, but in fact the H_v-structures appeared in the procedure. This method is the following. Let G be a structure and d be a property, which is not valid, and suppose that d is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an H_v-structure. Then quotient of G/d by the fundamental relation β^*, is a stricter structure $(G/d)/\beta^*$ for which d is valid. An application of the uniting elements is when more than one properties are desired. The reason for this is some of the properties lead straighter to the classes than others. The commutativity and the reproductivity are easily applicable. One can do this because the following statement is valid. Let $(G, \cdot)$ be a groupoid, and $F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$ a system of equations on G consisting of two subsystems $F_m = \{f_1, ..., f_m\}$ and $F_n = f_{m+1}, ..., f_{m+n}\}$. Let

σ and σ_m be the equivalence relations defined by the uniting elements using the F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G_m/\sigma_n)/\beta^*$. Then, we have $(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*$[19].

7 The e-hyperstructures

In 1996, Santilli and Vougiouklis point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. The e-hyperstructures are a special kind of hyperstructures and, they can be interpreted as a generalization of two important concepts for physics: Isotopies and Genotopies. On the other hand, biological systems such as cells or organisms at large are open and irreversible because they grow. The representation of more complex systems, such as neural networks, requires more advances methods, such as hyperstructures. In this manner, e-hyperstructures can play a significant role for the representation of complex systems in physics and biology, such as nuclear fusion, the reproduction of cells or neural systems. They are the most important tools in Lie-Santilli theory too [2, 11]. A hypergroupoid $(H, \cdot)$ is called an e-hypergroupoid if H contains a scalar identity (also called unit) e, which means that for all $x \in H,\ x{\cdot}e = e{\cdot}x = x$. In an e-hypergroupoid, an element x' is called inverse of a given element $x \in H$ if $e \in x \cdot x' \cap x' \cdot x$. Clearly, if a hypergroupoid contains a scalar unit, then it is unique, while the inverses are not necessarily unique. In what follows, we use some examples which are obtained as follows: Take a set where an operation "$\cdot$" is defined, then we "enlarge" the operation putting more elements in the products of some pairs. Thus a hyperoperation "$\circ$" can be obtained, for which we have $x \cdot y \in x \circ y,\ \forall x, y \in H$. Recall that the hyperstructures obtained in this way are H_b-structures. Consider the usual multiplication on the subset $\{1,\ -1,\ i,\ -i\}$ of complex numbers. Then, we can consider the hyperoperation $\circ$ defined in the following table:

$\circ$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	$i, -i$
i	i	$-i$	-1	1
$-i$	$-i$	i	$1, i$	$-1, i$

We enlarged the products $(-1){\cdot}(-i)$, $(-i){\cdot}i$ and $(-i){\cdot}(-i)$ by setting

$$(-1)\circ(-i)=\{i, -i\},\ (-i)\circ i=\{1, i\} \text{ and } (-i)\circ(-i)=\{-1, i\}.$$

We obtain an e-hypergroupoid, with the scalar unit 1. The inverses of the elements $-1, i, -i$ are $-1, -i, i$ respectively. Moreover, the above structure is an

H_v-abelian group, which means that the hyperoperation $\circ$ is weak associative, weak commutative and the reproductive axiom holds. The weak associativity is valid for all H_b-structures with associative basic operations [19]. We are interested now in another kind of an e-hyperstructure, which is the e-hyperfield. A set F, endowed with an operation "+", which we call addition and a hyperoperation, called multiplication "·", is said to be an e-hyperfield if the following axioms are valid: (1) $(F, +)$ is an abelian group where 0 is the additive unit; (2) the multiplication $\cdot$ is weak associative; (3) the multiplication $\cdot$ is weak distributive with respect to +,i.e., for all $x, y, z \in F$, $x(y+z)\cap(xy+xz) \neq \emptyset$, $(x+y)z\cap(xz+yz) \neq \emptyset$; (4) 0 is an absorbing element, i.e., for all $x \in F,\ 0{\cdot}x = x{\cdot}0 = 0$; (5) there exists a multiplicative scalar unit 1, i.e., for all $x \in F,\ 1\cdot x = x\cdot 1 = x$; (6) for every element $x \in F$ there exists an inverse x^{-1}, such that $1 \in x{\cdot}x^{-1}\cap x^{-1}{\cdot}x$. The elements of an e-hyperfield $(F, +, \cdot)$ are called e-hypernumbers. We can define the product of two e-matrices in an usual manner: the elements of product of two e-matrices $(a_{ij}), (b_{ij})$ are $c_{ij} = \sum a_{ik} \circ b_{kj}$, where the sum of products is the usual sum of sets. Let $(F, +, \cdot)$ be an e-hyperfield. An ordered set $a = (a_1, a_2, \ldots, a_n)$ of n e-hypernumbers of F is called an e-hypervector and the e-hypernumbers $a_i,\ i \in \{1, 2, \cdot, n\}$ are called components of the e-hypervector a. Two e-hypervectors are equals if they have equal corresponding components. The hypersums of two e-hypervectors a, b is defined as follows: $a + b = \{(c_1, c_2, \ldots, c_n) \mid c_i \in a_i + b_i,\ i \in \{1, 2, \cdot, n\}\}$. The scalar hypermultiplication of an e-hypervector a by an e-hypernumber λ is defined in a usual manner: $\lambda \circ a = \{(c_1, c_2, \ldots, c_n) \mid c_i \in \lambda \cdot a_i,\ i \in \{1, 2, \ldots, n\}\}$. The set F^n of all e-hypervectors with elements of F, endowed with the hypersum and the scalar hypermultiplication is called n-dimensional e-hypervector space. The set of $m \times n$ hypermatrices is an mn-dimensional e-hypervector space. We refer the readers to [5, 6, 7, 8] for more details.

8 Helix-hyperoperations

Algebraic hyperstructures are a generalization of the classical algebraic structures which, among others, are appropriate in two directions: (a) to represent a lot of application in an algebraic model, (b) to overcome restrictions ordinary structures usually have. Concerning the second direction the restrictions of the ordinary matrix algebra can be overcome by the helix-operations. More precisely, the helix addition and the helix-multiplication can be defined on every type of matrices [3, 23, 24]. Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s, t \in \mathbb{N}$ be two natural numbers such that $1 \leq s \leq m$ and $1 \leq t \leq n$. Then we define the characteristic-like map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{cst} = (a_{ij})$, where $1 \leq i \leq s$ and $1 \leq j \leq t$. We call this map cut-projection of type $\underline{st}$. In other words, $A\underline{cst}$ is a matrix obtained from A by cutting the lines and columns greater than s and t respectively. Let $A = (a_{ij}) \in M_{m\times n}$ be a matrix and $s, t \in \mathbb{N}$ be two natural numbers such that $1 \leq s \leq m$ and $1 \leq t \leq n$. Then we define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (A_{ij})$ which has as entries the sets $A_{ij} = \{a_{i+ks,j+\lambda t} \mid k, \lambda \in \mathbb{N},\ i + ks \leq m,\ j + \lambda t \leq n\}$, for $1 \leq i \leq s$ and $1 \leq j \leq t$. We call this multivalued map helix-projection of type $\underline{st}$. Therefore, $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in A_{ij}$ for all i, j. Obviously, $A\underline{mn} = A$. Let us consider the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 3 & 2 & 0 & 1 & 2 \\ 2 & 4 & 5 & 1 & -1 \\ 1 & -1 & 0 & 0 & 8 \end{bmatrix}.$$

Suppose that $s = 3$ and $t = 2$. Then

$$A\underline{c32} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}$$

and $A\underline{32} = (A_{ij})$, where

$$\begin{aligned} A_{11} &= \{a_{11}, a_{13}, a_{15}, a_{41}, a_{43}, a_{45}\} = \{2, 3, 2, 1, 0, 8\}, \\ A_{12} &= \{a_{12}, a_{14}, a_{42}, a_{44}\} = \{1, 4, -1, 0\}, \\ A_{21} &= \{a_{21}, a_{23}, a_{25}\} = \{3, 0, 2\}, \\ A_{22} &= \{a_{22}, a_{24}\} = \{2, 1\}, \\ A_{31} &= \{a_{31}, a_{33}, a_{35}\} = \{2, 5, -1\}, \\ A_{32} &= \{a_{32}, a_{34}\} = \{4, 1\}. \end{aligned}$$

Therefore,

$$\begin{aligned} A\underline{32} = (A_{ij}) \;\; &= \begin{bmatrix} \{2, 3, 1, 0, 8\} & \{1, 4, -1, 0\} \\ \{3, 0, 2\} & \{2, 1\} \\ \{2, 5, -1\} & \{4, 1\} \end{bmatrix} \\ &= \{(x_{ij}) \mid x_{11} \in \{0, 1, 2, 3, 8\}, x_{12} \in \{-1, 0, 1, 4\}, x_{21} \in \{0, 2, 3\}, \\ &\qquad x_{22} \in \{1, 2\}, x_{31} \in \{-1, 2, 5\}, x_{32} \in \{1, 4\}\}. \end{aligned}$$

Therefore $|A\underline{32}| = 720$.

Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(m, u)$, $t = \min(n, u)$. We define an addition, which we call cut-addition, as follows:

$$\begin{aligned} \oplus_c : \;\; & M_{m\times n} \times M_{u\times v} \longrightarrow M_{s\times t} \\ & (A, B) \mapsto A \oplus_c B = A\underline{cst} + B\underline{cst}. \end{aligned}$$

Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(n, u)$. Then we define a multiplication, which we call cut-multiplication, as follows:

$$\begin{aligned} \otimes_c : \quad & M_{m\times n} \times M_{u\times v} \longrightarrow M_{m\times v} \\ & (A, B) \mapsto A \otimes_c B = A\underline{cms} \cdot B\underline{csv}. \end{aligned}$$

The cut-addition is associative and commutative.

Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(m, u)$, $t = \min(n, v)$. We define a hyper-addition, which we call helix-addition or helix-sum, as follows:

$$\begin{aligned} \oplus : \quad & M_{m\times n} \times M_{u\times v} \longrightarrow \mathcal{P}(M_{s\times t}) \\ & (A, B) \mapsto A \oplus B = A\underline{st} +_h B\underline{st}, \end{aligned}$$

where $A\underline{st} +_h B\underline{st} = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in A_{ij}, b_{ij} \in B_{ij}\}$. For illustration, suppose that

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then

$$A\underline{22} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B\underline{22} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where

$$\begin{array}{ll} A_{11} = \{a_{11}, a_{31}\} = \{2\}, & B_{11} = \{b_{11}, b_{13}\} = \{1, 0\}, \\ A_{12} = \{a_{12}, a_{32}\} = \{1, 3\}, & B_{12} = \{b_{12}\} = \{4\}, \\ A_{21} = \{a_{21}\} = \{0\}, & B_{21} = \{b_{21}, b_{23}\} = \{2, 1\}, \\ A_{22} = \{a_{22}\} = \{1\}, & B_{22} = \{b_{22}\} = \{0\}. \end{array}$$

So

$$A\underline{22} = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \right\},$$

and

$$B\underline{22} = \left\{ \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \right\}.$$

Therefore, we have

$$\begin{aligned} A\underline{22} +_h B\underline{22} \; = & \left\{ \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 7 \\ 2 & 1 \end{bmatrix}, \right. \\ & \left. \begin{bmatrix} 3 & 7 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 7 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 7 \\ 1 & 1 \end{bmatrix} \right\}. \end{aligned}$$

The helix-addition is commutative. Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (a_{ij}) \in M_{u\times v}$ be two matrices and $s = \min(n, u)$. Then we define a hyper-multiplication, which we call helix hyperoperation, as follows:

$$\begin{array}{ll}\otimes: & M_{m\times n}\times M_{u\times v}\longrightarrow \mathcal{P}(M_{m\times v})\\ & (A,B)\mapsto A\otimes B = A\underline{ms}\cdot_h B\underline{sv},\end{array}$$

where $A\underline{ms}\cdot_h B\underline{sv} = \{(c_{ij}) = (\sum a_{it}b_{tj}) \mid a_{ij}\in A_{ij}, b_{ij}\in B_{ij}\}$. We consider the matrices A and B as follows:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0\\ 3 & 1 & 3 & 2\end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1\\ 0 & 2\end{bmatrix}.$$

Then

$$A\underline{22} = \begin{bmatrix} \{1,2\} & 0\\ 3 & \{1,2\}\end{bmatrix}.$$

Therefore,

$$A\otimes B = \begin{bmatrix} \{-1,-2\} & \{1,2\}\\ -3 & \{5,7\}\end{bmatrix}$$

The cut-multiplication $\otimes_c$ is associative, and the helix-multiplication $\otimes$ is weak associative [23]. Note that the helix-multiplication is not distributive (not even weak) with respect to the helix-addition [23]. But if all matrices which are used in the distributivity are of the same type $M_{m\times n}$, then we have $A\otimes(B\oplus C) = A\otimes(B+C)$ and $(A\otimes B)\oplus(A\otimes C) = (A\otimes B)+(A\otimes C)$. Therefore, the weak distributivity is valid and more precisely the inclusion distributivity is valid.

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