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On P-H_v-Structures in a Two-Dimensional Real Vector Space

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Abstract

In this paper we study P-H_v-structures in connection with H_v-structures, arising from a specific P-hope in a two-dimensional real vector space. The visualization of these P-H_v-structures is our priority, since visual thinking could be an alternative and powerful resource for people doing mathematics. Using position vectors into the plane, abstract algebraic properties of these P-H_v-structures are gradually transformed into geometrical shapes, which operate, not only as a translation of the algebraic concept, but also, as a teaching process.

Keywords: Hyperstructures; H_v-structures; hopes; P-hyperstructures.
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1 Introduction

In a set $H \neq \varnothing$, a *hyperoperation* (abbr. *hyperoperation=hope*) $(\cdot)$ is defined:

$$\cdot : H \times H \to \mathbf{P}(H) - \{\varnothing\} : (x, y) \mapsto x \cdot y \subset H$$

and the $(H, \cdot)$ is called *hyperstructure*.
It is abbreviated by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The largest class of hyperstructures is the one which satisfy the weak properties. These are called H_v-structures introduced by T. Vougiouklis in 1990 [13], [14] and they proved to have a lot of applications on several applied sciences such as linguistics, biology, chemistry, physics, and so on. The H_v-structures satisfy the weak axioms where the non-empty intersection replaces the equality. The H_v-structures can be used in models as an organized devise.

The hyperstructure $(H, \cdot)$ is called H_v-*group* if it is WASS and the reproduction axiom is valid, i.e., $xH = Hx = H, \forall x \in H$.
It is called *commutative* H_v-*group* if the commutativity is valid and it is called H_v- *commutative group* if it is COW.

The motivation for the H_v-structures [13] is that the quotient of a group with respect to any partition (or equivalently to any equivalence relation), is an H_v-group. The fundamental relation $\beta*$ is defined in H_v-groups as the smallest equivalence so that the quotient is a group [14].

In a similar way more complicated hyperstructures are defined [14].

One can see basic definitions, results, applications and generalizations on both hyperstructure and H_v-structure theory in the books and papers [1], [2], [3], [10], [12], [14], [18].

The element $e \in H$, is called *left unit element* if $x \in ex, \forall x \in H$, *right unit element* if $x \in xe, \forall x \in H$ and *unit element* if $x \in xe \cap x \in ex, \forall x \in h$.
An element $x' \in H$ is called *left inverse* of $x \in H$ if there exists a unit $e \in H$, such that $e \in x'x$, *right inverse* of $x \in H$ if $e \in xx'$ and inverse of $x \in H$ if $e \in x'x \cap xx'$.
By E^l_* is denoted the set of the left unit elements, by E^r_* the set of the right unit elements and by E_* the set of the unit elements with respect to hope (*) [7].
By $I^l_*(x, e)$ is denoted the set of the left inverses, by $I^r_*(x, e)$ the set of the right inverses and by $I_*(x, e)$ the set of the inverses of the element $x \in H$ associated with the unit $e \in H$ with respect to hope (*) [7].

The class of P-hyperstructures was appeared in 80's to represent hopes of constant length [16], [18]. Then many applications appeared [1], [2], [4], [5], [6], [8], [9], [15].
Vougiouklis introduced the following definition:

Definition 1.1. *Let $(G, \cdot)$ be a semigroup and $P \subset GP \neq \varnothing$. Then the following hyperoperations can be defined and they are called P-hyperoperations: $\forall x, y \in G$*

$$P^* : xP^*y = xPy,$$

$$P_r^* : xP_r^*y = (xy)P$$

$$P_l^* : xP_l^*y = P(xy).$$

The (G, P^*),(G, P_r^*), (G, P_l^*) are called P-hyperstructures.
One, combining the above definitions gets that the most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = xP^*y = xPy$ and $(G, \underline{P})$ is a semihypergroup, but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS. $(G, \underline{P})$, $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

In this paper we define in the $I\mathbb{R}^2$ a hope which is originated from geometry. This geometrically motivated hope in $I\mathbb{R}^2$ constructs H_v-structures and P-HV-structures in which the existence of units and inverses are studied. One using the above H_v-structures and P-H_v-structures into the plane can easily combine abstract algebraic properties with geometrical figures [11].

2 P-H_v-structures on $I\mathbb{R}^2$

Let us introduce a coordinate system into the $I\mathbb{R}^2$. We place a given vector $\overrightarrow{p}$ so that its initial point P determines an ordered pair (a_1, a_2). Conversely, a point P with coordinates (a_1, a_2) determines the vector $\overrightarrow{p} = \overrightarrow{OP}$, where O the origin of the coordinate system. We shall refer to the elements $x, y, z, \ldots$ of the set $I\mathbb{R}^2$, as vectors whose initial point is the origin. These vectors are very well known as position vectors.
In [7] Dramalidis introduced and studied a number of hyperoperations originated from geometry. Among them he introduced in $I\mathbb{R}^2$ the hyperoperation $(\oplus)$ as follows:

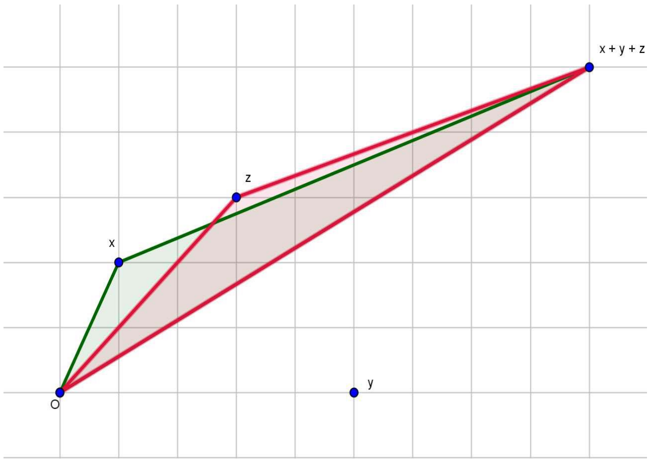
Definition 2.1. *For every $x, y \in I\mathbb{R}^2$*

$$\oplus : I\mathbb{R}^2 \times I\mathbb{R}^2 \to \mathbf{P}(I\mathbb{R}^2) - \{\varnothing\} : (x, y) \mapsto x \oplus y =$$

$$= [0, x + y] = \{\mu(x + y)/\mu \in [0, 1]\} \subset I\mathbb{R}^2$$

From geometrical point of view and for x, y linearly independent position vectors, the set $x \oplus y$ is the main diagonal of the parallelogram having vertices $0, x, x+y, y$.

Proposition 2.1. *The hyperstructure $(I\mathbb{R}^2, \oplus)$ is a commutative H_v-group.*



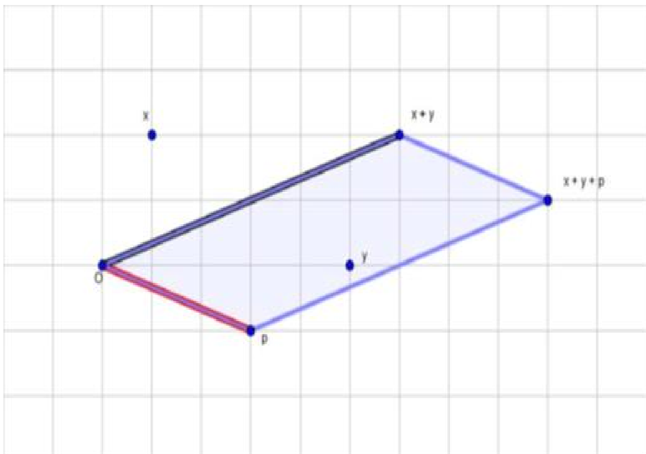
Now, let P be the set $P = [0, p] = \{\lambda p / \lambda \in [0, 1]\} \subset I\mathbb{R}^2$, where p is a fixed point of the plane. Geometrically, P is a line segment.

Consider the P-hyperoperation $(P^*_{r(\oplus)})$:

Definition 2.2. *For every $x, y \in I\mathbb{R}^2$*

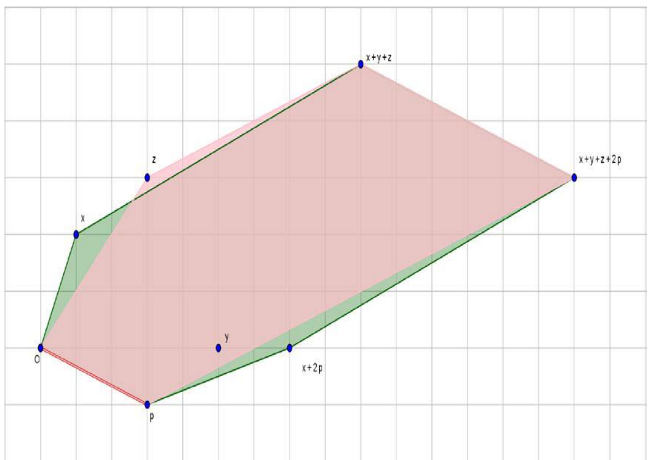
$$P^*_{r(\oplus)} : I\mathbb{R}^2 \times I\mathbb{R}^2 \to \mathbf{P}(I\mathbb{R}^2) - \{\varnothing\} : (x, y) \mapsto xP^*_{r(\oplus)}y = (x \oplus y) \oplus P \subset I\mathbb{R}^2$$

Obviously, $(P^*_{r(\oplus)})$ is commutative and geometrically, for x,y linearly independent position vectors, the set $xP^*_{r(\oplus)}y$ is the closed region of the parallelogram with vertices $0, x + y, x + y + p, p$.



Proposition 2.2. *The hyperstructure $(\mathbb{R}^2, P^*_{r(\oplus)})$ is a commutative P-H_v-group.*

Proof. Obviously, $xP^*_{r(\oplus)}\mathbb{R}^2 = \mathbb{R}^2 P^*_{r(\oplus)}x = \mathbb{R}^2, \forall x \in \mathbb{R}^2$.
For $x, y, z \in \mathbb{R}^2$
$(xP^*_{r(\oplus)}y)P^*_{r(\oplus)}z = \{[(x \oplus y)P] \oplus z\} \oplus P = [0, z, x+y+z, x+y+z+2p, p]$
$xP^*_{r(\oplus)}(yP^*_{r(\oplus)}z) = \{x \oplus [(y \oplus z) \oplus P]\} \oplus P =$
$= [0, x+y+z, x+y+z+2p, x+y+2p, x+2p, p]$

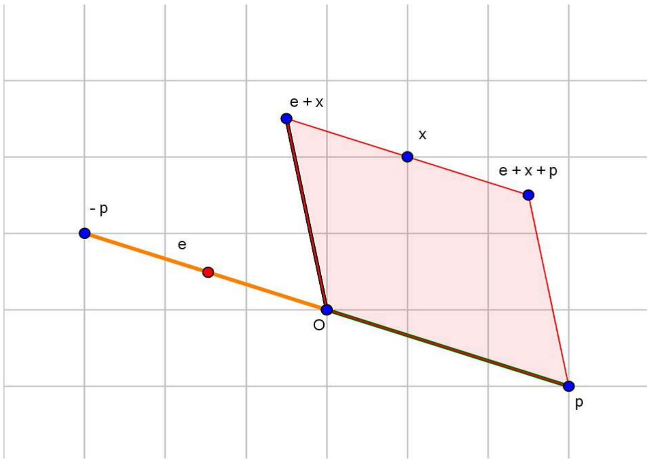


So,

$$(xP^*_{r(\oplus)}y)P^*_{r(\oplus)}z \cap xP^*_{r(\oplus)}(yP^*_{r(\oplus)}z) \neq \varnothing, \forall x, y, z \in \mathbb{R}^2. \square$$

Proposition 2.3. $E_{P^*_{r(\oplus)}} = [-p, 0] = \{-\lambda p / \lambda \in [0, 1]\}$

Proof. Let $e \in E^l_{P^*_{r(\oplus)}} \Leftrightarrow xeP^*_{r(\oplus)}x, \forall x \in \mathbb{R}^2 \Leftrightarrow x\{\mu\lambda e + \mu\lambda x + \mu\nu p / \mu, \nu, \lambda[0, 1]\}$.
That means that,
$\mu\lambda = 1$ and $\mu\lambda e + \mu\nu p = 0 \Leftrightarrow e + \mu\nu p = 0 \Leftrightarrow e = -\mu\nu p, -1 \leq -\mu\nu \leq 0$, then $e \in [-p, 0]$. So, $E^l_{P^*_{r(\oplus)}} = [-p, 0]$ and according to commutativity $E^r_{P^*_{r(\oplus)}} = [-p, 0] = E_{P^*_{r(\oplus)}} = [-p, 0]$.

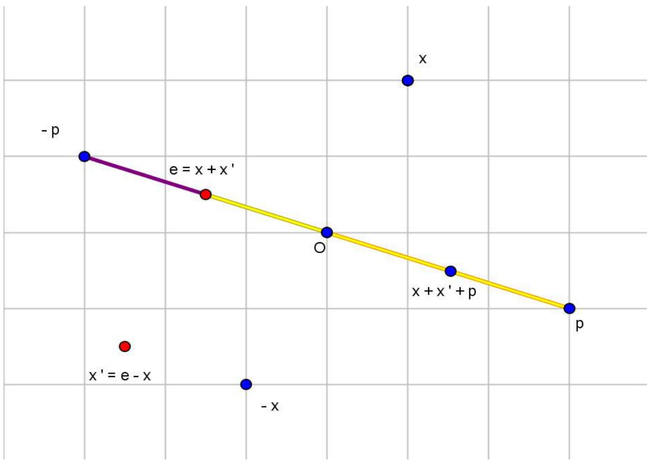


Proposition 2.4. $I_(P^*_{r(\oplus)})(x,e) = \{\frac{1}{\mu\lambda}e - x - \frac{\nu}{\lambda}p/\mu, \lambda \in (0,1], \nu \in [0,1]\}$, *where* $e \in E_{P^*_{r(\oplus)}}$.

Proof. Let $e \in E_{P^*_{r(\oplus)}}$ and $x' \in I^l_{P^*_{r(\oplus)}}(x,e) \Leftrightarrow e \in x'P^*_{r(\oplus)}x \Leftrightarrow e\{\mu\lambda x' + \mu\lambda x + \mu\nu p/\lambda, \mu, \nu[0,1]\}$.
That means there exist $\lambda_1, \mu_1, \nu_1[0,1]$:

$$e = \mu_1\lambda_1 x' + \mu_1\lambda_1 x + \mu_1\nu_1 p \Rightarrow x' = \frac{1}{\mu_1\lambda_1}e - x - \frac{\nu_1}{\lambda_1}p, \mu_1, \lambda_1 \neq 0.$$

So, $x' \in \{\frac{1}{\mu\lambda}e - x - \frac{\nu}{\lambda}p/\mu, \lambda \in (0,1], \nu \in [0,1]\}$.
Since $(P^*_{r(\oplus)})$ is commutative, we get $I_(P^*_{r(\oplus)})(x,e) = \{\frac{1}{\mu\lambda}e - x - \frac{\nu}{\lambda}p/\mu, \lambda \in (0,1], \nu \in [0,1]\}$.



The P-hyperoperation $P^*_{l(\oplus)} = P \oplus (x \oplus y)$ is identical to $(P^*_{r(\oplus)})$. But the P-hyperoperation $P^*_{(\oplus)} = x \oplus P \oplus y$ is different and even more $P^{*l}_{(\oplus)} = (x \oplus P) \oplus y \neq x \oplus (P \oplus y) = xP^{*r}_{(\oplus)}y$, since $(\oplus)$ is not associative.□

Definition 2.3. *For every $x, y \in I\mathbb{R}^2$*

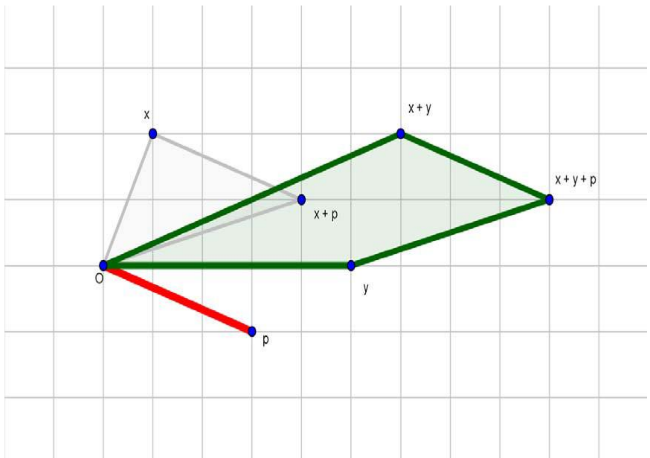
$$P^{*l}_{(\oplus)} : \mathbb{R}^2 \times \mathbb{R}^2 \to (\mathbb{R}^2) : (x, y) \mapsto xP^{*l}_{(\oplus)}y = (x \oplus P) \oplus y$$

More specifically,

$$xP^{*l}_{(\oplus)}y = \{\lambda\kappa x + \lambda y + \lambda\kappa\mu p / \lambda, \kappa, \mu \in [0, 1]\}, \forall x, y \in \mathbb{R}^2.$$

Geometrically, for x,y linearly independent position vectors, the set $xP^{*l}_{(\oplus)}$ y is the closed region of the quadrilateral with vertices $0, x + y, x + y + p, y$. On the other hand the set $yP^{*l}_{(\oplus)}x$ is the closed region of the quadrilateral with vertices $0, x, x + y, x + y + p$. So,

$$(xP^{*l}_{(\oplus)}y) \cap (yP^{*l}_{(\oplus)}x) = [0, x + y, x + y + p] \neq \varnothing, \forall x, y \in \mathbb{R}^2.$$

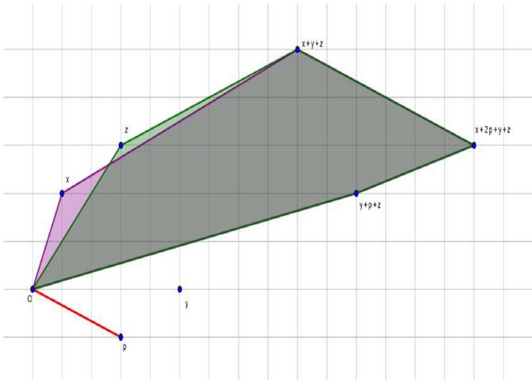


Proposition 2.5. *The hyperstructure $(\mathbb{R}^2, P^{*l}_{(\oplus)})$ is a P-H_v- commutative group.*

Proof. Obviously, $xP^{*l}_{(\oplus)}\mathbb{R}^2 = \mathbb{R}^2 P^{*l}_{(\oplus)}x = \mathbb{R}^2, \forall x \in \mathbb{R}^2$.
For $x, y, z\mathbb{R}^2$

$$(xP^{*l}_{(\oplus)}y)P^{*l}_{(\oplus)}z = \{[(x \oplus P) \oplus y]P\} \oplus z \equiv [O, z, x+y+z, x+2p+y+z, y+p+z]$$

$$xP^{*l}_{(\oplus)}(yP^{*l}_{(\oplus)}z) = (x \oplus P) \oplus [(y \oplus P) \oplus z] \equiv [O, x, x+y+z, x+2p+y+z, y+p+z]$$



So,

$$(xP^{*l}_{(\oplus)}y)P^{*l}_{(\oplus)}z \cap xP^{*l}_{(\oplus)}(yP^{*l}_{(\oplus)}z) \neq \varnothing, x, y, z \in \mathbb{R}^2.$$

$\square$

Proposition 2.6. *i)* $E^l_{P^{*l}_{(\oplus)}} = \mathbb{R}^2$

ii) $E^r_{P^{*l}_{(\oplus)}} = [0, -p] = \{-\nu p / \nu \in [0,1]\} = E_{P^{*l}_{(\oplus)}}$

Proof.

i) Notice that $x \in eP^{*l}_{(\oplus)}x = [0, e+x, e+x+p, x], \forall x, e \in \mathbb{R}^2$. So, $E^l_{P^{*l}_{(\oplus)}} = \mathbb{R}^2$.

ii) Let $e \in E^r_{P^{*l}_{(\oplus)}} \Leftrightarrow x \in xP^{*l}_{(\oplus)}e, \forall x \in \mathbb{R}^2 \Leftrightarrow x \in \{\lambda\kappa x + \lambda e + \lambda\kappa\mu p / \lambda, \kappa, \mu \in [0,1]\}$. Then, there exist $\mu_1, \lambda_1, \kappa_1 \in [0,1] : x = \lambda_1\kappa_1 x + \lambda_1 e + \lambda_{11}\mu_1 p \Leftrightarrow e = \frac{1}{l}\lambda_1[(1-\lambda_{11})x - \lambda_1\kappa_1\mu_1 p], \lambda_1 \neq 0$. The last one is valid $\forall x \in \mathbb{R}^2$, so by setting $x = 0$ we get $e = -\kappa_1\mu_1 p$. Since $\mu_1, \kappa_1 \in [0,1]$ there exists $\nu_1 \in [0,1] : \nu_1 = \kappa_1\mu_1 \Rightarrow e = -\nu_1 p \Rightarrow$

$$e \in \{-\nu p / \nu \in [0,1]\} = [0, -p].$$

Since $E^r_{P^{*l}_{(\oplus)}} \subset \mathbb{R}^2 = E^l_{P^{*l}_{(\oplus)}}$ we get $E^l_{P^{*l}_{(\oplus)}} \cap E^r_{P^{*l}_{(\oplus)}} = \{-\nu p / \nu[0,1]\} = E_{P^{*l}_{(\oplus)}}$.$\square$

Proposition 2.7. $\alpha)$ $I^r_{P^{*l}_{(\oplus)}}(x, e) = \{-\kappa x - (\frac{\nu}{\lambda} + \kappa\mu)p / \kappa, \mu, \nu \in [0,1], \lambda \in (0,1]\}, e \in E^r_{P^{*l}_{(\oplus)}}$.

$\beta)$ $I^r_{P^{*l}_{(\oplus)}}(x, e) = \{\frac{e}{\lambda} - \kappa x - \kappa\mu p / \kappa, \mu \in [0,1], \lambda \in (0,1]\}, e \in E^l_{P^{*l}_{(\oplus)}}$

$\gamma)$ $I^l_{P^{*l}_{(\oplus)}}(x, e) = \{-\frac{x}{\kappa} - (\frac{\nu}{\lambda\kappa} + \mu)p / \kappa, \lambda \in (0,1], \mu \in (0,1]\}, e \in E^r_{P^{*l}_{(\oplus)}}$.

δ) $I^l_{P^{*l}_{(\oplus)}}(x,e) = \{\frac{1}{\kappa}(\frac{e}{\lambda} - x) - \mu p/\kappa, \lambda \in (0,1], \mu \in [0,1]\}, e \in E^l_{P^{*l}_{(\oplus)}}$

Proof.

α) Let $e \in E^r_{P^{*l}_{(\oplus)}} = [0, -p]$ and $x' \in I^r_{P^{*l}_{(\oplus)}}(x,e)$, then
$e \in xP^{*l}_{(\oplus)}x' \Rightarrow e \in \{\lambda\kappa x + \lambda x' + \lambda\kappa\mu p/\kappa, \lambda, \mu \in [0,1]\}$. That means there exist $\kappa_1, \lambda_1, \mu_1 \in [0,1]$:
$e = \lambda_1\kappa_1 x + \lambda_1 x' + \lambda_1\kappa_1\mu_1 p \Rightarrow x' = \frac{e}{\lambda_1} - \kappa_1 x - \kappa_1\mu_1 p, \lambda_1 \neq 0.$
But, $e \in \{-\nu p/\nu[0,1]\} \Rightarrow \exists\ \nu_1 \in [0,1] : e = -\nu_1 p$.
So, $x' = -\frac{\nu_1}{\lambda_1}p - \kappa_1 x - \kappa_1\mu_1 p, \lambda_1 \neq 0 \Rightarrow x' = -\kappa_1 x(\frac{\nu_1}{\lambda_1} + \kappa_1\mu_1)p, \lambda_1 \neq 0.$
Then we get $x' \in \{-\kappa x - (\frac{\nu}{\lambda} + \kappa\mu)p/\kappa, \mu, \nu \in [0,1], \lambda \in (0,1]\}$.

β) Similarly as above.

γ) Similarly as above.

δ) Similarly as above.

□

Definition 2.4. *For every $x, y \in I\mathbb{R}^2$*

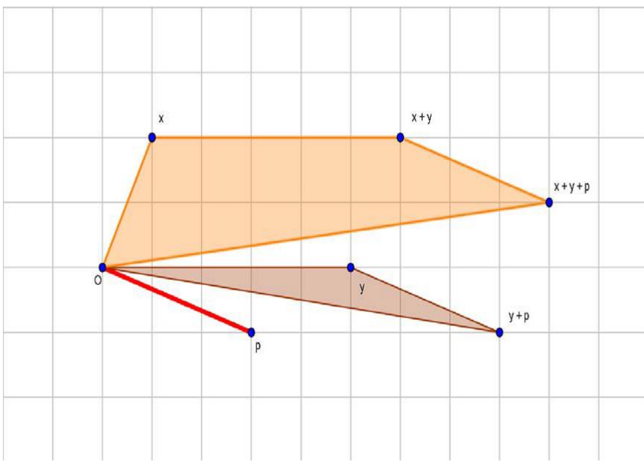
$$^{*r}_{(\oplus)} : \mathbb{R}^2 \times \mathbb{R}^2 \to (\mathbb{R}^2) : (x,y) \mapsto x^{*r}_{(\oplus)})y = x \oplus (P \oplus y)$$

More specifically,

$$x^{*r}_{(\oplus)}y = \{\lambda x + \lambda\kappa y + \lambda\kappa\mu p/\lambda, \kappa, \mu \in [0,1]\}, \forall x, y \in \mathbb{R}^2$$

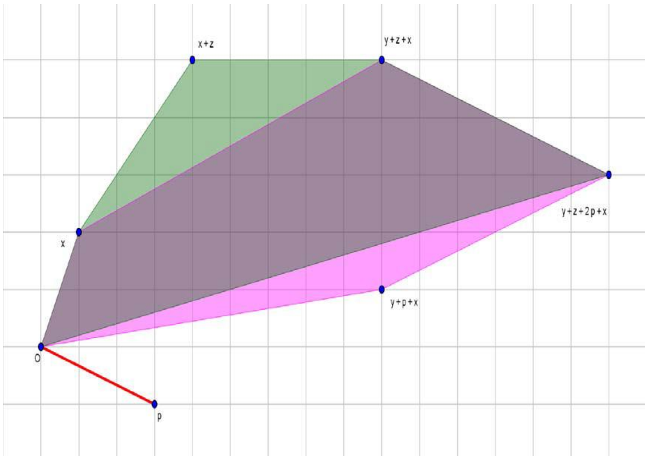
Geometrically, for x,y linearly independent position vectors, the set $xP^{*r}_{(\oplus)}y$ is the closed region of the quadrilateral with vertices $0, x, x+y, x+y+p$. On the other hand the set $yP^{*r}_{(\oplus)}x$ is the closed region of the quadrilateral with vertices $0, x+y, x+y+p, y$. So,

$$(xP^{*r}_{(\oplus)}y) \cap (yP^{*r}_{(\oplus)}x) = [0, x+y, x+y+p] \neq \varnothing, \forall x, y \in \mathbb{R}^2.$$



Proposition 2.8. *The hyperstructure* $(\mathbb{R}^2, P^{*r}_{(\oplus)})$ *is a P-*H_v*- commutative group.*

Proof. Obviously, $xP^{*r}_{(\oplus)}\mathbb{R}^2 = \mathbb{R}^2 P^{*r}_{(\oplus)}x = \mathbb{R}^2, \forall x \in \mathbb{R}^2$.
For $x, y, z \in R^2$
$(xP^{*r}_{(\oplus)}y)P^{*r}_{(\oplus)}z = [(x\oplus(P\oplus y)]\oplus(P\oplus z) \equiv [O, x, x+z, x+y+z, x+y+z+2p]$
$xP^{*r}_{(\oplus)}(yP^{*r}_{(\oplus)}z) = x\oplus\{P\oplus[y\oplus(P\oplus z)]\} \equiv [O, x, x+y+z, x+y+z+2p, y+p+x]$



So,
$$[(xP^{*r}_{(\oplus)}y)P^{*r}_{(\oplus)}z] \cap [xP^{*r}_{(\oplus)}(yP^{*r}_{(\oplus)}z)] \neq \varnothing, \forall x, y, z \in \mathbb{R}^2.\square$$

The following, are respective propositions of the Propositions 2.6. and 2.7. :

Proposition 2.9. *i)* $E^r_{P^{*r}_{(\oplus)}} = \mathbb{R}^2$

ii) $E^l_{P^{*r}_{(\oplus)}} = [0, -p] = \{-\nu p/\nu \in [0,1]\} = E_{P^{*r}_{(\oplus)}}$.

Proposition 2.10. $\alpha)$ $I^r_{P^{*r}_{(\oplus)}}(x, e) = \{\frac{1}{\kappa}(\frac{e}{\lambda}-x)-\mu p/\kappa, \lambda \in (0,1], \mu \in [0,1]\}, e \in E^r_{P^{*r}_{(\oplus)}}$

$\beta)$ $I^r_{P^{*r}_{(\oplus)}}(x, e) = \{-\frac{x}{\kappa} - (\frac{\nu}{\lambda\kappa} + \mu)p/\kappa, \lambda \in (0,1], \mu \in (0,1]\}, e \in E^l_{P^{*r}_{(\oplus)}}$.

$\gamma)$ $I^l_{P^{*r}_{(\oplus)}}(x, e) = \{\frac{e}{\lambda} - \kappa x - \kappa\mu p/\kappa, \mu \in [0,1], \lambda \in (0,1]\}, e \in E^r_{P^{*r}_{(\oplus)}}$

$\delta)$ $I^l_{P^{*r}_{(\oplus)}}(x, e) = \{-\kappa x - (\frac{\nu}{\lambda} + \kappa\mu)p/\kappa, \mu, \nu \in [0,1], \lambda \in (0,1]\}, e \in E^l_{P^{*r}_{(\oplus)}}$.

Remark 2.1. *Notice that,*

$$\alpha) x^{*l}_{(\oplus)}y = y^{*r}_{(\oplus)}x, \forall x, y \in \mathbb{R}^2$$
$$\beta) x^{*r}_{(\oplus)}y = y^{*l}_{(\oplus)}x, \forall x, y \in \mathbb{R}^2$$

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