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On a Geometric Representation of Probability Laws and of a Coherent Prevision-Function According to Subjectivistic Conception of Probability

Pierpaolo Angelini*, Angela De Sanctis†

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Abstract

We distinguish the two extreme aspects of the logic of certainty by identifying their corresponding structures into a linear space. We extend probability laws $\mathbf{P}$ formally admissible in terms of coherence to random quantities. We give a geometric representation of these laws $\mathbf{P}$ and of a coherent prevision-function $\mathbf{P}$ which we previously defined. This work is the foundation of our next and extensive study concerning the formulation of a geometric, well-organized and original theory of random quantities.

Keywords: metric; collinearity; vector subspace; convex set; linear dependence
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*MIUR, Roma, Italia. pierpaolo.angelini@istruzione.it
†DEA, Univerity "G. D'Annunzio" of Chieti-Pescara, Pescara, Italia angela.desanctis@unich.it

1 Introduction

An event E is conceptually a mental separation between subjective sensations: it is actually a proposition or statement such that, by betting on it, we can establish in an unmistakable fashion whether it is true or false, that is to say, whether it has occurred or not and so whether the bet has been won or lost ([9], [15]). For any individual who does not certainly know the true value of a quantity X, which is random in a non-redundant usage for him, there are two or more than two possible values for X. The set of these values is denoted by $I(X)$. In any case, only one is the true value of each random quantity and the meaning that you have to give to random is the one of unknown by the individual of whom you consider his state of uncertainty or ignorance. Thus, random does not mean undetermined but it means established in an unequivocal fashion, so a supposed bet based upon it would unmistakably be decided at the appropriate time. When one wonders if infinite events of a set are all true or which is the true event among an infinite number of events, one can never verify if such statements are true or false. These statements are infinite in number, so they do not coincide with any mental separation between subjective sensations: they are conceptually meaningless. Hence, we can understand the reason for which it is not a logical restriction to define a random quantity as a finite partition of incompatible and exhaustive events, so one and only one of the possible values for X belonging to the set $I(X) = \{x_1, \ldots, x_n\}$ is necessarily true. A random quantity is dealt with by the logic of certainty as well as by the logic of probable ([8]). We recognize two different and extreme aspects concerning the logic of certainty. At first we distinguish a more or less extensive class of alternatives which appear objectively possible to us in the current state of our information: when a given individual outlines the domain of uncertainty he does not use his subjective opinions on what he does not know because the possible values of X depend only on what he objectively knows or not. Afterwards we definitively observe which is the true alternative to be verified among the ones logically possible. The probability is an additional notion, so it comes into play after constituting the range of possibility and before knowing which is the true alternative to be verified: the logic of probable will fill in this range in a coherent way by considering a probabilistic mass distributed upon it. An individual correctly makes a prevision of a random quantity when he leaves the objective domain of the logically possible in order to distribute his subjective sensations of probability among all the possible alternatives and in the way which will appear most appropriate to him ([7], [12], [13]). Given an evaluation of probability p_i, $i = 1, \ldots, n$, a prevision of X turns out to be $\mathbf{P}(X) = x_1 p_1 + \ldots + x_n p_n$, where we have $0 \leq p_i \leq 1, i = 1, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$: it is rendered as a function of the probabilities of the possible values for X and it is admissible in terms of coherence because it is a barycenter of these values. It is usually called the mathematical expectation of X or its mean value ([14]). It is certainly possible to extend this result by using more advanced mathematical tools such as Stieltjes integrals. Nevertheless, such an extension adds nothing from conceptual and operational point of view and for this reason we will not consider it. Conversely, the possible values of any possible event are only two: 0 and 1. Therefore, each event is a specific random quantity. The same symbol $\mathbf{P}$ denotes both prevision of a random quantity and probability of an event ([10]).

2 Space of alternatives as a linear space

When we consider one random quantity X, each possible value of it, for a given individual at a certain instant, is a real number in the space $\mathcal{S}$ of alternatives coinciding with a line on which an origin, a unit of length and an orientation are chosen. Every point of this line is assumed to correspond to a real number and every real number to a point: the real line is a vector space over the field $\mathbb{R}$ of real numbers, that is to say, over itself, of dimension 1. When we consider two random quantities, X_1 and X_2, a Cartesian coordinate plane is the space $\mathcal{S}$ of alternatives: possible pairs (x_1, x_2) are the Cartesian coordinates of a possible point of this plane. Every point of a Cartesian coordinate plane is assumed to correspond to an ordered pair of real numbers and vice versa: $\mathbb{R}^2$ is a vector space over the field $\mathbb{R}$ of real numbers of dimension 2 and it is called the two-dimensional real space. When we consider three random quantities, X_1, X_2 and X_3, the three-dimensional real space $\mathbb{R}^3$ corresponds to the set $\mathcal{S}$ of alternatives and possible triples (x_1, x_2, x_3) are the Cartesian coordinates of a possible point of this linear space. There is a bijection between the points of the vector space $\mathbb{R}^3$ over the field $\mathbb{R}$ of real numbers and the ordered triples of real numbers. More generally, in the case of n random quantities, where n is an integer > 3, one can think of the Cartesian coordinates of the n-dimensional real space $\mathbb{R}^n$. There is a bijection between the points of the vector space $\mathbb{R}^n$ over the field $\mathbb{R}$ and the ordered n-tuples of real numbers. It is always essential that different pairs of real numbers are made to correspond to distinct points or different triples of real numbers are made to correspond to different points or, more generally, distinct n-tuples of real numbers are made to correspond to dissimilar points ([11]). Those alternatives which appear possible to us are elements of the set $\mathcal{Q}$ and they are embedded in the space $\mathcal{S}$ of alternatives. Such a space is conceptually a set of points whose subset $\mathcal{Q}$ consists of those possible points non-themselves subdivisible for the purposes of the problem under consideration. Sometimes, the set $\mathcal{Q}$ coincides with $\mathcal{S}$. There is a very meaningful point among the points of $\mathcal{Q}$: it represents the true alternative, that is to say, the one which will turn out to be verified "a posteriori". It is a random point "a priori" and it expresses everything there is to be said.

3 Two different aspects of the logic of certainty into a linear space

We study the two aspects of the logic of certainty into a linear space coinciding with the n-dimensional real space $\mathbb{R}^n$ where we consider n random quantities $X_1, \ldots, X_n$. Therefore, we have n orthogonal axes to each other: a same Cartesian coordinate system is chosen on every axis. Thus, the real space $\mathbb{R}^n$ has a Euclidean structure and it is evidently our space $\mathcal{S}$ of alternatives. Into the logic of certainty exist certain and impossible and possible as alternatives with respect to the temporary knowledge of each individual: each random quantity justifies itself "a priori" because every finite partition of incompatible and exhaustive events referring to it shows the possible ways in which a certain reality may be expressed. A multiplicity of possible values for every random quantity is only a formal construction that precedes the empirical observation by means of which a single value among the ones of the set $\mathcal{Q}$ is realized. The set $\mathcal{Q}$ of every random quantity is a subset of a vector subspace of dimension 1 into the n-dimensional real space $\mathbb{R}^n$. In general, given X, we have $\mathcal{Q} = I(X) = \{x_1, \ldots, x_n\}$. It is absolutely the same thing if every possible value of each random quantity is viewed as a particular n-tuple of real numbers or as a single real number. Every possible value for a random quantity definitively becomes 0 or 1 when we make an empirical observation referring to it: into the logic of certainty also exist true and false as final answers ([2], [3]). Logical operations are applicable to idempotent numbers 0 and 1. If A and B are events, the negation of A is $\bar{A} = 1 - A$ and such an event is true if A is false, while if A is true it is false; the negation of B is similarly $\bar{B} = 1 - B$. The logical product of A and B is $A \wedge B = AB$ and such an event is true if A is true and B is true, otherwise it is false; the logical sum of A and B is $(A \vee B) = \left(\overline{\bar{A} \wedge \bar{B}}\right) = 1 - (1 - A)(1 - B)$, from which it follows that such an event is true if at least one of events is true, where we have $A \vee B = A + B$ when A and B are incompatible events because it is impossible for them both to occur. Concerning the logical product and the logical sum, we have evidently the same thing when we consider more than two events. An algebraic structure $(L, \wedge, \vee)$, where the logical product $\wedge$ and the logical sum $\vee$ are two binary operations on the set L whose elements are 0 and 1, is a Boolean algebra because commutative laws, associative laws, absorption laws, idempotent laws and distributive laws hold for 0 and 1 of L. It admits both an identity element with respect to the logical product and an identity element with respect to the logical sum, so we have

$$(x \wedge 1) = x, (x \vee 0) = x,$$

for all x of L. It admits that every x of L has a unique complement $\bar{x}$, so we have

$$(x \wedge \bar{x}) = 0, (x \vee \bar{x}) = 1.$$

We can extend the logical operations into the field of real numbers when we make the following definitions: $x \wedge y = min(x, y)$, $x \vee y = max(x, y)$, $\bar{x} = 1 - x$. Therefore, it is not true that the logical operations are applicable only to idempotent numbers 0 and 1 because they are also applicable to all real numbers. On the other hand, it is not true that the arithmetic operations are applicable only to natural, rational, real, complex numbers or integers because they are also applicable to idempotent numbers 0 and 1 identifying events. For instance, the arithmetic sum of many events coincides with the random number of successes given by $Y = E_1 + \ldots + E_n$. Therefore, we observe that the set $\mathcal{Q}$ of every random quantity considered into a linear space becomes a Boolean algebra whose two idempotent numbers are on every axis of $\mathbb{R}^n$. These two numbers are elements of a subset of a vector subspace of dimension 1 into the n-dimensional real space $\mathbb{R}^n$ over the field $\mathbb{R}$ of real numbers. That being so, it is evident that to postulate that the field over which the probability is defined be a σ-algebra is not natural. Hence, what we will later say is conceptually and mathematically well-founded.

4 Probability laws P formally admissible in terms of coherence

The probability $\mathbf{P}$ of an event E, in opinion of a given individual, is operationally a price in terms of gain and a bet is the real or conceptual experiment to be made oneself in order to obtain its measurement ([4]). If $p = \mathbf{P}(E)$ is a coherent assessment expressed by this individual, then such a bet is fair because it is acceptable in both senses indifferently. Therefore, he considers as fair an exchange, for any S positive or negative, between a certain sum pS and the right to a sum S dependent on the occurrence of E ([5]). From notion of fairness it follows that the two possible values of the random quantity $G' = (\lambda - p)S$, where λ is a random quantity whose possible values are 0 and 1, do not have the same sign. Given S, if these values of G' are only positive or negative, then we have an incoherent assessment and the bet on the event under consideration is not fair. If E is a certain event, then we have $p = 1$ in a coherent fashion. If E is an impossible event, then we have $p = 0$ in a coherent fashion. If E is a possible event because it is not either certain or impossible, then we have $0 \leq p \leq 1$ in a coherent fashion. Even the probability $\mathbf{P}$ of the trievent $E = E''|E'$ is a price in terms of gain. It is the price to be paid for a bet that can be won or lost or annulled if E' does not occur. Nevertheless, we will not consider the notion of conditional probability from now on, because it is not essential to this context. Given n events $E_1, \ldots, E_n$ of the set $\mathcal{E}$ of events, a certain individual assigns to them, respectively, the probabilities $p_1 = \mathbf{P}(E_1), \ldots, p_n = \mathbf{P}(E_n)$ in a coherent way. Thus, by betting on $E_1, \ldots, E_n$, this individual considers as fair an exchange, for any $S_1, \ldots, S_n$ positive or negative, between a certain sum $p_1S_1 + \ldots + p_nS_n$ and the right to a sum $E_1S_1 + \ldots + E_nS_n$ dependent on the occurrence of $E_1, \ldots, E_n$, where we have $E_i = 1$ or $E_i = 0$, $i = 1, \ldots, n$, whether E_i occurs or not. Evidently, if $p_1 = \mathbf{P}(E_1), \ldots, p_n = \mathbf{P}(E_n)$ are not coherent assessments, then the possible values of the random quantity $G = (\lambda_1 - p_1)S_1 + \ldots + (\lambda_n - p_n)S_n$ are all positive or negative. Probability laws $\mathbf{P}$ formally admissible in terms of coherence allow to extend in a logical or coherent way the probabilities of the events of $\mathcal{E}$ which are already evaluated in a subjective way. These laws allow to determine which is the most general set of events whose probabilities are uniquely determined, in accordance with theorems of probability calculus, because one knows the probability of each event of $\mathcal{E}$. Moreover, probability laws $\mathbf{P}$ allow to determine which is the most general set of events for which their probabilities lie between two numbers, which are not 0 and 1, after evaluating the probability of each event of $\mathcal{E}$ in a subjective way, while for the remaining events can be said nothing in addition to the banal observation that their probabilities are included between 0 and 1. If $\mathcal{E}$ is a finite set of incompatible and exhaustive events $E_1, \ldots, E_n$, then $\mathbf{P}$ is a probability law formally admissible in terms of coherence with regard to events of $\mathcal{E}$ if and only if the theorem of total probability is valid, so we have $\mathbf{P}(E_1) + \ldots + \mathbf{P}(E_n) = 1$. Probability laws $\mathbf{P}$ formally admissible are evidently ∞^n and a given individual may subjectively choose one of these laws depending on the circumstances. Given A, its probability $\mathbf{P}(A)$ is uniquely determined when A is a logical sum of two or more than two incompatible events of $\mathcal{E}$: A is linearly dependent on these events. Otherwise, we can only say that $\mathbf{P}(A)$ is greater than or equal to the sum of the probabilities of the events E_i which imply A and less than or equal to the sum of the probabilities of the events E_i which are compatible with A. If $\mathcal{E}$ is a finite set of events $E_1, \ldots, E_n$ whatsoever, then the 2^n constituents $C_1, \ldots, C_s$ form a finite set of incompatible and exhaustive events for which it is certain that one and only one of them occurs. These constituents are elementary or atomic events and they are obtained by the logical product $E_1 \wedge \ldots \wedge E_n$: each time we substitute in an orderly way one event E_i, $i = 1, \ldots, n$, or more than one event with its negation $\bar{E}_i$ or their negations, we obtain one constituent of the set of constituents generated by $E_1, \ldots, E_n$. It is possible that some constituent is impossible, so the number of possible constituents is $s \leq 2^n$. The most general probability law assigns to the possible constituents $C_1, \ldots, C_s$ the probabilities $q_1, \ldots, q_s$ which sum to 1, while the probability of an impossible constituent is always 0. Conversely, every probability law which is valid for the events of $\mathcal{E}$ can be extended to the constituents $C_1, \ldots, C_s$, so the probabilities $p_1 = \mathbf{P}(E_1), \ldots, p_n = \mathbf{P}(E_n)$ are admissible if and only if the non-negative numbers $q_1, \ldots, q_s$ satisfy a system of $n+1$ linear equations in the s variables $q_1, \ldots, q_s$ expressed by

$$\begin{cases} \sum_i^{(1)} q_i = p_1 \\ \vdots \\ \sum_i^{(n)} q_i = p_n \\ \sum_{i=1}^{s} q_i = 1. \end{cases}$$

The notation $\sum_i^{(h)} q_i$ represents the sum concerning those indices i for which C_i is an event implying E_h. If A is a logical sum of some constituent, then we have $x = \sum_i^{(A)} q_i$ and we can say that the probability of A is uniquely determined because $x = \sum_i^{(A)} q_i$ is linearly dependent on the $n+1$ linear equations of the system under consideration. If A is not a logical sum of constituents, then A' is the greatest logical sum of the ones which are contained in A and A'' is the lowest logical sum of the ones which contain A, so we have $x' \leq x \leq x''$, where x' is the lowest admissible probability of A', while x'' is the greatest admissible probability of A''. If $\mathcal{E}$ is an infinite set of events, then $\mathbf{P}$ is a probability law formally admissible with regard to events of $\mathcal{E}$ if and only if $\mathbf{P}$ is a probability law formally admissible with regard to any finite subset of $\mathcal{E}$. Therefore, given A, its probability $\mathbf{P}(A)$ is uniquely determined or bounded from above and below or absolutely undetermined because we have $0 \leq \mathbf{P}(A) \leq 1$. Now we extend probability laws $\mathbf{P}$ formally admissible in terms of coherence to random quantities we defined in the beginning. The set $\mathcal{X}$ can be a finite set of n random quantities $X_1, \ldots, X_n$ or it can be an infinite set of random quantities. In general, given X, $I(X) = \{x_1, \ldots, x_n\}$ is the set of its possible values. Thus, after assigning to every possible value x_i of X its subjective and corresponding probability p_i, with $\sum_{i=1}^{n} p_i = 1$, we have $inf I(X) \leq \mathbf{P}(X) \leq sup I(X)$ in accordance with convexity property of $\mathbf{P}$. Given $Z = X_1 + \ldots + X_n$ which is a linear combination of n random quantities $X_1, \ldots, X_n$ of $\mathcal{X}$, $I(Z) = \{z_1, \ldots, z_n\}$ is the set of its possible values. Therefore, its coherent prevision must satisfy convexity property of $\mathbf{P}$, so we have $inf I(Z) \leq \mathbf{P}(Z) \leq sup I(Z)$, where it turns out to be $\mathbf{P}(Z)$ = $\mathbf{P}(X_1)$ + …+ $\mathbf{P}(X_n)$ in accordance with linearity property of $\mathbf{P}$. Linearity property can clearly be of interest to any linear combination of n random quantities. We may also consider less than n random quantities. The possibility of certain consequences whose unacceptability appears recognizable to everyone is excluded when convexity property of $\mathbf{P}$ and its linearity property are valid. They are the foundation of the whole theory of probability because they are necessary and sufficient conditions for coherence: decisions under conditions of uncertainty lead to a certain loss when linearity and convexity of $\mathbf{P}$ are broken ([1]). The probabilities of every possible value of a given random quantity belonging to a finite or infinite set of random quantities sum to 1 in a coherent way according to probability laws $\mathbf{P}$ formally admissible in terms of coherence with regard to these possible values.

5 A coherent prevision-function $\mathbb{P}$

From mathematical point of view, $\mathbf{P}$ is a function. We define it by taking into account its objective coherence. The domain of $\mathbf{P}$ is the arbitrary set $\mathcal{X} = \{X_1, \ldots, X_n\}$ consisting of a finite number of random quantities: for each of them, the set of possible values is $I(X_i) = \{x_{i1}, \ldots, x_{in}\}$, with $x_{i1} < \ldots < x_{in}$, $i = 1, \ldots, n$. Moreover, we suppose $x_{i1} \neq x_{j1}$ and $x_{in} \neq x_{jn}$, with $i \neq j$, $i, j = 1, \ldots, n$. The codomain of $\mathbf{P}$ is the set $\mathcal{Y}$ consisting of as many intervals as random quantities are found into the set $\mathcal{X}$ of $\mathbf{P}$, with $infI(X_i) \leq \mathbf{P}(X_i) \leq supI(X_i)$ for every interval referring to the random quantity X_i, $i = 1, \ldots, n$. Therefore, both $\mathcal{X}$ and $\mathcal{Y}$ are sets whose elements are themselves sets. The coherent function $\mathbf{P}$ is called prevision-function and it is a bijective function because each element of $\mathcal{X}$, $X_i \in \mathcal{X}$, is paired with exactly one element of $\mathcal{Y}$, for which it turns out to be $infI(X_i) \leq \mathbf{P}(X_i) \leq supI(X_i)$, and each element of $\mathcal{Y}$ is paired with exactly one element of $\mathcal{X}$. There are no unpaired elements, with $\mathbf{P}(X_i)$ which is a prevision of X_i on the basis of the state of information of a certain individual at a given instant. Given the set $I(X) = \{x_1, \ldots, x_n\}$, with $x_1 < \ldots < x_n$, the image of X under $\mathbf{P}$ is $\mathbf{P}(X) = x_1 p_1 + \ldots + x_n p_n$, with $0 \leq p_i \leq 1, i = 1, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$: such an image coincides with all weighted arithmetic means calculated in a coherent fashion when p_i varies while x_i is constant. All coherent previsions of X satisfy the inequality $infI(X) \leq \mathbf{P}(X) \leq supI(X)$. The image of the entire domain $\mathcal{X}$ of $\mathbf{P}$ is the image of $\mathbf{P}$ and it coincides with the entire codomain $\mathcal{Y}$. If $\mathcal{X}$ is an infinite set of random quantities, we can always consider a restriction of the prevision-function $\mathbf{P}$ which is a new function obtained by choosing a smaller and finite domain. Therefore, the above observations remain unchanged because such a new function coincides with $\mathbf{P}$ whose domain is a finite set of random quantities. In the case in which the domain of $\mathbf{P}$ is the arbitrary set $\mathcal{E} = \{E_1, \ldots, E_n\}$ consisting of a finite number of possible events, its codomain is the set $\mathcal{Y}$ consisting of as many intervals as events are found into the set $\mathcal{E}$ of $\mathbf{P}$, with $infE_i \leq \mathbf{P}(E_i) \leq supE_i$, $i = 1, \ldots, n$, for each of such intervals. Nevertheless, since we have $infE_i = 0$ and $supE_i = 1$, $i = 1, \ldots, n$, it turns out to be $0 \leq \mathbf{P}(E_i) \leq 1$ for every interval of $\mathcal{Y}$. The coherent function $\mathbf{P}$ is called probability-function and it is a bijective function because each element of $\mathcal{E}$, $E_i \in \mathcal{E}$, is paired with exactly one element of $\mathcal{Y}$, for which it turns out to be $0 \leq \mathbf{P}(E_i) \leq 1$, and each element of $\mathcal{Y}$ is paired with exactly one element of $\mathcal{E}$.

There are no unpaired elements, with $\mathbf{P}(E_i)$ which is an evaluation of probability of E_i. The image of E_i under $\mathbf{P}$ is an interval. If $\mathcal{E}$ is an infinite set of events, we can always consider a restriction of the probability-function $\mathbf{P}$ as above. We admit that $\mathbf{P}$ can be evaluated by anybody for every event E or random quantity X. Thus, it is not true that it would make sense to speak of probability only when all events under consideration are repeatable, as well as it is not true that it would make sense to speak of prevision only when all random quantities under consideration belong to a measurable set $\mathcal{I}$. We cannot pretend that $\mathbf{P}$ is actually imagined as determined, by any individual, for all events or random quantities which could be considered in the abstract. We must recognize if $\mathbf{P}$ includes or not any incoherence. If so the individual, when made aware of such an incoherence, should eliminate it. Thus, the subjective evaluation is objectively coherent and can be extended to any larger set of events or random quantities. It is necessary to interrogate a given individual in order to force him to reveal his evaluation of elements of the codomain $\mathcal{Y}$ of $\mathbf{P}$, $\mathbf{P}(X_i)$ or $\mathbf{P}(E_i)$, $i = 1, \ldots, n$: both prevision of a random quantity and probability of an event always express what an individual chooses in his given state of ignorance, so it is wrong to imagine a greater degree of ignorance which would justify the refusal to answer. If a prevision-function is not understood as an expression of the opinion of a certain individual, we can interrogate many individuals in order to study their common opinion which is denoted by $\mathbf{P}$. Therefore, $\mathbf{P}$ will exist in the ambit of those random quantities X for which all evaluations $\mathbf{P}_i(X)$, $i = 1, \ldots, n$, coincide. Such evaluations will define $\mathbf{P}(X)$ in this way. Evidently, $\mathbf{P}$ will not exist elsewhere, for other random quantities X for which the subjective evaluations $\mathbf{P}_i(X)$ do not coincide. The above observations remain valid when a given individual confines himself to evaluations which conform to more restrictive criteria coinciding with classical definition of probability and with the statistical one ([6], [16]).

6 Geometric representation of P

Given the set $\mathcal{X} = \{X_1, \ldots, X_n\}$ or the set $\mathcal{E} = \{E_1, \ldots, E_n\}$, the possible values of each random quantity or random event can geometrically be represented on n lines for which a Cartesian coordinate system has been chosen. Such lines belong to the vector space $\mathbb{R}^n$ over the field $\mathbb{R}$ of real numbers. $\mathbb{R}^n$ has a Euclidean structure characterized by a metric. Hence, the standard basis of $\mathbb{R}^n$ is given by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, where we have $\mathbf{e}_1 = (1, \ldots, 0)$, ..., $\mathbf{e}_n = (0, \ldots, 1)$, and it consists of orthogonal vectors to each other having a Euclidean norm equal to 1. The point of $\mathbb{R}^n$ where n lines meet is the origin of $\mathbb{R}^n$ given by $(0, \ldots, 0)$. We have an one-to-one correspondence between the points of $\mathbb{R}^n$ and the n-tuples of real numbers. We consider n coordinate subspaces of dimension 1 in the vector space $\mathbb{R}^n$. In fact, when we project every point of $\mathbb{R}^n$ referring to $(X_1, X_2, \ldots, X_n)$ and expressed by $(x_1, x_2, \ldots, x_n)$ onto the coordinate axis x_1, it becomes $(x_1, \ldots, 0)$. When we project the same point onto the coordinate axis x_2, it becomes $(0, x_2, \ldots, 0)$ and so on. After projecting all the possible points of X_1 onto the coordinate axis x_1, ..., all the possible points of X_n onto the coordinate axis x_n, every point onto the coordinate axis x_1, coinciding with a particular n-tuple of real numbers of $\mathbb{R}^n$, can be viewed as a real number of $\mathbb{R}$, ..., every point onto the coordinate axis x_n, coinciding with a particular n-tuple of real numbers of $\mathbb{R}^n$, can be viewed as a real number of $\mathbb{R}$. It is finally clear that n projected points onto the coordinate axis x_1 can be viewed as n real numbers of an one-dimensional vector space, ..., n projected points onto the coordinate axis x_n can be viewed as n real numbers of an one-dimensional vector space. The possible points of E_i projected onto the coordinate axis $x_i, i = 1, \ldots, n$, are evidently only two. For instance, if n = 3, we have the points $(1, 0, 0)$ and $(0, 0, 0)$ onto the coordinate axis x_1 referring to E_1 which can respectively be viewed as 1 and 0, ..., the points $(0, 0, 1)$ and $(0, 0, 0)$ onto the coordinate axis x_3 referring to E_3 which can respectively be viewed as 1 and 0. Nevertheless, we have always three real lines, so we do not get confused. In any case, it is conceptually the same thing if we make use only of particular n-tuples of real numbers of $\mathbb{R}^n$ without seeing them as real numbers of $\mathbb{R}$. The codomain of $\mathbf{P}$ is the set $\mathcal{Y}$ consisting of n intervals which coincide with n line segments belonging to n different real lines. These line segments could become increasingly larger by virtue of linearity of $\mathbf{P}$ extended to any finite number of random quantities considered on a same line. Indeed, we observe that all weights or probabilistic masses, which are non-negative and sum to 1, remain unchanged with respect to starting point characterized by only one random quantity. Nevertheless, they are paired with real numbers whose absolute values are evidently greater. Such numbers can be interpreted as the possible values of one random quantity considered on a same line. It is evident that the set of all coherent previsions of every random quantity X_i as well as the set of all coherent probabilities of every random event E_i, $i = 1, \ldots, n$, is a subset of a vector subspace of dimension 1. Such a subset is however a convex set while the set of the possible values for every random quantity considered into $\mathbb{R}^n$ is not a convex set. The same thing goes when we consider the set of the possible values for every random event represented into $\mathbb{R}^n$. We already saw that it is always possible to consider a finite number of events or random quantities in order to study probability laws $\mathbf{P}$ formally admissible in terms of coherence. As a first step we refer to events. Given n events $E_1, \ldots, E_n$ of $\mathcal{E}$, we represent them by means of n axes of $\mathbb{R}^n$. Nevertheless, instead of concentrating our attention on n axes of $\mathbb{R}^n$ as above, we consider only one of them which we choose in an arbitrary fashion. Such an axis is an one-dimensional vector subspace of $\mathbb{R}^n$. It is generated by a vector of the standard basis of $\mathbb{R}^n$. Every point of $\mathbb{R}^n$ on a same line is obtained multiplying by a real number the vector of the standard basis of $\mathbb{R}^n$ which we have arbitrarily chosen. Therefore, we can always multiply by any real number a same n-tuple of real numbers in order to obtain points of $\mathbb{R}^n$ which are said to be collinear. Now the real number or coefficient of the linear combination under consideration, characterized by only one scalar, represents the probability of an event A into our geometric scheme of representation. Given $\mathbf{P}(E_1)$, ..., $\mathbf{P}(E_n)$, we know that $\mathbf{P}(A)$ can be uniquely determined or bounded from above and below or absolutely undetermined depending on the circumstances. If it is uniquely determined, then we have a precise point of $\mathbb{R}^n$ on the axis under consideration. If it is bounded from above and below, then we have two points of $\mathbb{R}^n$ on this axis and an admissible probability is found between them. If it is absolutely undetermined, then we have a larger interval on this axis which is included between the lowest admissible probability of any event and the greatest admissible one. In particular, given $\mathbf{P}(E_1) = p_1$, ..., $\mathbf{P}(E_n) = p_n$, after choosing the vector $\mathbf{e}_n$ of the standard basis of $\mathbb{R}^n$, by means of the linear combination given by $[(\lambda_1 - p_1)S_1 + \ldots + (\lambda_n - p_n)S_n]\mathbf{e}_n$, with $S_i \neq 0$, $i = 1, \ldots, n$, we can obtain the possible values of the random quantity $G = (\lambda_1 - p_1)S_1 + \ldots + (\lambda_n - p_n)S_n$ referring to n bets concerning n events as special n-tuples of $\mathbb{R}^n$. The same thing goes if we choose another vector of the standard basis of $\mathbb{R}^n$. Thus, we even represent n bets concerning n events into a linear space. By examining n random quantities $X_1, \ldots, X_n$ of $\mathcal{X}$, we similarly represent them by means of n axes of $\mathbb{R}^n$. Nevertheless, by considering another random quantity Z which is again bounded from above and below, instead of concentrating our attention on n axes of $\mathbb{R}^n$, we consider only one of them which we choose in an arbitrary way. After individuating two points of $\mathbb{R}^n$ on this axis which are respectively the lowest possible value of the random quantity under consideration and the greatest possible one, $\mathbf{P}(Z)$ can be viewed as a point of $\mathbb{R}^n$ coherently included between the two points of $\mathbb{R}^n$ already individuated. Probability laws $\mathbf{P}$ formally admissible in terms of coherence are those laws for which the probabilities of the possible values of the random quantity under consideration sum to 1.

7 Conclusions

We distinguished the two extreme aspects of the logic of certainty by identifying their corresponding structures into a linear space. We extended probability laws $\mathbf{P}$ formally admissible in terms of coherence to random quantities. We proposed a geometric representation of these laws and of a coherent prevision-function $\mathbf{P}$ which we previously defined. We connected the convex set of all coherent previsions of a random quantity as well as the convex set of all coherent probabilities of an event with a specific algebraic structure: such a structure is an one-dimensional vector subspace over the field $\mathbb{R}$ of real numbers because events of any finite set of events can be viewed as special points of a vector space of dimension n over the field $\mathbb{R}$ of real numbers. It is exactly the linear space of random quantities having a Euclidean structure characterized by a metric coinciding with the dot product in a natural way. Overall, we pointed out that linearity is the most meaningful concept concerning probability calculus whose laws gain a more extensive rigour by means of the geometric scheme of representation we showed. On the other hand, it is possible to extend linearity concept in order to formulate a geometric, well-organized and original theory of random quantities: we will make this into our next works.

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