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Sums of Generalized Harmonic Series with Periodically Repeated Numerators (a, b) and (a, a, b, b)

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Abstract

This paper deals with certain generalization of the alternating harmonic series – the generalized convergent harmonic series with periodically repeated numerators (a, b) and (a, a, b, b). Firstly, we find out the value of the numerators b of the first series, for which the series converges, and determine the formula for the sum $s(a)$ of this series. Then we determine the value of the numerators b of the second series, for which this series converges, and derive the formula for the sum $s(a, a)$ of this second series. Finally, we verify these analytically obtained results and compute the sums of these series by using the computer algebra system Maple 16 and its basic programming language.

Keywords: harmonic series, alternating harmonic series, sequence of partial sums, computer algebra system Maple.
2010 AMS subject classifications: 40A05, 65B10.

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1 Introduction

Let us recall the basic terms, concepts and notions. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$.
The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each n as the sum of the sequence $\{a_k\}$ from a_1 to a_n, i.e. $s_n = a_1 + a_2 + \cdots + a_n$.
The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$.
The *harmonic series* is the sum of reciprocals of all natural numbers (except zero), so this is the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots .$$

The divergence of this series can be easily proved e.g. by using the integral test or the comparison test of convergence.

In this paper we will deal with the series of the form

$$\sum_{k=1}^{\infty} \left(\frac{a}{2k-1} + \frac{b}{2k} \right) \quad \text{and} \quad \sum_{k=1}^{\infty} \left(\frac{a}{4k-3} + \frac{a}{4k-2} + \frac{b}{4k-1} + \frac{b}{4k} \right),$$

where a, b are such numbers that these series converge.

If we know, these two types of infinite series has not yet been studied in the literature. The author has previously in the papers [1], [2], and [3] dealt with the series of the form

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k-1} + \frac{a}{2k} \right), \quad \sum_{k=1}^{\infty} \left(\frac{1}{3k-2} + \frac{1}{3k-1} + \frac{a}{3k} \right),$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{3k-2} + \frac{a}{3k-1} + \frac{b}{3k} \right), \text{ and } \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} + \frac{a}{4k-2} + \frac{b}{4k-1} + \frac{c}{4k} \right),$$

so that this contribution is a free follow-up to these three papers. Let us note that in the previous issues of Ratio Mathematica, for example, papers [4] and [5] also deal with the topic infinite series and their convergence.

2 The sum of generalized harmonic series with periodically repeated numerators (a, b)

We deal with the numerical series of the form

$$\sum_{k=1}^{\infty}\left(\frac{a}{2k-1}+\frac{b}{2k}\right)=\frac{a}{1}+\frac{b}{2}+\frac{a}{3}+\frac{b}{4}+\frac{a}{5}+\frac{b}{6}+\cdots, \tag{1}$$

where $a, b \in \mathbb{R}$ are appropriate constants for which the series (1) converges. This series we shall call *generalized harmonic series with periodically repeated numerators* (a, b). We determine the value of the numerators b, for which the series (1) converges, and the sum $s(a)$ of this series.

The power series corresponding to the series (1) has evidently the form

$$\sum_{k=1}^{\infty}\left(\frac{ax^{2k-1}}{2k-1}+\frac{bx^{2k}}{2k}\right)=\frac{ax}{1}+\frac{bx^2}{2}+\frac{ax^3}{3}+\frac{bx^4}{4}+\frac{ax^5}{5}+\frac{bx^6}{6}+\cdots. \tag{2}$$

We denote its sum by $s(x)$. The series (2) is for $x \in (-1, 1)$ absolutely convergent, so we can rearrange it and rewrite it in the form

$$s(x)=a\sum_{k=1}^{\infty}\frac{x^{2k-1}}{2k-1}+b\sum_{k=1}^{\infty}\frac{x^{2k}}{2k}. \tag{3}$$

If we differentiate the series (3) term-by-term, where $x \in (-1, 1)$, we get

$$s'(x)=a\sum_{k=1}^{\infty}x^{2k-2}+b\sum_{k=1}^{\infty}x^{2k-1}. \tag{4}$$

After reindexing and fine arrangement the series (4) for $x \in (-1, 1)$ we obtain

$$s'(x)=a\sum_{k=0}^{\infty}x^{2k}+bx\sum_{k=0}^{\infty}x^{2k},$$

that is

$$s'(x)=(a+bx)\sum_{k=0}^{\infty}\left(x^2\right)^k. \tag{5}$$

When we summate the convergent geometric series on the right-hand side of (5) with the first term 1 and the ratio x^2, where $\left|x^2\right| < 1$, i.e. for $x \in (-1, 1)$, we get

$$s'(x)=\frac{a+bx}{1-x^2}.$$

We convert this fraction using the CAS Maple 16 to partial fractions and get

$$s'(x) = \frac{a-b}{2(x+1)} - \frac{a+b}{2(x-1)} = \frac{a-b}{2(1+x)} + \frac{a+b}{2(1-x)}\,,$$

where $x \in (-1, 1)$. The sum $s(x)$ of the series (2) we obtain by integration in the form

$$s(x) = \int\left(\frac{a-b}{2(1+x)} + \frac{a+b}{2(1-x)}\right)\mathrm{d}x = \frac{a-b}{2}\ln(1+x) - \frac{a+b}{2}\ln(1-x) + C.$$

From the condition $s(0) = 0$ we obtain $C = 0$, hence

$$s(x) = \frac{a-b}{2}\ln(1+x) - \frac{a+b}{2}\ln(1-x)\,. \tag{6}$$

Now, we will deal with the convergence of the series (2) in the right point $x = 1$. After substitution $x = 1$ to the power series (2) – it can be done by the extended version of Abel's theorem (see [6], p. 23) – we get the numerical series (1). By the integral test we can prove that the series (1) converges if and only if $b + a = 0$. After simplification the equation (6), where $b = -a$, we have

$$s(x) = \frac{a+a}{2}\ln(1+x) = a\ln(1+x)\,.$$

For $x = 1$ and after re-mark $s(1)$ as $s(a)$, we obtain a very simple formula

$$s(a) = a\ln 2\,, \tag{7}$$

which is consistent with the well-known fact that the sum of the *alternating harmonic series* $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ is equal to $\ln 2$.

3 The sum of generalized harmonic series with periodically repeated numerators (a, a, b, b)

Now, we deal with the numerical series of the form

$$\sum_{k=1}^{\infty}\left(\frac{a}{4k-3} + \frac{a}{4k-2} + \frac{b}{4k-1} + \frac{b}{4k}\right) = \frac{a}{1} + \frac{a}{2} + \frac{b}{3} + \frac{b}{4} + \cdots\,, \tag{8}$$

where $a, b \in \mathbb{R}$ are appropriate constants for which the series (8) converges. This series we shall call *generalized harmonic series with periodically repeated numerators* (a, a, b, b). We determine the value of the numerators b, for which the series (8) converges, and the sum $s(a, a)$ of this series.

The power series corresponding to the series (8) has evidently the form

$$\sum_{k=1}^{\infty}\left(\frac{ax^{4k-3}}{4k-3}+\frac{ax^{4k-2}}{4k-2}+\frac{bx^{4k-1}}{4k-1}+\frac{bx^{4k}}{4k}\right)=\frac{ax}{1}+\frac{ax^2}{2}+\frac{bx^3}{3}+\frac{bx^4}{4}+\cdots. \quad (9)$$

We denote its sum by $s(x)$. The series (9) is for $x \in (-1,1)$ absolutely convergent, so we can rearrange it and rewrite it in the form

$$s(x)=a\sum_{k=1}^{\infty}\frac{x^{4k-3}}{4k-3}+a\sum_{k=1}^{\infty}\frac{x^{4k-2}}{4k-2}+b\sum_{k=1}^{\infty}\frac{x^{4k-1}}{4k-1}+b\sum_{k=1}^{\infty}\frac{x^{4k}}{4k}. \quad (10)$$

If we differentiate the series (10) term-by-term, where $x \in (-1,1)$, we get

$$s'(x)=a\sum_{k=1}^{\infty}x^{4k-4}+a\sum_{k=1}^{\infty}x^{4k-3}+b\sum_{k=1}^{\infty}x^{4k-2}+b\sum_{k=1}^{\infty}x^{4k-1}. \quad (11)$$

After reindexing and fine arrangement the series (11) for $x \in (-1,1)$ we obtain

$$s'(x)=a\sum_{k=0}^{\infty}x^{4k}+ax\sum_{k=0}^{\infty}x^{4k}+bx^2\sum_{k=0}^{\infty}x^{4k}+bx^3\sum_{k=0}^{\infty}x^{4k},$$

that is

$$s'(x)=(a+ax+bx^2+bx^3)\sum_{k=0}^{\infty}\left(x^4\right)^k. \quad (12)$$

After summation the convergent geometric series on the right-hand side of (12) with the first term 1 and the ratio x^4, where $\left|x^4\right|<1$, i.e. for $x \in (-1,1)$, we get

$$s'(x)=\frac{a+ax+bx^2+bx^3}{1-x^4}=\frac{(a+bx^2)(1+x)}{(1+x^2)(1-x)(1+x)}=\frac{a+bx^2}{(1+x^2)(1-x)}.$$

We convert this fraction using the CAS Maple 16 to partial fractions and get

$$s'(x)=\frac{a+b}{2(1-x)}+\frac{a-b+ax-bx}{2(1+x^2)},$$

where $x \in (-1,1)$. The sum $s(x)$ of the series (9) we obtain by integration:

$$s(x)=\int\left(\frac{a+b}{2(1-x)}+\frac{a-b}{2(1+x^2)}+\frac{(a-b)x}{2(1+x^2)}\right)\mathrm{d}x=$$

$$=-\frac{a+b}{2}\ln(1-x)+\frac{a-b}{2}\arctan x+\frac{a-b}{4}\ln(1+x^2)+C.$$

From the condition $s(0) = 0$ we obtain $C = 0$, hence

$$s(x) = -\frac{a+b}{2}\ln(1-x) + \frac{a-b}{2}\arctan x + \frac{a-b}{4}\ln(1+x^2). \tag{13}$$

Now, we will deal with the convergence of the series (9) in the right point $x = 1$. After substitution $x = 1$ to the power series (9) we get the numerical series (8). By the integral test we can prove that the series (8) converges if and only if $a + b = 0$. After simplification the equation (13), where $b = -a$, we have

$$s(x) = a\arctan x + \frac{a}{2}\ln(1+x^2) = \frac{a}{2}\big[2\arctan x + \ln(1+x^2)\big].$$

For $x = 1$ and after re-mark $s(1)$ as $s(a, a)$, we obtain a simple formula

$$s(a, a) = \frac{a}{4}(\pi + 2\ln 2)\,. \tag{14}$$

4 Numerical verification

We have solved the problem to determine the sums $s(a)$ and (a, a) above of the convergent numerical series (1) and (8) for several values of a (and for $b = -a$) by using the basic programming language of the computer algebra system Maple 16. They were used two following very simple procedures `sumab` and `sumaabb`:

```
sumab=proc(t,a)
   local r,k,s; s:=0; r:=0;
   for k from 1 to t do
       r:=a*(1/(2*k-1) - 1/(2*k)); s:=s+r;
   end do;
   print("s(",a,")=",evalf[9](s),
       "f=",evalf[9](a*ln(2)));
end proc:
```

```
sumaabb:=proc(t,a)
   local r,k,s; s:=0; r:=0;
   for k from 1 to t do
       r:=a*(1/(4*k-3) + 1/(4*k-2) - 1/(4*k-1) - 1/(4*k));
       s:=s+r;
   end do;
   print("s(",a,a,")=",evalf[9](s),
       "f=",evalf[9](a*(Pi+2*ln(2))/4);
end proc:
```

For evaluation the sums $s(10^6, a)$ and $s(10^6, a, a)$ and the corresponding values $s(a)$ and $s(a, a)$ defined by the formulas (7) and (14) it was used this for-loop statement:

```
for a from 1 to 10 do
sumab(1000000,a); sumaabb(1000000,a);
end do;
```

The approximative values of the sums $s(10^6, a)$, $s(a)$, $s(10^6, a, a)$, and $s(a, a)$ rounded to seven decimals obtained by these procedures are written into the following Table 1. Let us note that the computation of 20 values $s(a)$ and $s(a, a)$ took almost 51 hours 26 minutes. The relative quantification accuracies $r(a) = \dfrac{|s(10^6, a) - s(a)|}{s(10^6, a)}$ of the sum $s(a, 10^6)$ and $r(a, a) = \dfrac{|s(10^6, a, a) - s(a, a)|}{s(10^6, a, a)}$ of the sum $s(a, a, 10^6)$ are stated in the fourth and eighth columns of Table 1. These relative quantification accuracies are approximately between $4 \cdot 10^{-7}$ and $2 \cdot 10^{-7}$.

a	$s(10^6, a)$	$s(a)$	$r(a)$	a	$s(10^6, a, a)$	$s(a, a)$	$r(a, a)$
1	0.6931469	0.6931472	$4{\cdot}10^{-7}$	1	1.1319715	1.1319718	$3{\cdot}10^{-7}$
2	1.3862939	1.3862944	$4{\cdot}10^{-7}$	2	2.2639430	2.2639435	$2{\cdot}10^{-7}$
3	2.0794408	2.0794415	$3{\cdot}10^{-7}$	3	3.3959145	3.3959153	$2{\cdot}10^{-7}$
4	2.7725877	2.7725887	$4{\cdot}10^{-7}$	4	4.5278860	4.5278870	$2{\cdot}10^{-7}$
5	3.4657347	3.4657359	$3{\cdot}10^{-7}$	5	5.6598575	5.6598588	$2{\cdot}10^{-7}$
6	4.1588816	4.1588831	$4{\cdot}10^{-7}$	6	6.7918290	6.7918305	$2{\cdot}10^{-7}$
7	4.8520285	4.8520303	$4{\cdot}10^{-7}$	7	7.9238005	7.9268023	$2{\cdot}10^{-7}$
8	5.5451754	5.5451774	$4{\cdot}10^{-7}$	8	9.0557720	9.0557740	$2{\cdot}10^{-7}$
9	6.2383224	6.2383246	$4{\cdot}10^{-7}$	9	10.1877435	10.1877458	$2{\cdot}10^{-7}$
10	6.9314693	6.9314718	$4{\cdot}10^{-7}$	10	11.3197150	11.3197175	$2{\cdot}10^{-7}$

Table 1: The approximate values of the sums of the generalized harmonic series with periodically repeating numerators $(a, -a)$ and $(a, a, -a, -a)$ for $a = 1, 2, \ldots, 10$

5 Conclusions

In this paper we dealt with the generalized harmonic series with periodically repeated numerators (a, b) and (a, a, b, b), i.e. with the series

$$\sum_{k=1}^{\infty} \left(\frac{a}{2k-1} + \frac{b}{2k} \right) = \frac{a}{1} + \frac{b}{2} + \frac{a}{3} + \frac{b}{4} + \frac{a}{5} + \frac{b}{6} + \cdots$$

with the sum $s(a)$ and with series

$$\sum_{k=1}^{\infty}\left(\frac{a}{4k-3}+\frac{a}{4k-2}+\frac{b}{4k-1}+\frac{b}{4k}\right)=\frac{a}{1}+\frac{a}{2}+\frac{b}{3}+\frac{b}{4}+\cdots$$

with the sum $s(a,a)$, where $a, b \in \mathbb{R}$.

We derived that the only value of the numerators $b \in \mathbb{R}$, for which these series converge, are $b=-a$, and we also derived that the sums of these series are determined by the formulas

$$s(a)=a\ln 2$$

and

$$s(a,a)=\frac{a}{4}(\pi+2\ln 2)\,.$$

So, for example, the series $\sum_{k=1}^{\infty}\left(\frac{5}{2k-1}-\frac{5}{2k}\right)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)k}$ has the sum $s(5)\doteq 3.4657$ and the series $\sum_{k=1}^{\infty}\left(\frac{5}{4k-3}+\frac{5}{4k-2}-\frac{5}{4k-1}-\frac{5}{4k}\right)=\frac{5}{4}\sum_{k=1}^{\infty}\frac{32k^2-24k+3}{(4k-3)(2k-1)(4k-1)k}$ has the sum $s(5,5)\doteq 5.6599$.

Finally, we verified these two main results by computing some sums by using the CAS Maple 16 and its basic programming language. These generalized harmonic series so belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically and also presented by means of a simple numerical expression. From the derived formulas for $s(a)$ and $s(a,a)$ above it follows that

$$a=\frac{s(a)}{\ln 2}\qquad\text{and}\qquad a=\frac{4s(a,a)}{\pi+2\ln 2}\,.$$

These relations allow calculate the value of the numerators a for a given sum $s(a)$ or $s(a,a)$, as illustrates the following Table 2:

$s(a)$	a	$s(a,a)$	a
1	1.4427	1	0.8834
$\ln 0.5 \doteq -0.6391$	-1	$(-\pi-2\ln 2)/4 \doteq -1.1320$	-1
$\ln 2 \doteq 0.6391$	1	$(\pi+2\ln 2)/4 \doteq 1.1320$	1

Table 2: The approximate values of the numerators a for some sums $s(a)$ and $s(a,a)$

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