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Algebraic Spaces and Set Decompositions.

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Abstract

The contribution is growing up from certain parts of scientific work by professor Borůvka in several ways. Main focus is on the decomposition theory, especially algebraized decompositions of groups. Professor Borůvka in his excellent and well-known book [3] has developped the decomposition (partition) theory, where the fundamental role belongs to so called generating decompositions. Furthermore, the contribution is also devoted to hypergroups, to algebraic spaces called also quasi-automata or automata without outputs. There is attempt to develop more fresh view point on this topic.

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1 Introduction

The present contribution is growing up from certain parts of scientific work by professor Borůvka in several ways. First of all is the decomposition theory, especially algebraized decompositions of groups. Professor Borůvka in his excellent and well-known book [3] has developped the decomposition (partition) theory, in (and on) sets which is applied to decompositions on groupoids and groups where the fundamental role belongs to so called generating decompositions. It is to be noted that a decomposition $\overline{A}$ in a groupoid $(G, \cdot)$ is called *generating* if there exists, to any two-membered sequence of the elements $\bar{a}, \bar{b} \in \overline{A}$ an element $\bar{c} \in \overline{A}$ such that $\bar{a}\bar{b} \subset \bar{c}$. With the decomposition $\overline{A}$ in a groupoid $(G, \cdot)$ there can be uniquely associate a groupoid denoted (in the mentioned book) by $\mathcal{U}$ and defined such a way that the carrier set of $\mathcal{U}$ is the decomposition $\overline{A}$ and the multiplication is defined by $\bar{a} \circ \bar{b} = \bar{c}$, where $\bar{a}, \bar{b}, \bar{c} \in \overline{A}$ are such elements (i. e. cosets) that $\bar{a} \cdot \bar{b} \subset \bar{c}$ in the groupoid $(G, \cdot)$. A special and important case of generating decompositions on a group $(G, \cdot)$ created by left on right cosets of an invariant (normal) subgroup $(H, \cdot)$ of $(G, \cdot)$ is the carrier of a factor-group G/H which is a factoroid created by cosets of the form $a \cdot H$ (or which is the same $H \cdot a$) for an invariant subgroup H of G. On the other hand if left or right decompositions generated by a subgroup H which is not invariant in a noncommutative group G are algebraized in a similar way as above, we get multivalued binary operations on these decompositions which determine a structure called a *multigroups* or a *hypergroup* by the latest terminology. This one has been done by Marty in 1934 and since the time these structures were investigated by many mathematicians in France, Italy, Greece, Roumania, USA, Canada , Czechoslovakia and elsewhere.

2 Preliminaries

A hypergroup in the sense of Marty is a pair $(H, \cdot)$ where H is a non-empty set and $\cdot : H \times H \to \mathcal{P}'(H)$ (the system of all non-empty subsets off H) is an associative multioperation (called also a hyperoperation) satisfying the reproduction axiom: $a \cdot H = H = H \cdot a$ for any $a \in H$ [11, 12].

A commutative hypergroup $(H, \cdot)$ is called a *join hypergroup* or a *join space* if it satisfies the *exchange condition*: For any quadruple $a, b, c, d \in H$ such that $a/b \cap c/d \neq \emptyset$ (where $a/b = \{x \in H; a \in x \cdot b\}$ and similarly for c/d) we have

$$(a \cdot d) \cap (b \cdot c) \neq \emptyset.$$

In the last years investigations of hypergroups which are determined by binary relations (i.e. the binary hyperoperation $\cdot$ is derived by a certain standard way from a given relation on its carrier set) are of certain interests in investigations on this field. The notion of a join space has been introduced by W. Prenowitz and used by him and afterwards together with J. Jantosciak to built again several branches of geometry. In the opinion of professor P. Corsini - which is one of present leading personalities in the hypergroup theory - the presentation and development of geometry in the context of join spaces is an important moment in the recent history of mathematics. There are also close connections of the, mentioned structure to ternary spaces, especially formed by sets endowed by ternary betweenness relations, here.

It is to be noted that any abelian group is a join space with single-valued operations. A simple example of a non-trivial join space or a join hypergmup can be constructed from arbitrary (non-extremal) decomposition of a set: Let $\overline{A}$ be a decomposition on a non-empty set A. For any pair of elements $x, y \in A$ let us define $x \cdot y = \bar{a} \cup \bar{b}$, where $\bar{a}, \bar{b} \in \overline{A}$ are blocks of the given decomposition such that $x \in \bar{a}, y \in \bar{b}$. Then it is easy to see that $(A, \cdot)$ is a join hypergroup (a join space) in which for a pair $x, y \in A$ the fraction x/y is either a block of A containing x or $x/y = A$ whenever x, y belong to the same block of A.

The algebraic theory of automata is widely elaborated classical discipline; the golden age or which can be designated from the beginning of sixties up to the end of tne last century. Nevertheless fundamental publications from the earlier time due to N. Wiener, J. von Neumann, S. Ginsburg, M. A. Arbib, V. M. Gluškov, R. E. Kálmán, M. O. Rabin, D. Scott, S. Greibach, K. B. Krohn, J. L. Rhodes, E. F. More and others, have massive influence on the development of the automata and artificial languages theory. In spite of studies devoted to finite automata also infinite automata and their generalizations have been of some interests (cf. Ferenc Gécseg, István Peák nad others). It is to be noted that various concepts of a product of automata (the basic of which has been introduced and studied by M. V. Gluškov in 1961 as an abstract model of electronic cirquits) are treated in a large collection of studies devoted to this topics. During the years of investigations of the mentioned thema, there occure various modifications; most of them can be generalized to the case of multiautomata or to actions of multistructures. Investigations of automata in connection with multistructures yield more new impulses. It is evident that infinite antomata without outputs called also quasi-automata are in fact discrete modifications or “algebraic skelets” of dynamical systems. Objects of investigations of the mentioned theories can be also considered as special general systems and they are close to the control theory.

The other connection of this contribution to the research of professor Borůvka consists in investigations of group and semigroup actions on sets which are substantial parts of the algebraic concept of an automaton, namely if we concentrate on changes of states rather than outputs which has been used by professor Borůvka in his two-parted paper [4]. Automata without outputs are termed also algebraic spaces (according to Dubreil, Dubreil - Jacobin and Borůvka). So, we can use also this terminology. In accordance with [4] we define an *algebraic space with operators* as a triad $\boldsymbol{E} = (E, G, \alpha)$, where $E \neq \emptyset$ (a *state set* or a *phase set*), G is a monoid the identity e (in a special case G is supposed to be a group) called also an *input* or *phase monoid* and $\alpha : G \times E \to E$ is an action (called also a *transition function*) wich satisfies two conditions:

1. Identity condition $\alpha(e, x) = x$ for any $x \in E$,
2. Condition of mixed associativity $\alpha(b, (\alpha(a, x)) = \alpha(ab, x)$ for any $a, b \in G$, $x \in E$.

An algebraic space $\boldsymbol{E} = (E, G, \alpha)$ is said to be *homogenous* if G is acting on the set E transitively, i.e. for any pair of elements $x, y \in E$ there exists $a \in G$ such that $\alpha(a, x) = y$. Usually an algebraic space $\boldsymbol{E}$ is called homogenous if G is a group transitively acting on E, which we can called *strong homogeneous* or shortly *s-homogeneous.*

3 Algebraic spaces and hypergroups

We assign to every algebraic space $\boldsymbol{E} = (E, G, \alpha)$ a commutative hypergroup $\mathbf{H}(\boldsymbol{E}) = (E, \bullet)$ in this way: For any pair $x, y \in E$ we define

$$x \bullet y = \alpha(G, x) \cup \alpha(G, y),$$

where $\alpha(G, x) = \{\alpha(a, x); a \in G\}$ is the trajectory of the element x over the monoid G. Then the hypergroup $\mathbf{H}(\boldsymbol{E})$ is called a *state hypergroup* of the algebraic space $\boldsymbol{E}$. It is clear that on the state set of any algebraic space $\boldsymbol{E} = (E, G, \alpha)$ there are defined two totally additive closure operations:

$$\boldsymbol{S}_+, \boldsymbol{S}_- : \mathcal{P}(E) \to \mathcal{P}(E)$$

in this way: $\boldsymbol{S}_+(X) = \alpha(G, X)$, $\boldsymbol{S}_-(X) = \{x \in X; \alpha(a, x) \in X$ for some $a \in G\}$ if X is a non-empty subset of the set E and $\boldsymbol{S}_+(\emptyset) = \boldsymbol{S}_-(\emptyset) = \emptyset$ (caled a *source* and an *successor* closure operation, respectively).

The above defined transfer can be extended into functorial if we consider suitable morphism between hypergroups (where we use mostly homomorphisms and good homomorphisms).

By [18] a hypergroup H is said to be *cyclic* if for some $h \in H$ we have $H = \bigcup_{k \in \mathbf{N}} h^k$ and it is called single-power cyclic (more exactly *n-singIe-power cyclic*) if there exist $h \in H$, $n \in \mathbf{N}$ such that $H = h^n$. In this case the element his called *n-generating*. From the above definition of a state hypergroup we get:

Proposition 1. *An algebraic space* $\boldsymbol{E}$ *is homogeneous if and only if its state hypergroup* $\mathbf{H}(\boldsymbol{E})$ *is 2-single-power cyclic and each element* $x \in E$ *is a 2- generating element of this hypergroup.*$\square$

The following theorem gives necessary and sufficient conditions under which the state hypergroup of an algebraic space is a join hypergroup:

Theorem 1. *Let* $(E, \bullet)$ *be a state hypergroup of an algebraic space* (E, G, α)*. Then the following conditions are equivalent:*

1. $(E, \bullet)$ *is a join hypergroup.*
2. *For any pair* $(x, y) \in E \times E$ *such that* $x \bullet y \subseteq u^2$ *for a suitable element* $u \in E$*, there exists an element* $v \in E$ *with the property* $v^2 \subseteq x^2 \cap y^2$.
3. *For any pair* $(x, y) \in E \times E$ *such that there exists a pair* $(a, b) \in G \times G$ *and an element* $u \in E$ *with* $\alpha(a, u) = x,\ \alpha(b, u) = y$*, we have* $\alpha(c, x) = \alpha(d, y)$ *for some pair* $(c, d) \in G \times G$.

$\square$

On the contrary to the case of algebraic structures with single-valued operations in the case of hypergroups there are possible various modifications of the concept of generating decomposition of the carrier set of a hypergroup. It depends on the various approaches to the congruence concept for hyperstructures. One of them is the following notion:

Definition. Let $(G, \cdot)$ be a hypergroupoid (i. e. $\cdot : G \times G \to \mathcal{P}(G)$ is an arbitrary mapping) Let $\overline{G}$ be such a decomposition on the set G that for any quadruple $a, b, c, d \in G$ with the property $a, c \in \bar{a},\ b, d \in \bar{b}$ for some $a, b \in G$ we have $(a \cdot b \lceil \overline{G}) = (c \cdot d \lceil \overline{G})$; here $X \lceil \overline{G}$ denotes the closure of the set X in the decomposition $\overline{G}$ ([3], 2. 3). Then the decomposition $\overline{G}$ is called *generating (on the hypergroupoid* $(G, \cdot)$) or *h-generating.*

Example 1. Let X be a nonempty set, $f : X \to X$ be a mapping. For $x, y \in X$ we put

$$x \cdot y = \{f^n(u); u \in \{x, y\}, n \in \mathbf{N}_0\},$$

where f^n is the *n*-th iteration of the mapping f. Then it is easy to verify that $(X, \cdot)$ is a commutative hypergroup in the above considered sense. Then the decomposition $\overline{X}_f$ corresponding to a KW-equivalence (Kuratowski -Whyburn - equivalence) $\mathbf{r}$ on X is defined by $x\,\mathbf{r}\,y$ iff $f^m(x) = f^n(x)$ for some pair $m, n \in \mathbf{N}_0$ (the set of all non-negative integers) Then the decomposition $\overline{X}_f$ is generating on the hypergroup $(X, \cdot)$.

Example 2. By a deformation of one hypergroupoid $(G, \cdot)$ onto another one hypergroupoid $(H, \cdot)$ we mean a good (also called strong) homomorphism $f : (G, \cdot) \to (H, \cdot)$, i.e. for any pair $x, y \in G$ we have $f(x \cdot y) = f(x) \cdot f(y)$. Then the decomposition $\overline{G}$ of the hypergroupoid $(G, \cdot)$ corresponding to deformation f (i.e. elements $x, y \in G$ belong to some element $\bar{a} \in \overline{G}$ if an only if $f(x) = f(y)$) i.e. the decomposition corresponding to f is h-generating.

4 h-genenerating and Levine‘s decompositions

Now we define a hyperoperation on an h-genenerating decomposition $\overline{G}$ on a hypergroupoid $(G, \cdot)$. For arbitrary pair of elements $\bar{a}, \bar{b} \in \overline{G}$ we put

$$\bar{a} \cdot \bar{b} = (x.y)[\overline{G},$$

where $(x, y) \in \bar{a} \times \bar{b}$ is an arbitrary pair.
It is easy to prove that then $(G, \cdot)$ is a hypergroupoid and that the definition is correct (it is independent on the choice of elements x, y). The hypergroupoid $(\overline{G}, \cdot)$ is then called a *factor - hypergroupoid* on $(G, \cdot)$ or a *hyperfactoroid on* $(G, \cdot)$ or a *hyperfactoroid of* $(G, \cdot)$. Moreover we have:

Theorem 2. *Let $\overline{G}$ be an h-generating decomposition on a hypergroup $(G, \cdot)$. Then the hyperfactoroid $(\overline{G}, \cdot)$* of $(G, \cdot)$ *is a hypergroup.* □

Now consider an algebraic space with operators $\boldsymbol{E} = (E, G, \alpha)$ with a monoid G of operators. On the system $\mathcal{P}(E)$ of all subsets of E, i.e. the power set of E, we define a decomposition in this way: Denote $\mathcal{S}(E) = \{K \in \mathcal{P}(E); S + K) = K\}$, i.e. $K \in \mathcal{S}(E)$ whenever $\alpha(G, K) = K$. Now suppose $\overline{\mathcal{P}(E)}$ is a decomposition of $\mathcal{P}(E)$ such that sets $X, Y, \in \mathcal{P}(E)$ belong to some element of $\overline{\mathcal{P}(E)}$ if for any set $M \in \mathcal{P}(E)$ such that $M = E \setminus K$(a complement) for some $K \in \mathcal{S}(E)$ we have $X \subseteq M$ if and only if $Y \subseteq M$. Then the decomposition $\overline{\mathcal{P}(E)}$ is called a *decomposition of the Levine‘s type* or a *Levine‘s decomposition* of the power set $\mathcal{P}(E)$.

Proposition 2. *Let $\boldsymbol{E} = (E, G, \alpha)$ be an algebraic space with operators, $\overline{\mathcal{P}(E)}$ be the Levine‘s decomposition of power set $\mathcal{P}(E)$. Then sets $X, Y \in \mathcal{P}(E)$ belong to the same element of $\overline{\mathcal{P}(E)}$ if and only if $x \in X$ implies $\alpha(G, x) \cap Y \neq \emptyset$ and $y \in Y$ implies $\alpha(G, y) \cap X \neq \emptyset$.*

Denote by $\mathcal{CS}(\boldsymbol{E}) = \{M; M \subseteq E, E \setminus M \in \mathcal{S}(\boldsymbol{E})\}$ and $\mathcal{U}_E(X) = \{M; X \subseteq M, M \in \mathcal{CS}(\mathbf{E})\}$ for any $X \in \mathcal{P}(E)$. Then we get:

Theorem 3. *Let* $\boldsymbol{E} = (E, G, \alpha)$ *be an algebraic space with operators. For any pair of sets* $A, B \in \mathcal{P}(E)$ *we define* $A \bullet B = \mathcal{U}_E(A) \cup \mathcal{U}_E(B) \cup \{A, B\}$. *Then* $(\mathcal{P}(E), \bullet)$ is a *commutative extensive join hypergroup and the Levine's decomposition* $\overline{\mathcal{P}(E)}$ *is h-generating* on $(\mathcal{P}(E), \bullet)$.

Let $f : X \to Y$ be a mapping. We denote by $f_+ : \mathcal{P}(X) \to \mathcal{P}(Y)$ its lifing into power sets, i.e. we define $f_+(A) = f(A) = \{f(a); a \in A\}$ for any non-empty set $A \in \mathcal{P}(X)$ and $f_+(\emptyset) = \emptyset$ Then we have

Theorem 4. *Let* $\boldsymbol{E}_i = (E_i, G_i, \alpha_i)$, $i = 1, 2$, *be algebraic spaces with operators,* $f : E_1 \to E_2$. *be a mapping preserving* $\mathcal{CS}$ *- systems of spaces* $\boldsymbol{E}_i$, i.e. $X \in \mathcal{CS}(\boldsymbol{E}_1)$ *implies* $f(X) \in \mathcal{CS}(\boldsymbol{E}_2)$. *Then* f_+ *is a homomorphism of the hypergroup* $(\mathcal{P}(\boldsymbol{E}_1), \bullet)$ *into the hypergroup* $(\mathcal{P}(\boldsymbol{E}_2), \bullet)$. *If moreover the mapping* f *is surjective and reflects* $\mathcal{CS}$*- systems, ie.* $Y \in \mathcal{CS}(\boldsymbol{E}_2)$ *implies* $f^-(Y) \in \mathcal{CS}(\boldsymbol{E}_1)$ *(where* $f^-(Y)$ *is the preimage of the set* Y*) we have*

$$f_+ : (\mathcal{P}(\boldsymbol{E}_1), \bullet) \to (\mathcal{P}(\boldsymbol{E}_2), \bullet)$$

is a deformation, i. e. a good homomorphism of hypergroups and determines a homomorphism f_{++} *of corresponding factor hypergroups*

$$f_{++} = (\overline{\mathcal{P}(\boldsymbol{E}_1)}, \bullet) \to (\overline{\mathcal{P}(\boldsymbol{E}_2)}.$$

Remark. The closure operations $\mathcal{S}_+, \mathcal{S}_- : \mathcal{P}(E) \to \mathcal{P}(E)$ determine a quasidiscrete or Alexandroff discrete topologies on the state set E of the algebraic space **E**, thus some of the above constructions can be expressed in terms of the topological spaces theory with the use of their special morphisms. Language of the decomposition theory is in certain sense parallel to algebra of equivalence relations, however the first approach is useful in the context with coverings of spaces and with non-associative hyperstructures which are determined by the mentioned coverings of sets.

There are many papers devoted to hyperstructures - hypergroups and some of their generalizations in connection with automata and multiautomata. We mention at least papers [6,7,8,9,10] and [12, 13, 14, 15, 16, 17] from references of this contribution. The mentioned papers contain investigation of transposition hypergroups and application of these multistructures for the constructing of actions and multiactions in connection with some other mathematical concepts.

5 Conclusion

Considering the class of all quasiautomata (algebraic spaces) with pointed monoids as input alphabets (i.e. monoids with distinguished elements) we can construct multiautomata in such a way that input alphabets are centralizers of distinguished elements within the given monoids. Hyperoperations on mentioned alphabets are defined by products of elements using powers of distinguished elements. Then we obtain a class of multiautomata, where the mentioned construction - described exactly e. g. in paper [10], page 5 - is functorial, which means that it preserves homomorphisms; more precisely homomorphisms of quasiautomata (of algebraic spaces with input monoids) turn out into good homomorphisms of multiautomata. It is to be noted that multiautomata are serving as suitable tools for modelling of various processes concernig important mathematical objects and structures.

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