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A Proof of Descartes' Rule of Signs

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Abstract

In 1637 Descartes, in his famous Géométrie, gave the rule of the signs without a proof. Later many different proofs appeared of algebraic and analytic nature. Among them in 1828 the algebraic proof of Gauss. In this note, we present a proof of Descartes' rule of signs that use the roots of the first derivative of a polynomial and that can be presented to the students of the last year of a secondary school.

Keywords: roots of a polynomial; derivative of a polynomial.
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1 Introduction

Descartes' Rule of signs first appeared in 1637 in Descartes' famous *Géométrie* [1], where also analytic geometry was given for the first time. Descartes gave the rule without a proof. Later several discussions appear trying to understand which one was the first proof of the Rule. It seems that a first proof of the Rule was given in Segner's degree thesis in 1728 and it is contained in a letter that Segner sent to Hamberger [3]. In 1828 Gauss [2] gave a purely algebraic and very simple proof. Many other proofs, both

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algebraic and analytic in nature have been given later. One of the possible statements giving Descartes' Rule of signs is the following:

Theorem 1.1 *If a polynomial with real coefficients in one unknown has all of its roots being real number, then the number of positive roots, counted with their multiplicity, equals the number of variations of signs among the ordered sequence of his coefficients.*

A more general statement covers the case when one does not know if all roots are real and it is given in the following:

Theorem 1.2 *The number of variations of sign is the maximum number of positive roots of a polynomial with real coefficients. The number of positive roots equals either the maximum or the maximum minus an even number.*

The previous theorems do not give results on the number of negative roots. The negative roots of $p(x)$ are in number equal to the number of positive roots of $p(-x)$ and hence in order to count the number of negative roots of $p(x)$ one can count the number of positive roots of $p(-x)$ by applying Descartes' Rule of signs.

Let $p(x)$ be a polynomial whose monomials are given either in increasing or in decreasing order. Consider the sequence of its coefficients in the same order. One says that there is a "change of sign" if two consecutive terms have opposite sign.

For example if $p(x)=x^6-3x^5+4x^3+x^2-5x+9$, then the sequence of its coefficients is:

$1,-3,4,1,-5,9$ and the number of variations is 4. Hence the number of positive roots of $p(x)$ is either 4 or 2 or 0.

Observe that if the number of variations is even, then the Rule cannot say that the polynomial has a positive root. If the number of variations is odd, the Rule says that there is at least a positive root.

2 The Proof

2.1 The Derivative of a Polynomial

We first start with some results giving information between the roots of a polynomial $p(x)$ and the roots of its derivative $p'(x)$.

Lemma 2.1 *A roots of $p(x)$ is also a root of $p'(x)$ with multiplicity one less.*

Proof. Let $p(x) = (x - a)^k q(x)$ with $(x - a)$ that does not divide $q(x)$. Hence $q(a) \neq 0$. Then k is the multiplicity of a as root of $p(x)$. It is

$$p'(x) = k(x - a)^{k-1} q(x) + (x - a)^k q'(x) =$$

$$= (x - a)^{k-1}[kq(x) + (x - a)q'(x)]$$

$$= (x - a)^{k-1} F(x).$$

Since $F(a) = kq(a) \neq 0$, $(x - a)$ does not divide the polynomial $F(x)$. Hence $k - 1$ is the multiplicity of a as root of $p'(x)$.

Next results follows from Rolle's theorem.

Lemma 2.2 *If all roots of a polynomial $p(x)$ are real numbers, then also all roots of $p'(x)$ are real numbers. Moreover between to consecutive roots of $p(x)$ there is a simple (multiplicity 1) root of $p'(x)$.*

Proof. Let $x_1 < x_2 < \cdots\cdots < x_k$ be the roots of $p(x)$ with multiplicity $m_1, m_2, \ldots\ldots\ldots, m_k$, respectively. Since all roots are real numbers we have that

$$m_1 + m_2 + \cdots\cdots + m_k = n = \deg(p(x)).$$

From the previous lemma $p'(x)$ has roots $x_1 < x_2 < \cdots\cdots < x_k$ with multiplicity $m_1 - 1, m_2 - 1, \ldots\ldots\ldots, m_k - 1$. Moreover, from Rolle's theorem between two real roots of $p(x)$ there is at least a real root of $p'(x)$. Hence $p'(x)$ has at least other $k - 1$ real roots. From

$$(m_1 - 1) + (m_2 - 1) + \cdots + (m_k - 1) + k - 1 = n - 1 = \deg(p'(x)),$$

it follows that $p'(x)$ cannot have other roots. The assertion follows.

Lemma 2.3 *If all roots of a polynomial* $p(x)$ *are real numbers and* k *of them are positive numbers, then* $p'(x)$ *has either* k *or* $k-1$ *positive roots.*

Proof. Let $x_1 < x_2 < \cdots\cdots < x_s$ be the positive roots of $p(x)$ with multiplicity $m_1, m_2, \dots\dots\dots, m_s$, respectively. From the hypothesis we have

$$m_1 + m_2 + \cdots\cdots + m_s = k.$$

The derivate $p'(x)$ will have as positive roots $x_1 < x_2 < \cdots\cdots < x_s$ with multiplicity

$$m_1 - 1, m_2 - 1, \dots\dots\dots, m_s - 1,$$

the simple roots $y_1, y_2, \dots\dots\dots, y_{s-1}$ between consecutive positive roots and, possibly, another simple root y_0 between the maximum negative root and x_1.

So the total number of positive roots of $p'(x)$ is either

$$(m_1 - 1) + (m_2 - 1) + \cdots\cdots + (m_s - 1) + s - 1 = k - 1$$

if y_0 is not a positive number or

$$(m_1 - 1) + (m_2 - 1) + \cdots\cdots + (m_s - 1) + s - 1 + 1 = k$$

if y_0 a positive number.

2.2 Proof of Theorem 1.1

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_0$ be a degree n polynomial. Hence $a_n \neq 0$. We may assume that $a_n > 0$. In what follows we assume that all roots of $p(x)$ are real numbers.

Lemma 2.4 *If* $p(x)$ *has* k *positive roots, then the sign of the last non zero coefficient of* $p(x)$ *is* $(-1)^k$.

Proof. Let a_h be the last non zero coefficient. Since all roots of $p(x)$ are real numbers we can factorize the polynomial as

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_h x^h =$$

$$= a_n x^h (x - x_1) \cdots\cdots (x - x_k)(x - x_{k+1}) \cdots\cdots (x - x_{n-h})$$

where $x_1, \dots\dots\dots, x_k$ are the positive roots and $x_{k+1}, \dots\dots\dots, x_{n-k}$ are the negative roots.

It follows that $a_h = a_n \cdot (-1)^k \cdot x_1 x_2 \cdots\cdot x_k \cdot (-x_{k+1}) \cdots (-x_{n-h}\ )$ and since all numbers are positive the sign is $(-1)^k$.

We will now give the proof of the Theorem 1.1 by induction on $n = \deg(p(x))$.

If $n = 1$, the assertion holds. Indeed $p(x) = a_1 x + a_0$ has a unique root $x_1 = -a_0/a_1$. It is a positive root if and only if a_1 and a_0 have opposite sign, that is there is a variation.

Suppose the assertion holds for all polynomials of degree $n-1$ with all real roots. Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots\cdots + a_0$$

be a polynomial of degree n.

If $a_0 = 0$, then $p(x) = xq(x)$ and the polynomials $p(x)$ and $q(x)$ have the same number of positive roots and the same number of variations of sign. Since $\deg\big(q(x)\big) = n-1$ and the assertion holds for $q(x)$, then it also holds for $p(x)$.

If $a_0 \neq 0$, then

$$p'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots\cdots\cdots + a_1.$$

The last non zero coefficient in $p'(x)$ is the non zero coefficient consecutive to a_0 in $p(x)$. If the sign of a_0 and the last non zero coefficient in $p'(x)$ coincide, then $p(x)$ and $p'(x)$ have the same number of variations of sign, otherwise $p'(x)$ has one less variations of sign compared with $p(x)$. Since the sign of the last non zero coefficient determines the parity of the number of positive roots (Lemma 2.4) in the first case the parity of the number of roots of $p(x)$ and $p'(x)$ is the same in the second case it is different.

On the other hand from Lemma 2.3 the number of roots of $p(x)$ and $p'(x)$ is different for at most 1. Hence either $p(x)$ and $p'(x)$ have the same number of positive roots or this number is different for 1.

From the inductive hypothesis $p'(x)$ has the same number of positive roots as the number of variations of sign. Since from $p(x)$ to $p'(x)$ the number of positive roots and the number of variations of sign either remains the same for both or it is 1 more for both, the assertion follows also for $p(x)$.

3 Conclusions

The proof of Descartes rule of signs is a good example of math reasoning and it should be taught to the students of last year of secondary schools. Contrary to this in many schools it is given the Rule without a proof. In particular it is a good example for understanding the relation between the roots of a polynomials and its first derivative. It also uses Rolle's theorem, that is one of the most important result shown to the students of last year of secondary schools. Moreover Descartes' rule of signs is one of the Math results that puts together analysis and algebra and it doesn't happen so often in curricula of secondary school. In Math, except for axioms, everything should be demonstrated.

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