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The Sum of the Series of Reciprocals of the Quadratic Polynomials with Complex Conjugate Roots

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Abstract

This contribution is a follow-up to author's papers [1], [2], [3], [4], [5], [6], [7], and in particular [8] dealing with the sums of the series of reciprocals of quadratic polynomials with different positive integer roots, with double non-positive integer root, with different negative integer roots, with double positive integer root, with one negative and one positive integer root, with purely imaginary conjugate roots, with integer roots, and with the sum of the finite series of reciprocals of the quadratic polynomials with integer purely imaginary conjugate roots respectively. We deal with the sum of the series of reciprocals of the quadratic polynomials with complex conjugate roots, derive the formula for the sum of these series and verify it by some examples evaluated using the basic programming language of the computer algebra system Maple 16. This contribution can be an inspiration for teachers of mathematics who are teaching the topic Infinite series or as a subject matter for work with talented students.

Keywords: sum of the series, harmonic number, imaginary conjugate roots, hyperbolic cotangent, computer algebra system Maple.
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1 Introduction

Let us recall the basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \ .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum\limits_{k=1}^{\infty} a_k$ is defined for each n as the sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \,.$$

The series $\sum\limits_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim\limits_{n\to\infty} s_n = s$. We say that the series $\sum\limits_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum\limits_{k=1}^{\infty} a_k = s$.

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum of the reciprocals of the square numbers (the *Basel problem*) is $\pi^2/6$:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \doteq 1.644934 \,.$$

The *nth harmonic number* is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \,,$$

H_0 being defined as 0. The *generalized harmonic numbers* of order n in power r is the sum

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r} \,,$$

where $H_{n,1} = H_n$ are harmonic numbers.

Generalized harmonic number of order n in power 2 can be written as a function of harmonic numbers using formula (see [9])

$$H_{n,2} = \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n} \,.$$

From formulas for $H_{n,r}$, where $r = 1, 2$ and $n = 1, 2, \ldots, 9$, we get the following table:

n	1	2	3	4	5	6	7	8	9
H_n	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$
$H_{n,2}$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	$\frac{1077749}{705600}$	$\frac{771817}{352800}$

Table 1: Nine first harmonic numbers H_n and generalized harmonic numbers $H_{n,2}$

The *hyperbolic cotangent* is defined as a ratio of hyperbolic cosine and hyperbolic sine

$$\coth x = \frac{\cosh x}{\sinh x}, \qquad x \neq 0.$$

Because hyperbolic cosine and hyperbolic sine can be defined in terms of the exponential function

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} + 1}{2\,\mathrm{e}^x}, \qquad \sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} - 1}{2\,\mathrm{e}^x},$$

we get

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} = \frac{\mathrm{e}^{2x} + 1}{\mathrm{e}^{2x} - 1}, \qquad x \neq 0.$$

2 The sum of the series of reciprocals of the quadratic polynomials with integer roots

As regards the sum of the series of reciprocals of the quadratic polynomials with different positive integer roots a and b, $a < b$, i.e. of the series

$$\sum_{\substack{k=1 \\ k \neq a,b}}^{\infty} \frac{1}{(k-a)(k-b)},$$

in the paper [1] it was derived that the sum $s(a,b)^{++}$ is given by the following formula using the nth harmonic numbers H_n

$$s(a,b)^{++} = \frac{1}{b-a}\left(H_{a-1} - H_{b-1} + 2H_{b-a} - 2H_{b-a-1}\right). \tag{1}$$

In the paper [2] it was shown that the sum $s(a,b)^{--}$ of the series of reciprocals of the quadratic polynomials with different negative integer roots a and b, $a < b$, i.e. of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)},$$

is given by the simple formula

$$s(a,b)^{--} = \frac{1}{b-a}\left(H_{-a} - H_{-b}\right). \tag{2}$$

The sum of the series

$$\sum_{\substack{k=1 \\ k\neq b}}^{\infty} \frac{1}{(k-a)(k-b)},$$

of reciprocals of the quadratic polynomials with integer roots $a < 0$, $b > 0$ was derived in the paper [4]. This sum $s(a,b)^{-+}$ is given by the formula

$$s(a,b)^{-+} = \frac{(b-a)\left(H_{-a} - H_{b-1}\right) + 1}{(b-a)^2}. \tag{3}$$

In the paper [3] it was derived that the sum $s(a,a)^{--}$ of the series of reciprocals of the quadratic polynomials with double non-positive integer root a, i.e. of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^2},$$

is given by the following formula using the generalized harmonic number $H_{-a,2}$ of order $-a$ in power 2

$$s(a,a)^{--} = \frac{\pi^2}{6} - H_{-a,2}. \tag{4}$$

The sum of the series

$$\sum_{\substack{k=1 \\ k\neq a}}^{\infty} \frac{1}{(k-a)^2},$$

of reciprocals of the quadratic polynomials with double positive integer root a, was derived in the paper [5]. This sum $s(a,a)^{++}$ is given by the formula with the generalized harmonic number in power 2

$$s(a,a)^{++} = \frac{\pi}{2} + H_{a-1,2}. \tag{5}$$

The formula for the sum $s(a,0)^{-0}$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k-a)}$$

of reciprocals of the quadratic polynomials with one zero and one negative integer root a and also the formula for the sum $s(0,b)^{0+}$ of the series

$$\sum_{\substack{k=1\\k\neq b}}^{\infty} \frac{1}{k(k-b)}$$

of reciprocals of the quadratic polynomials with one zero and one positive integer root b were derived in the paper [7]. These sums are given by the simple formulas

$$s(a,0)^{-0} = \frac{H_{-a}}{-a}\,, \tag{6}$$

$$s(0,b)^{0+} = \frac{1-bH_{b-1}}{b^2}\,. \tag{7}$$

3 The sum of the series of reciprocals of the quadratic polynomials with complex conjugate roots

We deal with the problem to determine the sum $S(a,b)$, where a,b are non-zero integers, of the infinite series of reciprocals of the quadratic polynomials with complex conjugate roots $a \pm b\mathrm{i}$, i.e. the series of the form

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2}\,. \tag{8}$$

The quadratic trinomial $(k-a)^2+b^2 = k^2-2ak+a^2+b^2$ can be in the complex domain written in the product form $\big[k-(a+b\mathrm{i})\big]\cdot\big[k-(a-b\mathrm{i})\big]$, so the quadratic trinomial $k^2-2ak+a^2+b^2$ has complex conjugate roots $k_1 = a+b\mathrm{i}$, $k_2 = a-b\mathrm{i}$.

The series (8) is convergent because $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2} \leq \sum_{k=1}^{\infty} \frac{1}{(k-a)^2}$. For non-positive integer a we get by (4) an equality $S(a,b) < \pi^2/6 \doteq 1.6449$ and for positive integer a we have by (5) $S(a,b) < \pi/2 + \pi^2/6 \doteq 3.2157$ (see [5]).

Because it obviously does not matter the sign of an imaginary part b, let us deal further with two cases of the series (8) – with a positive real part a and with a negative one. If the integer real part $a > 0$, then the sum $S(a,b)$ has the form

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2} &= \sum_{k=1}^{\infty} \frac{1}{(a-k)^2+b^2} = \\
&= \frac{1}{(a-1)^2+b^2} + \frac{1}{(a-2)^2+b^2} + \cdots + \frac{1}{1^2+b^2} + \\
&+ \frac{1}{b^2} + \frac{1}{1^2+b^2} + \frac{1}{2^2+b^2} + \frac{1}{3^2+b^2} + \cdots = s(a-1,b) + \frac{1}{b^2} + s(b)\,,
\end{aligned}$$

where

$$s(b) = \sum_{k=1}^{\infty} \frac{1}{k^2 + b^2}$$

is the sum that was derived in the paper [6] and which is given by the formula

$$s(b) = \frac{\pi}{2b} \cdot \frac{\mathrm{e}^{2\pi b} + 1}{\mathrm{e}^{2\pi b} - 1} - \frac{1}{2b^2} = \frac{\pi}{2b} \coth \pi b - \frac{1}{2b^2} \tag{9}$$

and where

$$s(a-1, b) = \sum_{k=1}^{a-1} \frac{1}{k^2 + b^2} \tag{10}$$

is the sum of the finite series, which was in the paper [8] derived by means of the trapezoidal rule and which is given by the approximate formula

$$s(a-1, b) \doteq \frac{1}{b} \arctan \frac{a}{b} - \frac{1}{2} \left(\frac{1}{b^2} + \frac{1}{a^2 + b^2} \right). \tag{11}$$

In this paper it was shown that this approximate formula is a suitable approximation of the sum $s(a-1, b)$, because one hundred results obtained by means of this formula when modelling in Maple 16 have very small relative errors (in the range of 0.60% to 0.05%). In total, we get

$$\begin{aligned} S(a, b) &= s(a-1, b) + \frac{1}{b^2} + s(b) \doteq \\ &\doteq \frac{1}{b} \arctan \frac{a}{b} - \frac{1}{2} \left(\frac{1}{b^2} + \frac{1}{a^2 + b^2} \right) + \frac{1}{b^2} + \frac{\pi}{2b} \coth \pi b - \frac{1}{2b^2}, \end{aligned}$$

so after simple arrangement we have the following result:

$$S(a, b) \doteq \frac{1}{b} \arctan \frac{a}{b} - \frac{1}{2(a^2 + b^2)} + \frac{\pi}{2b} \coth \pi b, \qquad a > 0.$$

If the integer real part $a < 0$, then the sum $S(a, b)$ has for $A = -a > 0$ the form

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{(k-a)^2 + b^2} = \sum_{k=1}^{\infty} \frac{1}{(k+A)^2 + b^2} = \\ &= \frac{1}{(1+A)^2 + b^2} + \frac{1}{(2+A)^2 + b^2} + \frac{1}{(3+A)^2 + b^2} + \cdots = \\ &= \frac{1}{1^2 + b^2} + \frac{1}{2^2 + b^2} + \cdots + \frac{1}{A^2 + b^2} + \frac{1}{(1+A)^2 + b^2} + \frac{1}{(2+A)^2 + b^2} + \cdots \\ &\cdots - \left(\frac{1}{1^2 + b^2} + \frac{1}{2^2 + b^2} + \cdots + \frac{1}{(A-1)^2 + b^2} \right) - \frac{1}{A^2 + b^2} = \\ &\qquad = s(b) - s(A-1, b) - \frac{1}{A^2 + b^2} = s(b) - s(-a-1, b) - \frac{1}{a^2 + b^2}. \end{aligned}$$

So we have

$$S(a,b) = s(b) - s(-a-1,b) - \frac{1}{a^2+b^2} \doteq$$
$$\doteq \frac{\pi}{2b}\coth \pi b - \frac{1}{2b^2} - \left[\frac{1}{b}\arctan\frac{-a}{b} - \frac{1}{2}\left(\frac{1}{b^2} + \frac{1}{a^2+b^2}\right)\right] - \frac{1}{a^2+b^2}$$

and after simple arrangement we get the same result as above for $a > 0$:

$$S(a,b) \doteq \frac{1}{b}\arctan\frac{a}{b} - \frac{1}{2(a^2+b^2)} + \frac{\pi}{2b}\coth \pi b, \qquad a < 0.$$

Therefore for every integer a including zero we get the main result

$$S(a,b) \doteq \frac{1}{b}\arctan\frac{a}{b} - \frac{1}{2(a^2+b^2)} + \frac{\pi}{2b}\coth \pi b. \tag{12}$$

4 Numerical verification

We solve the problem to determine the values of the sum $S(a,b)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)^2+b^2}$$

for $a = -5, -4, \ldots, 5$ and for $b = 1, 2, \ldots, 10$. We use on the one hand an approximative direct evaluation of the sum

$$s(a,b,t) = \sum_{k=1}^{t} \frac{1}{(k-a)^2+b^2},$$

where $t = 10^5$, using the basic programming language of Maple 16, and on the other hand the formula (12) for evaluation the sum $S(a,b)$. We compare 110 pairs of these two ways obtained sums $s(a,b,10^5)$ and $S(a,b)$ to verify the formula (12). We use following procedure `sumsab` and succeeding for-loop statement:

```
sumsab=proc(a,b,t)
   local k,s,S; s:=0;
   for k from 1 to t do
       s:=s+1/((k-a)*(k-a)+b*b);
   end do;
   print("s(",a,b,t,")=",evalf[6](s);
   S:=evalf[6]((1/b)*arctan(a/b)-1/(2*(a*a+b*b))
       +(Pi/(2*b))*coth(Pi*b));
   print("S(",a,b,")=",evalf[6](S));
   print("relerr(S)=",evalf[10](100*abs(s-S)/s),"%");
end proc:
```

```
for a from -5 to 5 do
    for b from 1 to 10 do
        sumsab(100000,a,b);
    end do;
end do;
```

Forty of these one hundred and ten approximative values of the sums $s(a, b, 10^5)$ and $S(a, b)$ rounded to four decimals obtained by these procedure and the relative quantification accuracies

$$r(a, b) = \frac{|s(a, b, 10^5) - S(a, b)|}{s(a, b, 10^5)}$$

of the sums $s(a, b, 10^5)$ (expressed as a percentage) are written into Table 2 below. Let us note that the computation of 110 values $s(a, b, 10^5)$ (abbreviated in Table 2 as $s(a, b)$) and $S(a, b)$ took over 5 hours 24 minutes. The relative quantification accuracies are approximately between 6.14% and 0.0006%, 96 of these 110 approximative values have the relative quantification accuracy smaller than 0.5%.

$s \mid S \mid r$	$a = -3$	$a = -2$	$a = -1$	$a = 0$	$a = 1$	$a = 2$	$a = 3$	$a = 4$
$s(a, 1)$	0.2766	0.3767	0.5767	1.0767	2.0767	2.5767	2.7767	2.8767
$S(a, 1)$	0.2776	0.3695	0.5413	1.0767	2.1121	2.5838	2.7757	2.8731
$r(a, 1)$	0.34	1.90	6.14	0.0006	1.70	0.28	0.04	0.12
$s(a, 2)$	0.2585	0.3354	0.4604	0.6604	0.9104	1.1104	1.2354	1.3123
$S(a, 2)$	0.2555	0.3302	0.4536	0.6604	0.9172	1.1156	1.2383	1.3140
$r(a, 2)$	1.13	1.55	1.48	0.002	0.75	0.47	0.24	0.13
$s(a, 3)$	0.2356	0.2911	0.3680	0.4680	0.5791	0.6791	0.7561	0.8116
$S(a, 3)$	0.2340	0.2891	0.3663	0.4680	0.5808	0.6811	0.7576	0.8127
$r(a, 3)$	0.65	0.38	0.46	0.002	0.29	0.29	0.21	0.13
$s(a, 4)$	0.2126	0.2526	0.3026	0.3614	0.4239	0.4828	0.5328	0.5728
$S(a, 4)$	0.2118	0.2518	0.3020	0.3614	0.4245	0.4836	0.5335	0.5734
$r(a, 4)$	0.37	0.33	0.19	0.003	0.14	0.18	0.15	0.12
$s(a, 5)$	0.1918	0.2212	0.2557	0.2941	0.3341	0.3726	0.4071	0.4365
$S(a, 5)$	0.1914	0.2208	0.2554	0.2942	0.3344	0.3730	0.4075	0.4369
$r(a, 5)$	0.22	0.18	0.09	0.003	0.08	0.11	0.11	0.09

Table 2: Some approximate values of the sums $s(a, b, 10^5)$, $S(a, b)$ and the relative quantification accuracies $r(a, b)$ of the sums $s(a, b, 10^5)$ for some values of a and b

5 Conclusions

We dealt with the problem to determine the sum $S(a,b)$, where a,b are non-zero integers, of the series

$$\sum_{k=1}^{\infty}\frac{1}{(k-a)^2+b^2}$$

of reciprocals of the quadratic polynomials with complex conjugate roots $a \pm b\mathrm{i}$.

We derived that the sum $S(a,b)$ is given by the approximate formula

$$S(a,b) \doteq \frac{1}{b}\arctan\frac{a}{b} - \frac{1}{2(a^2+b^2)} + \frac{\pi}{2b}\coth \pi b.$$

We verified this result by computing 110 various sums by using the computer algebra system Maple 16. This result also includes a special case, when $b=a$. In this case we get the approximate formula

$$S(a,a) \doteq \frac{\pi}{4a} - \frac{1}{4a^2} + \frac{\pi}{2a}\coth \pi a.$$

Because for integer $a \geq 1$ it holds $\coth \pi a \to 1$ (e.g. $\coth \pi \doteq 1.004$, $\coth 2\pi \doteq 1.000007$, $\coth 3\pi \doteq 1.00000001$), we have the simple aproximate formula

$$S(a,a) \doteq \frac{3\pi a - 1}{4a^2}.$$

Let us note that this consequence of the main result corresponds to the numeric values in Table 2: $S(1,1) \doteq (3\pi-1)/4 \doteq 2.1062$, $S(2,2) \doteq (6\pi-1)/16 \doteq 1.1156$, $S(3,3) \doteq (9\pi-1)/36 \doteq 0.7576$, $S(4,4) \doteq (12\pi-1)/64 \doteq 0.5734$.

The series of the quadratic polynomials with complex conjugate roots $a \pm b\mathrm{i}$ so belong to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically and also presented by means of a simple numerical expression.

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