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Fuzzy hyperalgebras and direct product

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Abstract

We introduce and study the direct product of a family of fuzzy hyperalgebras of the same type and present some properties of it.

Key words: Fuzzy hyperalgebras, Term function, Direct product.

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1 Introduction

In this section we present some definitions and simple properties of hyperalgebras which will be used in the next section. In the sequel H is a fixed nonvoid set, $P^*(H)$ is the family of all nonvoid subsets of H, and for a positive integer n we denote for H^n the set of n-tuples over H (for more see [1]).

Recall that for a positive integer n a n-ary *hyperoperation* β on H is a function $\beta : H^n \rightarrow P^*(H)$. We say that n is the *arity* of β. A subset S of H is *closed* under the n-ary hyperoperation β if $(x_1, \dots, x_n) \in S^n$ implies that $\beta(x_1, \dots, x_n) \subseteq S$. A *nullary hyperoperation* on H is just an element of $P^*(H)$; i.e. a nonvoid subset of H.
A *hyperalgebra* $\mathbb{H} = \langle H, (\beta_i, \mid i \in I) \rangle$ (which is called *hyperalgebraic system* or a *multialgebra*) is the set H with together a collection $(\beta_i, \mid i \in I)$ of hyperoperations on H.

A subset S of a hyperalgebra $\mathbb{H}$=$\langle H, (\beta_i, : i \in I)\rangle$ is a *subhyperalgebra* of $\mathbb{H}$ if S is closed under each hyperoperation β_i, for all $i \in I$, that is $\beta_i(a_1, ..., a_{n_i}) \subseteq S$, whenever $(a_1, ..., a_{n_i}) \in S^{n_i}$. The *type* of $\mathbb{H}$ is the map from I into the set $\mathbb{N}^*$ of nonnegative integers assigning to each $i \in I$ the arity of β_i. Two hyperalgebras of the same type are called *similar* hyperalgrbras.
For $n > 0$ we extend an n-ary hyperoperation β on H to an n-ary operation $\overline{\beta}$ on $P^*(H)$ by setting for all $A_1, ..., A_n \in P^*(H)$

$$\overline{\beta}(A_1, ..., A_n) = \bigcup\{\beta(a_1, ..., a_n)|a_i \in A_i(i = 1, ..., n)\}$$

It is easy to see that $\langle P^*(H), (\overline{\beta}_i : i \in I)\rangle$ is an algebra of the same type of $\mathbb{H}$.

Definition 1.1 *Let* $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ *and* $\overline{\mathbb{H}}$=$\langle \overline{H}, (\overline{\beta}_i : i \in I)\rangle$ *be two similar hyperalgebras. A map* h *from* $\mathbb{H}$ *into* $\overline{\mathbb{H}}$ *is called a*
(i) A homomorphism if for every $i \in I$ *and all* $(a_1, ..., a_{n_i}) \in H^{n_i}$ *we have that*

$$h(\beta_i((a_1, ..., a_{n_i})) \subseteq \overline{\beta}_i(h(a_1), ..., h(a_{n_i}));$$

(ii) a good homomorphism if for every $i \in I$ *and all* $(a_1, ..., a_{n_i}) \in H^{n_i}$ *we have that*

$$h(\beta_i((a_1, ..., a_{n_i})) = \overline{\beta}_i(h(a_1), ..., h(a_{n_i})).$$

Definition 1.2 *Let* H *be a nonempty set. A fuzzy subset* μ *of* H *is a function*

$$\mu : H \to [0, 1].$$

Definition 1.3 *A fuzzy* n*-ary hyperoperation* f^n *on* S *is a map* $f^n : S \times \cdots \times S \longrightarrow F^*(S)$*, which associated a nonzero fuzzy subset* $f^n(a_1, \ldots, a_n)$ *with any* n*-tuple* $(a_1, \ldots, a_n)$ *of elements of* S*. The couple* (S, f^n) *is called a fuzzy* n*-ary hypergroupoid. A fuzzy nullary hyperoperation on* S *is just an element of* $F^*(S)$*; i.e. a nonzero fuzzy subset of* S*.*

Definition 1.4 *Let* H *be a nonempty set and for every* $i \in I$*,* β_i *be a fuzzy* n_i*-ary hyperoperation on* H*, Then* $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ *is called fuzzy hyperalgebra, where* $(n_i : i \in I)$ *is type of this fuzzy hyperalgebra.*

Definition 1.5 *If* $\mu_1, \ldots, \mu_{n_i}$ *be* n_i *nonzero fuzzy subsets of a fuzzy huperalgebra* $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$*, we define for all* $t \in H$

$$\beta_i(\mu_1, \ldots, \mu_{n_i})(t) = \bigvee_{(x_1,\ldots,x_{n_i})\in H^{n_i}} (\mu_1(x_1) \bigwedge \cdots \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, \ldots, x_{n_i})(t))$$

Finally, for nonempty subsets $A_1, \ldots, A_{n_k}$ *of* H*, set* $A = A_1 \times \ldots \times A_{n_i}$*. Then for all* $t \in H$

$$\beta_k(A_1, \ldots, A_{n_k})(t) = \vee_{(a_1,\ldots,a_{n_k})\in A}(\beta_k(a_1, \ldots, a_{n_k})(t)).$$

For nonempty subset A of H, χ_A denote the characteristic function of A . Note that, if $f : H_1 \longrightarrow H_2$ is a map and $a \in H_1$, then $f(\chi_a) = \chi_{f(a)}$.

Definition 1.6 *Let $\mathbb{H} = \langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{H}' = \langle H', (\beta'_i : i \in I)\rangle$ be two fuzzy hyperalgebras with the same type, and $f : H \longrightarrow H'$ be a map. We say that f is a homomorphism of fuzzy hyperalgebras if for every $i \in I$ and every $a_1, \ldots, a_{n_i} \in H$ we have*

$$f(\beta_i(a_1, \ldots, a_{n_i})) \leq \beta'_i(f(a_1), \ldots, f(a_{n_i})).$$

Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ be a fuzzy hyperalgebra then, the set of the nonzero fuzzy subsets of H denoted by $F^*(H)$, can be organized as a universal algebra with the operations;

$$\beta_i(\mu_1, \ldots, \mu_{n_i})(t) = \bigvee_{(x_1,\ldots,x_{n_i})\in H^{n_i}} (\mu_1(x_1) \bigwedge \ldots \bigwedge \mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, \ldots, x_{n_i})(t))$$

for every $i \in I$, $\mu_1, \ldots, \mu_{n_i} \in F^*(H)$ and $t \in H$. We denote this algebra by $\mathtt{F}^*(\mathbb{H})$.

In [3] Gratzer presents the algebra of the term functions of a universal algebra. If we consider an algebra $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ we call $n-$ary term functions on $\mathbb{B}$ ($n \in \mathbb{N}$) those and only those functions from B^n into B, which can be obtained by applying (i) and (ii) from bellow for finitely many times:
(i) the functions $e^n_i : B^n \to B$, $e^n_i(x_1, \ldots, x_n) = x_i$, $i = 1, \ldots, n$ are $n-$ary term functions on $\mathbb{B}$;
(ii) if $p_1, \ldots, p_{n_i}$ are $n-$ary term functions on $\mathbb{B}$, then $\beta_i(p_1, \ldots, p_{n_i}) : B^n \to B$,
$\beta_i(p_1, \ldots, p_{n_i})(x_1, \ldots, x_n) = \beta_i(p_1(x_1, \ldots, x_n), \ldots, p_{n_i}(x_1, \ldots, x_n))$ is also a $n-$ary term function on $\mathbb{B}$.
We can observe that (ii) organize the set of $n-$ary term functions over $\mathbb{B}$ ($P^{(n)}(\mathbb{B})$) as a universal algebra, denoted by $\mathtt{B}^{(n)}(\mathbb{B})$.
If $\mathbb{H}$ is a fuzzy hyperalgebra then for any $n \in \mathbb{N}$, we can construct the algebra of $n-$ary term functions on $\mathtt{F}^*(\mathbb{H})$, denoted by $\mathtt{B}^{(n)}(\mathtt{F}^*(\mathbb{H})) = \langle P^{(n)}(F^*(\mathbb{H})), (\beta_i : i \in I)\rangle$.

2 On the Direct Product of Fuzzy Hyperalgebras

Proposition 2.1 *Let $\mathbb{H}$=$\langle H, (\beta_i : i \in I)\rangle$ and $\mathbb{B}$=$\langle B, (\beta_i : i \in I)\rangle$ are fuzzy hyperalgebras of the same type, $h : H \to B$ a fuzzy homomorphism and $p \in P^{(n)}(\mathtt{F}^*(\mathbb{H}))$. Then for all $a_1, \ldots, a_n \in H$ we have $h(p(\chi_{a_1}, \ldots, \chi_{a_n})) \subseteq p(h(\chi_{a_1}), \ldots, h(\chi_{a_n}))$.*

Proof. The prove is by induction over the steps of construction of a term.□

Remark 2.1 *If $h : H \to B$ be fuzzy good homomorphism then*
$$h(p(\chi_{a_1}, \ldots, \chi_{a_n})) = p(h(\chi_{a_1}), \ldots, h(\chi_{a_n})).$$

Remark 2.2 *We can easily construct the category of the fuzzy hyperalgebras of the same type, where the morphisms are considered to be the fuzzy homomorphisms and the composition of two morphisms is the usual mapping composition and we will denote it by* ***FHA***

Definition 2.1 *Let $q, p \in P^{(n)}(\boldsymbol{F}^*(\mathbb{H}))$. The $n-$ary (strong) identity $p = q$ is said to be satisfied on a fuzzy hyperalgebra $\mathbb{H}$ if*
$$p(\chi_{a_1}, \ldots, \chi_{a_n}) = q(\chi_{a_1}, \ldots, \chi_{a_n})$$
for all $a_1, \ldots, a_n \in H$. We can also consider that a weak identity $p \cap q \neq \varnothing$ is said to be satisfied on a fuzzy hyperalgebra $\mathbb{H}$ if
$$p(\chi_{a_1}, \ldots, \chi_{a_n}) \wedge q(\chi_{a_1}, \ldots, \chi_{a_n}) > 0$$
for all $a_1, \ldots, a_n \in H$.

Definition 2.2 *Let $((H_k, (\beta_i^k : i \in I)), k \in K)$ be an indexed family of fuzzy hyperalgebras with the same type. The direct product $\prod_{k\in K} H_k$ is a fuzzy hyperalgebra with univers $\Pi_{k\in K} H_k$ and for every $i \in I$ and $(a_k^1)_{k\in K}, \ldots, (a_k^{n_i})_{k\in K} \in \Pi_{k\in K} H_k$:*
$$\beta_i^{\prod}((a_k^1)_{k\in K}, \ldots, (a_k^{n_i})_{k\in K})(t_k)_{k\in K} = \bigwedge_{k\in K} \beta_i^k(a_k^1, \ldots, a_k^{n_i})(t_k)$$

Theorem 2.1 *The fuzzy hyperalgebra $\prod_{k\in K} H_k$ constructed this way, together with the canonical projections, is the product of the fuzzy hyperalgebras $(H_k, k \in K)$ in the category* ***FHA****.*

Proof. For any fuzzy hyperalgebra $(B, (\beta_i^B : i \in I))$ and for any family of fuzzy hyperalgebra homomorphisms $(\alpha_k : B \to H_k | k \in K)$ there is only one homomorphism $\alpha : B \to \Pi_{k\in K} H_k$ such that $\alpha_k = \pi_k^K \circ \alpha$ for any $k \in K$. Indeed, there exists only one mapping α such that the diagram is commutative.

$$\begin{tikzcd}
\Pi_{k\in K} H_k \arrow[r, "\pi_k^K"] & H_k \\
B \arrow[u, "\alpha"] \arrow[ur, "\alpha_k"'] &
\end{tikzcd}$$

This mapping is defined by $\alpha(b) = (\alpha_k(b))_{k\in K}$. Now we have to do is to verify that α is fuzzy hyperalgebra homomorphism. If we consider $i \in I$ and $b_1, \ldots, b_{n_i} \in B$, $(t_k)_{k\in K} \in \Pi_{k\in K} H_k$ then if $r \in \alpha^{-1}((t_k)_{k\in K})$ we have $\alpha(r) = (t_k)_{k\in K}$ and $\alpha(r) = (\alpha_k(r))_{k\in K}$, hence $\forall k \in K; t_k = \alpha_k(r)$, it means that $\forall k \in K; r \in \alpha_k^{-1}(t_k)$, therefore $\forall k \in K; \alpha^{-1}((t_k)_{k\in K}) \subseteq \alpha_k^{-1}(t_k)$. We have

$$\alpha(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)_{k\in K} = \bigvee_{r\in\alpha^{-1}((t_k)_{k\in K})} (\beta_i^B(b_1, \ldots, b_{n_i}))(r)$$

$$\leq \bigvee_{s\in\alpha_k^{-1}(t_k))} \beta_i^B(b_1, \ldots, b_{n_i}))(s) = \alpha_k(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)$$

then

$$\alpha(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)_{k\in K} \leq \bigwedge_{k\in K} \alpha_k(\beta_i^B(b_1, \ldots, b_{n_i}))(t_k)$$

$$\leq \bigwedge_{k\in K} \beta_i^k(\alpha_k(b_1), \ldots, \alpha_k(b_{n_i}))(t_k) = \beta_i^{\Pi}(\alpha(b_1), \ldots, \alpha(b_{n_i}))(t_k)_{k\in K}.$$

Which finishes the proof.□

Proposition 2.2 *For every $n \in \mathbb{N}$, $p \in P^{(n)}(\mathcal{F}^*(\mathbb{H}))$ and $(a_k^1)_{k\in K}, \ldots, (a_k^n)_{k\in K}$, we have*

$$p(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K} = \bigwedge_{k\in K} p(\chi_{a_k^1}, \ldots, \chi_{a_k^n})(t_k)$$

Proof. We will use the steps of construction of a term.

i. If $p = e_n^j (j = 1, 2, \ldots, n)$ then

$$\begin{aligned} p(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K} &= e_n^j(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K} \\ &= \chi_{(a_k^j)_{k\in K}}(t_k)_{k\in K} \\ &= \bigwedge_{k\in K} e_n^j(\chi_{a_k^1}, \ldots, \chi_{a_k^n})(t_k) \\ &= \bigwedge_{k\in K} p(\chi_{a_k^1}, \ldots, \chi_{a_k^n})(t_k) \end{aligned}$$

ii. Suppose that the statement has been proved for $p_1, \ldots, p_{n_i}$ and that $p = \beta_i(p_1, \ldots, p_{n_i})$. Then we have

$p(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K} = \beta_i(p_1, \ldots, p_{n_i})(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(t_k)_{k\in K}$

$= \beta_i(p_1(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}}), \ldots, p_{n_i}(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}}))(t_k)_{k\in K}$

$= \bigvee_{(s_k^1)_{k\in K}, \ldots, (s_k^{n_i})_{k\in K}} [p_1(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})(s_k^1)_{k\in K} \wedge \ldots \wedge p_{n_i}(\chi_{(a_k^1)_{k\in K}}, \ldots, \chi_{(a_k^n)_{k\in K}})$

$(s_k^{n_i})_{k\in K} \wedge \beta_i((s_k^1)_{k\in K}, \ldots, (s_k^{n_i})_{k\in K})(t_k)_{k\in K}]$

$$= \bigvee_{(s_k^1)_{k\in K},\ldots,(s_k^{n_i})_{k\in K}} [\bigwedge_{k\in K} p_1(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^1) \wedge \ldots \wedge \bigwedge_{k\in K} p_{n_i}(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^{n_i}) \wedge$$
$$\bigwedge_{k\in K} \beta_i(s_k^1,\ldots,s_k^{n_i})(t_k)]$$
$$= \bigwedge_{k\in K} [\bigvee_{(s_k^1)_{k\in K},\ldots,(s_k^{n_i})_{k\in K}} p_1(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^1) \wedge \ldots \wedge p_{n_i}(\chi_{a_k^1},\ldots,\chi_{a_k^n})(s_k^{n_i}) \wedge \beta_i(s_k^1,\ldots,s_k^{n_i})(t_k)]$$
$$= \bigwedge_{k\in K} \beta_i(p_1(\chi_{a_k^1},\ldots,\chi_{a_k^n}),\ldots,p_{n_i}(\chi_{a_k^1},\ldots,\chi_{a_k^n}))(t_k)$$
$$= \bigwedge_{k\in K} \beta_i(p_1,\ldots,p_{n_i})(\chi_{a_k^1},\ldots,\chi_{a_k^n})(t_k)$$
$$= \bigwedge_{k\in K} p(\chi_{a_k^1},\ldots,\chi_{a_k^n})(t_k).$$

which finishes the proof of the proposition.□

Theorem 2.2 *If $((H_k,(\beta_i^k : i\in I)), k\in K)$ be an indexed family of fuzzy hyperalgebras with the same type I such that $p\cap q\neq\varnothing$ is satisfied on each fuzzy hyperalgebra H_k, then is also satisfied on the fuzzy hyperalgebra $\prod_{k\in K} H_k$.*

Proof. Let $p,q\in P^{(n)}(\mathtt{F}^*(\mathbb{H}))$ and suppose that $p\cap q\neq\varnothing$ is satisfied on each fuzzy hyperalgebra H_k. This means that for all $k\in K$ and for any $a_k^1,\ldots,a_k^n\in H_k$ we have $p(\chi_{a_k^1},\ldots,\chi_{a_k^n})\wedge q(\chi_{a_k^1},\ldots,\chi_{a_k^n})>0$. By proposition 3.7 , we conclude that

$$\begin{aligned} p(\chi_{(a_k^1)_{k\in K}},\ldots,\chi_{(a_k^n)_{k\in K}}) \wedge r(\chi_{(a_k^1)_{k\in K}},\ldots,\chi_{(a_k^n)_{k\in K}}) = \\ = \bigwedge_{k\in K} p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \wedge \bigwedge_{k\in K} q(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \\ = \bigwedge_{k\in K} (p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \wedge q(\chi_{a_k^1},\ldots,\chi_{a_k^n})) > 0 \end{aligned}$$

and the proof is finished.□

Theorem 2.3 *If $((H_k,(\beta_i^k : i\in I)), k\in K)$ be an indexed family of fuzzy hyperalgebras with the same type I such that $p=q$ is satisfied on each fuzzy hyperalgebra H_k, then $p=q$ is also satisfied on the fuzzy hyperalgebra $\prod_{k\in K} H_k$.*

Proof. Let $p,q\in P^{(n)}(\mathtt{F}^*(\mathbb{H}))$ and suppose that $p=q$ is satisfied on each fuzzy hyperalgebra H_k. This means that for all $k\in K$ and for any $a_k^1,\ldots,a_k^n\in H_k$ we have $p(\chi_{a_k^1},\ldots,\chi_{a_k^n})=q(\chi_{a_k^1},\ldots,\chi_{a_k^n})$. By proposition 3.7 , we conclude that

$$\begin{aligned} p(\chi_{(a_k^1)_{k\in K}},\ldots,\chi_{(a_k^n)_{k\in K}}) &= \bigwedge_{k\in K} p(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \\ &= \bigwedge_{k\in K} q(\chi_{a_k^1},\ldots,\chi_{a_k^n}) \end{aligned}$$

$$= r(\chi_{(a_k^1)_{k\in K}}, \dots, \chi_{(a_k^n)_{k\in K}})$$

and the proof is finished.□

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