Ratio Mathematica Volume 31, 2019, pp. 79-88

The distinguishing number and the distinguishing index of co-normal product of two graphs

Saeid Alikhani*
Samaneh Soltani †

Abstract

The distinguishing number (index) $D(G)$ ($D'(G)$) of a graph G is the least integer d such that G has an vertex labeling (edge labeling) with d labels that is preserved only by a trivial automorphism. The co-normal product $G \star H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $\{\{(x_1, x_2), (y_1, y_2)\} | x_1y_1 \in E(G) \text{ or } x_2y_2 \in E(H)\}$. In this paper we study the distinguishing number and the distinguishing index of the co-normal product of two graphs. We prove that for every $k \geq 3$, the k-th co-normal power of a connected graph G with no false twin vertex and no dominating vertex, has the distinguishing number and the distinguishing index equal two.

Keywords: distinguishing number; distinguishing index; co-normal product.
2010 AMS subject classifications: 05C15, 05C60. [1]

*Department of Mathematics, Yazd University, Yazd, Iran; alikhani@yazd.ac.ir
†Department of Mathematics, Yazd University, Yazd, Iran; s.soltani1979@gmail.com
[1]Received on February 12th, 2019. Accepted on May 3rd, 2019. Published on June 30th, 2019. doi: 10.23755/rm.v36i1.452. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction and definitions

Let $G = (V, E)$ be a simple graph of order $n \geq 2$. We use the the following notations: The set of vertices adjacent in G to a vertex of a vertex subset $W \subseteq V$ is the *open neighborhood* $N(W)$ of W. Also $N(W) \cup W$ is called a *closed neighborhood* of W and denoted by $N[W]$. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, we call H a *spanning subgraph* of G. Any spanning subgraph of G can be obtained by deleting some of the edges from G. Two distinct vertices u and v are called *true twins* if $N[v] = N[u]$ and *false twins* if $N(v) = N(u)$. Two vertices are called *twins* if they are true or false twins. The number $|N(v)|$ is called the *degree* of v in G, denoted as $\deg_G(v)$ or $\deg(v)$. A vertex having degree $|V(G)| - 1$ is called a *dominating vertex* of G. Also, $\mathrm{Aut}(G)$ denotes the automorphism group of G, and graphs with $|\mathrm{Aut}(G)| = 1$ are called *rigid* graphs.

A labeling of G, $\phi : V \to \{1, 2, \ldots, r\}$, is said to be r*-distinguishing*, if no non-trivial automorphism of G preserves all of the vertex labels. The point of the labels on the vertices is to destroy the symmetries of the graph, that is, to make the automorphism group of the labeled graph trivial. Formally, ϕ is r-distinguishing if for every non-trivial $\sigma \in \mathrm{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(\sigma(x))$. The *distinguishing number* of a graph G is defined by

$$D(G) = \min\{r | \ G \text{ has a labeling that is } r\text{-distinguishing}\}.$$

This number has defined in [1]. Similar to this definition, the *distinguishing index* $D'(G)$ of G has defined in [8] which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. If a graph has no nontrivial automorphisms, its distinguishing number is 1. In other words, $D(G) = 1$ for the asymmetric graphs. The other extreme, $D(G) = |V(G)|$, occurs if and only if G is a complete graph. The distinguishing index of some examples of graphs was exhibited in [8]. For instance, $D(P_n) = D'(P_n) = 2$ for every $n \geq 3$, and $D(C_n) = D'(C_n) = 3$ for $n = 3, 4, 5$, $D(C_n) = D'(C_n) = 2$ for $n \geq 6$, where P_n denotes a path graph on n vertices and C_n denotes a cycle graph on n vertices. A graph and its complement, always have the same automorphism group while their graph structure usually differs, hence $D(G) = D(\overline{G})$ for every simple graph G.

Product graph of two graphs G and H is a new graph having the vertex set $V(G) \times V(H)$ and the adjacency of vertices is defined under some rule using the adjacency and the nonadjacency relations of G and H. The distinguishing number and the distinguishing index of some graph products has been studied in literature (see [2, 6, 7]). The *Cartesian product* of graphs G and H is a graph, denoted by $G \Box H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$.

In 1962, Ore [10] introduced a product graph, with the name Cartesian sum of graphs. Hammack et al. [4], named it co-normal product graph. The *co-normal product* of G and H is the graph denoted by $G \star H$, and is defined as follows:

$$V(G \star H) = \{(g, h)|g \in V(G) \text{ and } h \in V(H)\},$$
$$E(G \star H) = \{\{(x_1, x_2), (y_1, y_2)\}|x_1y_1 \in E(G) \text{ or } x_2y_2 \in E(H)\}.$$

We need knowledge of the structure of the automorphism group of the Cartesian product, which was determined by Imrich [5], and independently by Miller [9].

Theorem 1.1. [5, 9] *Suppose ψ is an automorphism of a connected graph G with prime factor decomposition $G = G_1 \Box G_2 \Box \ldots \Box G_r$. Then there is a permutation π of the set $\{1, 2, \ldots, r\}$ and there are isomorphisms $\psi_i : G_{\pi(i)} \to G_i$, $i = 1, \ldots, r$, such that*

$$\psi(x_1, x_2, \ldots, x_r) = (\psi_1(x_{\pi(1)}), \psi_2(x_{\pi(2)}), \ldots, \psi_r(x_{\pi(r)})).$$

Imrich and Klavžar in [7], and Gorzkowska et.al. in [3] showed that the distinguishing number and the distinguishing index of the square and higher powers of a connected graph $G \neq K_2, K_3$ with respect to the Cartesian product is 2.

The relationship between the automorphism group of co-normal product of two non isomorphic, non rigid connected graphs with no false twin and no dominating vertex is the same as that in the case of the Cartesian product.

Theorem 1.2. [12] *For any two non isomorphic, non rigid graphs G and H,* $\mathrm{Aut}(G \star H) = \mathrm{Aut}(G) \times \mathrm{Aut}(H)$ *if and only if both G and H have no false twins and dominating vertices.*

Theorem 1.3. [12] *For any two rigid isomorphic graphs G and H,* $\mathrm{Aut}(G \star H) \cong S_2$.

Theorem 1.4. [12]*The graph $G \star H$ is rigid if and only if $G \ncong H$ and both G and H are rigid graphs.*

In the next section, we study the distinguishing number of the co-normal product of two graphs. In section 3, we show that the distinguishing index of the co-normal product of two simple connected non isomorphic, non rigid graphs with no false twin and no dominating vertex cannot be more than the distinguishing index of their Cartesian product. As a consequence, we prove that all powers of a connected graph G with no false twin and no dominating vertex distinguished by exactly two edge labels with respect to the co-normal product.

2 Distinguishing number of co-normal product of two graphs

We begin this section with a general upper bound for the co-normal product of two simple connected graphs. We need the following theorem.

Theorem 2.1. [12] *Let G and H be two graphs and $\lambda : V(G \star H) \to V(G \star H)$ be a mapping.*

(i) If $\lambda = (\alpha, \beta)$ defined as $\lambda(g, h) = (\alpha(g), \beta(h))$, where $\alpha \in \mathrm{Aut}(G)$ and $\beta \in \mathrm{Aut}(H)$, then λ is an automorphism on $G \star H$.

(ii) If G is isomorphic to H and $\lambda = (\alpha, \beta)$ defined as $\lambda(g, h) = (\beta(h), \alpha(g))$, where α is an isomorphism on G to H and β is an isomorphism on H to G, then λ is an automorphism on $G \star H$.

Theorem 2.2. *If G and H are two simple connected graphs, then*

$$\max\big\{D(G\Box H), D(G), D(H)\big\} \leq D(G\star H) \leq \min\big\{D(G)|V(H)|, |V(G)|D(H)\big\}.$$

Proof. We first show that $\max\{D(G), D(H)\} \leq D(G\star H)$. By contradiction, we assume that $D(G \star H) < \max\{D(G), D(H)\}$. Without loss of generality we suppose that $\max\{D(G), D(H)\} = D(G)$. Let C be a $(D(G\star H))$-distinguishing labeling of $G \star H$. Then the set of vertices $\{(g, h^*) : g \in V(G)\}$, where $h^* \in V(H)$ have been labeled with less than $D(G)$ labels. Hence we can define the labeling C' with $C'(g) := C(g, h^*)$ for all $g \in V(G)$. Since $D(G \star H) < D(G)$, so C' is not a distinguishing labeling of G, and so there exists a nonidentity automorphism α of G preserving the labeling C'. Thus there exists a nonidentity automorphism λ of $G\star H$ with $\lambda(g, h) := (\alpha(g), h)$ for $g \in V(G)$ and $h \in V(H)$, such that λ preserves the distinguishing labeling C, which is a contradiction. Now we show that $D(G\Box H) \leq D(G \star H)$, and so we prove the left inequality. By Theorems 1.1 and 2.1, we can obtain that $\mathrm{Aut}(G\Box H) \subseteq \mathrm{Aut}(G \star H)$, and since $V(G\Box H) = V(G \star H)$, we have $D(G\Box H) \leq D(G \star H)$.

Now we show that $D(G \star H) \leq \min\{D(G)|V(H)|, |V(G)|D(H)\}$. For this purpose, we define two distinguishing labelings of $G \star H$ with $D(G)|V(H)|$ and $|V(G)|D(H)$ labels, respectively. Let C be a $D(G)$-distinguishing labeling of G and C' be a $D(H)$-distinguishing labeling of H. We suppose that $V(G) = \{g_1, \ldots, g_n\}$ and $V(H) = \{h_1, \ldots, h_m\}$, and define the two following distinguishing labelings L_1 and L_2 of $G \star H$ with $D(G)|V(H)|$ and $|V(G)|D(H)$ labels.

$$L_1(g_j, h_i) := (i - 1)D(G) + C(g_j),$$
$$L_2(g_j, h_i) := (j - 1)D(H) + C'(h_i).$$

We only prove that the labeling L_1 is a distinguishing labeling, and by a similar argument, it can be concluded that L_2 is a distinguishing labeling of $G \star H$. If f is an automorphism of $G \star H$ preserving the labeling L_1, then f maps the set $H_i := \{(g_j, h_i) \ : \ g_j \in V(G)\}$ to itself, setwise, for all $i = 1, \ldots, m$. Since the restriction of f to H_i can be considered as an automorphism of G preserving the distinguishing labeling C, so for every $1 \leq i \leq m$, the restriction of f to H_i is the identity automorphism. Hence f is the identity automorphism of $G \star H$. $\square$

The bounds of Theorem 2.2 are sharp. For the right inequality it is sufficient to consider the complete graphs as the graphs G and H. In fact, if $G = K_n$ and $H = K_m$, then $G \star H = K_{nm}$. For the left inequality we consider the non isomorphic rigid graphs as the graphs G and H. Then by Theorem 1.4, we conclude that $G \star H$ and $G \square H$ are a rigid graph and hence $\max\big\{D(G \square H), D(G), D(H)\big\} = D(G \star H)$.

With respect to Theorems 1.1 and 1.2, we have that the automorphism group of a co-normal product of connected non isomorphic, non rigid graphs with no false twin and no dominating vertex, is the same as automorphism group of the Cartesian product of them, so the following theorem follows immediately:

Theorem 2.3. *If G and H are two simple connected, non isomorphic, non rigid graphs with no false twin and no dominating vertex, then $D(G \star H) = D(G \square H)$.*

Since the path graph P_n ($n \geq 4$), and the cycle graph C_m ($m \geq 5$) are connected, graphs with no false twin and no dominating vertex, then by Theorem 2.3 we have $D(P_n \star P_q) = D(P_n \star C_m) = D(C_m \star C_p) = 2$ for any $q, n \geq 3$, where $q \neq n$ and $m, p \geq 5$, where $m \neq p$. (see [7] for the distinguishing number of Cartesian product of these graphs).

To prove the next result, we need the following lemmas.

Lemma 2.1. [13] *For any two distinct vertices (v_i, u_j) and (v_r, u_s) in $G \star H$, $N((v_i, u_j)) = N((v_r, u_s))$ if and only if*

(i) $v_i = v_r$ *in G and* $N(u_j) = N(u_s)$ *in H, or*

(ii) $u_j = u_s$ *in H and* $N(v_i) = N(v_r)$ *in G, or*

(iii) $N(v_i) = N(v_r)$ *in G and* $N(u_j) = N(u_s)$.

Lemma 2.2. [13] *A vertex (v_i, u_j) is a dominating vertex in $G \star H$ if and only if v_i and u_j are dominating vertices in G and H, respectively.*

Theorem 2.4. [12] *For a rigid graph G and a non rigid graph H, $|\mathrm{Aut}(G \star H)| = |\mathrm{Aut}(H)|$ if and only if G has no dominating vertex and H has no false twin.*

Now we are ready to state and prove the main result of this section.

Theorem 2.5. *Let G be a connected graph with no false twin and no dominating vertex, and $\star G^k$ the k-th power of G with respect to the co-normal product. Then $D(\star G^k) = 2$ for $k \geq 3$. In particular, if G is a rigid graph, then for $k \geq 2$, $D(\star G^k) = 2$.*

Proof. By Lemmas 2.1 and 2.2, we can conclude that $G \star G$ has no false twin and no dominating vertex. We consider the two following cases:

Case 1) Let G be a non rigid graph. If $H := G \star G$, then $D(\star G^3) = 2$ by Theorem 2.3. Now by induction on k, we have the result.

Case 2) Let G be a rigid graph. In this case, $|\mathrm{Aut}(G \star G)| = 2$, by Theorem 1.3, and so $D(G \star G) = 2$. If $H := G \star G$, then $|\mathrm{Aut}(G \star H)| = |\mathrm{Aut}(H)|$, by Theorem 2.4. Hence $|\mathrm{Aut}(\star G^3)| = 2$. By induction on k and using Theorem 2.4, we obtain $D(\star G^k) = 2$ for $k \geq 2$, where G is a rigid graph. □

3 Distinguishing index of co-normal product of two graphs

In this section we investigate the distinguishing index of co-normal product of graphs. Pilśniak in [11] showed that the distinguishing index of traceable graphs, graphs with a Hamiltonian path, of order equal or greater than seven is at most two.

Theorem 3.1. [11] *If G is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

We say that a graph G is almost spanned by a subgraph H if $G - v$, the graph obtained from G by removal of a vertex v and all edges incident to v, is spanned by H for some $v \in V(G)$. The following two observations will play a crucial role in this section.

Lemma 3.1. [11] *If a graph G is spanned or almost spanned by a subgraph H, then $D'(G) \leq D'(H) + 1$.*

Lemma 3.2. *Let G be a graph and H be a spanning subgraph of G. If $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Aut}(H)$, then $D'(G) \leq D'(H)$.*

Proof. Let to call the edges of G which are the edges of H, H-edges, and the others non-H-edges, then since $\mathrm{Aut}(G) \subseteq \mathrm{Aut}(H)$, we can conclude that each automorphism of G maps H-edges to H-edges and non-H-edges to non-H-edges. So assigning each distinguishing edge labeling of H to G and assigning non-H-edges a repeated label we make a distinguishing edge labeling of G. □

Since for two distinct simple non isomorphic, non rigid connected graphs, with no false twin and no dominating vertex we have $\mathrm{Aut}(G \star H) = \mathrm{Aut}(G \Box H)$, so a direct consequence of Lemmas 3.1 and 3.2 is as follows:

Theorem 3.2. *(i) If G and H are two simple connected graphs, then $D'(G \star H) \leq D'(G \Box H) + 1$.*

(ii) If G and H are two simple connected non isomorphic, non rigid graphs with no false twin and no dominating vertex, then $D'(G \star H) \leq D'(G \Box H)$.

Theorem 3.3. *Let G be a connected graph with no false twin and no dominating vertex, and $\star G^k$ the k-th power of G with respect to the co-normal product. Then for $k \geq 3$, $D'(\star G^k) = 2$. In particular, if G is a rigid graph, then for $k \geq 2$, $D'(\star G^k) = 2$.*

Proof. By Lemmas 2.1 and 2.2, we can conclude that $G \star G$ has no false twin and no dominating vertex. We consider the two following cases:

Case 1) Let G be a non rigid graph. If $H = G \star G$, then $D(\star G^3) = 2$ by Theorem 3.2(ii). Now by an induction on k, we have the result.

Case 2) Let G be a rigid graph. In this case, $|\mathrm{Aut}(G \star G)| = 2$, by Theorem 1.3, and so $D(G \star G) = 2$. If $H := G \star G$, then $|\mathrm{Aut}(G \star H)| = |\mathrm{Aut}(H)|$, by Theorem 2.4. Hence $|\mathrm{Aut}(\star G^3)| = 2$. By an induction on k and using Theorem 2.4, we obtain $D(\star G^k) = 2$ for $k \geq 2$, where G is a rigid graph. □

Theorem 3.4. *Let G be a connected graph of order $n \geq 2$. Then $D'(G \star K_m) = 2$ for every $m \geq 2$, except $D'(K_2 \star K_2) = 3$.*

Proof. Since $|\mathrm{Aut}(G \star K_m)| \geq 2$, so $D'(G \geq K_m) = 2$. With respect to the degree of vertices $G \star K_m$ we conclude that $G \star K_m$ is a traceable graph. We consider the two following cases:

Case 1) Suppose that $n \geq 2$. If $m \geq 3$, or $m = 2$, and $n \geq 4$, then the order of $G \star K_m$ is at least 7, and so the result follows from Theorem 3.1. If $m = 2$, $n = 3$, then $G = P_3$ or K_3. In each case, it is easy to see that $D'(G \star K_m) = 2$.

Case 2) Suppose that $n = 2$. Then $G = K_2$, and so $G \star K_m = K_{2m}$. Thus $D'(G \star K_m) = 2$ for $m \geq 3$, and $D'(K_2 \star K_2) = D'(K_4) = 3$. □

By the value of the distinguishing index of Cartesian product of paths and cycles graphs in [3] and Theorem 3.2, we can obtain this value for the co-normal product of them as the two following corollaries.

Corolary 3.1. (i) *The co-normal product $P_m \star P_n$ of two paths of orders $m \geq 2$ and $n \geq 2$ has the distinguishing index equal to two, except $D'(P_2 \star P_2) = 3$.*

(ii) *The co-normal product $C_m \star C_n$ of two cycles of orders $m \geq 3$ and $n \geq 3$ has the distinguishing index equal to two.*

(iii) *The co-normal product* $P_m \star C_n$ *of orders* $m \geq 2$ *and* $n \geq 3$ *has the distinguishing index equal to two.*

Proof.

(i) If $n, m \geq 4$, then the result follows from Theorem 3.2 (ii). If $n = 2$ or $m = 2$, then we have the result by Theorem 3.4. For the remaining cases, with respect to the degree of vertices in $P_m \star P_n$, we obtain easily the distinguishing index.

(ii) If $n, m \geq 5$, then the result follows from Theorem 3.2 (ii). If $n = 3$ or $m = 3$, then we have the result by Theorem 3.4. For the remaining cases we use of Hamiltonicity of $C_m \star C_n$ and Theorem 3.1.

(iii) If $n \geq 5$ and $m \geq 4$, then the result follows from Theorem 3.2 (ii). If $n = 3$ or $m = 2$, then we have the result by Theorem 3.4. The remaining cases are $C_n \star P_3$ and $C_4 \star P_m$. In the first case and with respect to the degree of vertices in $C_n \star P_3$, we obtain easily the distinguishing index. In the latter case, we use of Hamiltonicity of $C_4 \star P_m$ and Theorem 3.1. $\square$

4 Acknowledgements

The authors would like to express their gratitude to the referee for her/his careful reading and helpful comments.

References

[1] M.O. Albertson and K.L. Collins, *Symmetry breaking in graphs*, Electron. J. Combin. 3 (1996), #R18.

[2] S. Alikhani and S. Soltani, *The distinguishing number and distinguishing index of the lexicographic product of two graphs*, Discuss. Math. Graph Theory 38 (2018) 853-865.

[3] A. Gorzkowska, R. Kalinowski, and M. Pilśniak, *The distinguishing index of the Cartesian product of finite graphs*, Ars Math. Contem. 12 (2017), 77-87.

[4] R. Hammack, W. Imrich and S. Klavžar, *Handbook of product graphs (second edition)*, Taylor & Francis group 2011.

[5] W. Imrich, *Automorphismen und das kartesische Produkt von Graphen*, Oesterreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II 177 (1969), 203-214.

[6] W. Imrich, J. Jerebic and S. Klavžar, *The distinguishing number of Cartesian products of complete graphs*, European J. Combin. 29 (4) (2008), 922-929.

[7] W. Imrich and S. Klavžar, *Distinguishing Cartesian powers of graphs*, J. Graph Theory, 53.3 (2006), 250-260.

[8] R. Kalinowski and M. Pilśniak, *Distinguishing graphs by edge colourings*, European J. Combin. 45 (2015), 124-131.

[9] D.J. Miller, *The automorphism group of a product of graphs*, Proc. Amer. Math. Soc. 25 (1970), 24-28.

[10] O. Ore, *Theory of Graphs*, Amer. Math. Society 1962.

[11] M. Pilśniak, *Improving upper bounds for the distinguishing index*, Ars Math. Contemp. 13 (2017), 259-274.

[12] S. Rehman and I. Javaid, *Fixing number of co-noraml product of graphs*, arXiv:1703.00709 (2017).

[13] S. Rehman and I. Javaid, *Resolving, dominating and locating dominating sets in co-normal product of graphs*, submitted.