Ratio Mathematica Volume 36, 2019, pp. 43-52

The inclusion and exclusion principle in view of number theory

Viliam Ďuriš*

Tomáš Lengyelfalusy †

Abstract

The inclusion and exclusion (connection and disconnection) principle is mainly known from combinatorics in solving the combinatorial problem of calculating all permutations of a finite set or other combinatorial problems. Finite sets and Venn diagrams are the standard methods of teaching this principle. The paper presents an alternative approach to teaching the inclusion and exclusion principle from the number theory point of view, while presenting several selected application tasks and possible principle implementation into the Matlab computing environment.

Keywords: inclusion, exclusion, number theory, combinatorics, Matlab

2010 AMS subject classification: 11B75.‡

---

* Department of Mathematics, Faculty of Natural Sciences Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, 949 74 Nitra, Slovakia; vduris@ukf.sk.

† Department of Didactics, Technology and Educational Technologies, DTI University Sládkovičova 533/20, 018 41 Dubnica nad Váhom, Slovakia; lengyelfalusy@dti.sk.

‡Received on May 2nd, 2019. Accepted on June 3rd, 2019. Published on June 30th, 2019. doi: 10.23755/rm.v36i1.465. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

In traditional secondary school mathematics (in combinatorics, number theory or even in probability theory), the notion of factorial and combinatorial numbers is introduced [1]. If n and k are two natural numbers with $n \geq k$, then we call *a combinatorial number* the following notation

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)\ldots(n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k}$$

while (*factorial of the number n*) $n! = 1 \cdot 2 \cdot \cdots \cdot n,$ where $n > 1,$ $0! = 1,$ $1! = 1.$

For combinatorial numbers, the basic properties apply:

$$\binom{n}{1} = n \quad \binom{n}{0} = 1 \quad \binom{0}{0} = 1 \quad \binom{n}{k} = \binom{n}{n-k} \quad \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

The relation $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ is the basis for placing combinatorial numbers in the plane in the shape of a triangle (a so-called *Pascal's triangle*) [2], in which combinatorial numbers can be gradually calculated using the fact that $\binom{n}{0} = \binom{n}{n} = 1$ for each n.

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & & \cdots & & &
\end{array}$$

If n is a natural number, and if a, b are arbitrary complex numbers, then the *binomial theorem* can be applied by using the form:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

The special cases of the binomial theorem are as follows:

a) if $a = 1, b = -1$:

$$1 - \binom{n}{1} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n = 0$$

b) if $a = 1, b = 1$:

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Let us consider now N given objects and K properties $a_1, \dots, a_K$. Let us denote $N(0)$ as the number of objects that do not have either of these properties, $N(a_i)$ as the number of those that have the property a_i, $N(a_i a_j)$ as the number of those that have the property a_i as well as a_j etc. Then

$$N(0) = N - \sum N(a_i) + \sum N(a_i a_j) - \sum N(a_i a_j a_s) + \cdots + (-1)^K N(a_1 a_2 \dots a_K),$$

where, in the first addition, we sum up using numbers $i = 1, 2, \dots, K$, in the second addition, using all pairs of these numbers, in the third addition, using all threesomes of these numbers, etc. We call this relationship *the inclusion and exclusion principle* [3].
The validity of the inclusion and exclusion principle can be shown from the number theory point of view the way that if an object has no property from the properties $a_i, i = 1, \cdots, K$, so it contributes by the unit value to the left equality, though contributing at the same time to the right side, that is, to the number N (in the following additions it does not reappear). Let an object now have t properties ($t \geq 1$). Then, it does not contribute to the left side as there is a number of objects on the left side that do not have any of the properties. Let us calculate the contribution of this object to the right side. In the first addition, it appears t-times. In the second addition, it appears $\binom{t}{2}$–times because from t properties it is possible to choose pairs of the properties in $\binom{t}{2}$ ways. In the third addition, it appears $\binom{t}{3}$–times, etc., so the total contribution to the right side is as follows:

$$1 - t + \binom{t}{2} - \binom{t}{3} + \dots + (-1)^{t-1}\binom{t}{t-1} + (-1)^t = 0,$$

which is a special case of the binomial theorem. Thus, the total contribution of such an object to both sides is zero and the right side is actually equal to the number of objects that do not have any of the given properties.

2 Selected examples of the inclusion and exclusion principle

The first example requires some mathematical concepts to be recalled. By the *Cartesian product* of sets A, B we mean set $A \times B = \{[x, y]: x \in A \wedge y \in B\}$, with the symbol $|A|$ we denote the number of elements (so-called *cardinality*) of the finite set *A*. If $|A| = a$, $|B| = b$, the Cartesian product then contains $a \cdot b$ of ordered pairs. Since the Cartesian product contains ordered pairs, $A \times B$ is not the same set as $B \times A$. [4]

The relation *f* of set *A* to set *B* is called a function of set *A* to set *B* if $\forall x \in A \exists y \in B: [x, y] \in f$ and simultaneously if $[x, y] \in f \wedge [x, z] \in f$, so $y = z$. The symbol B^A denotes a set of all functions $A \to B$.

If *f* is a function of set *A* into set *B* and $\forall x_1, x_2 \in A: x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, the function *f* is called an injective function of set *A* into set *B* (or simply an *injection*; we also say that the function *f* is ordinary).

Let us now consider two finite sets *A*, *B*, where $|A| = n$ and $|B| = m$. Then the number of all injective functions from *A* into *B* is $m \cdot (m-1) \cdot \dots \cdot (m-n+1) = \prod_{i=0}^{n-1}(m-i)$. Injections from set $A = \{1, 2, \cdots, n\}$ into set *B*, where $|B| = m$, are called *variations without repetition (or simply variations) of the n-th class from m elements* (of the set *B*). For these functions, the term $V_n(m)$ is used in practice. It is easier to write the expression $m \cdot (m-1) \cdot \dots \cdot (m-n+1)$ with the following factorial notation $V_n(m) = \frac{m!}{(m-n)!}$.

Variations of the *n*-th class from *n* elements of the set B are bijective functions $A \to B$ and their number is $n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1 = n!$. They are called *permutations* (of set *B*) and denote $P(n) = n!$.

Let us now consider basic set *A* with the cardinality $|A| = n$. *Combinations (without repetition) of the k-th class (or k-combinations) from n elements* are *k*-element subsets of set *A*. We denote them as $C_k(n)$. If *A* is a finite set, with $|A| = n$, then, the number of *k*-combinations of elements of set *A* is $C_k(n) = \binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$. [5]

Example 2.1. A group of *N* men is to take part in a chess tournament. Before entering the room, they place their coats in the locker room. However, when they are about to leave, they are unable to recognize their coats. What is the probability that none of them will take their own coat?

Solution. Let us denote the coats $1, 2, \cdots, N$. Then the distribution of the coats on the chess players can be made $N!$, since these are the permutations of the set $\{1, 2, \cdots, N\}$. First, we determine the number $N(0)$ of permutations, for which there is no coat on the right player. The number of permutations that do not leave in place the k-element set of coats is $(N-k)!$ The number of k-sets can be chosen in $\binom{N}{k}$ ways.

Then, based on the inclusion and exclusion principle, there applies

$$N(0) = N - \binom{N}{1}(N-1)! + \binom{N}{2}(N-2)! - \cdots + (-1)^N \binom{N}{N}(N-N)!$$

$$N(0) = \sum_{k=0}^{N} (-1)^k \binom{N}{k}(N-k)!$$

Next, we get

$$N(0) = \sum_{k=0}^{N} (-1)^k \frac{N!}{k!\,(N-k)!}(N-k)! = N! \sum_{k=0}^{N} \frac{(-1)^k}{k!}$$

All permutations of *N* elements is *N!*, hence the likelihood that no chess player is wearing his coat when leaving the tournament is

$$\frac{N! \sum_{k=0}^{N} \frac{(-1)^k}{k!}}{N!} = \sum_{k=0}^{N} \frac{(-1)^k}{k!}$$

Example 2.2. A tennis centre has a certain number of players and 4 groups A, B, C, D. Each player trains in at least one group, while some players train in multiple groups at once according to the table.

A............26	AC..........18	ABC..........5
B............17	AD..........3	ABD..........0
C............58	BC..........9	ACD..........2
D............19	BD..........0	BCD..........0
AB..........7	CD..........5	ABCD........0

We will show how many players have a tennis centre.

Solution. Let us denote M_1 as the set of all players in group A, M_2 as the set of all players in group B, M_3 as the set of all players in group C and M_4 as the set of all players in group D. Then, set $N = M_1 \cup M_2 \cup M_3 \cup M_4$ is a set of all players in the centre.

Based on the inclusion and exclusion principle, there applies:

$$0 = |M_1 \cup M_2 \cup M_3 \cup M_4| - (26 + 17 + 59 + 19) + (7 + 18 + 3 + 9 + 5) - (5 + 2) + 0$$

From which $|M_1 \cup M_2 \cup M_3 \cup M_4| = 26 + 17 + 59 + 19 - 7 - 18 - 3 - 9 - 5 + 5 + 2 = 85$. As a result, the tennis centre has 85 players.

Example 2.3. Let $n > 1$ be a natural number. In number theory, the symbol $\varphi(n)$ denotes the number of natural numbers smaller than *n* and relatively prime s *n*, where $\varphi(n)$ is called Euler's function [3]. Let $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be a canonical decomposition of the number *n*. We will show that the following relation applies:

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right)$$

Solution. Once more, we will use the inclusion and exclusion principle. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ is a canonical decomposition of the number *n*. The natural numbers that are relatively prime with the number *n* are those that are not divisible by either of the prime numbers $p_1, p_2, \ldots, p_k$. So, let a_i mean the property that "the number *m* is divisible by the prime number $p_i, i = 1, \ldots, k$". The number of numbers that are smaller or equal to the number *n* and are divisible by the number p_i is $N(a_i) = \frac{n}{p_i}$. It is an integer since $p_i \mid n$. Next, we get $N(a_i a_j) = \frac{n}{p_i p_j}$ and other members of the notation.

Then:

$$\varphi(n) = n - \sum \frac{n}{p_i} + \sum \frac{n}{p_i p_j} - \sum \frac{n}{p_i p_j p_s} + \cdots + (-1)^k \frac{n}{p_1 p_2 \cdots p_k}$$

This expression can be simplified to the form:

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right)$$

Several other interesting tasks and applications of the inclusion and exclusion principle can be found e.g. in the resources [6], [7].

3 Implementation of the inclusion and exclusion principle in the Matlab computing environment

When solving various practical tasks with pupils, it is possible and appropriate to use some computing environment, e.g. Matlab. We will now solve a simple task of divisibility.

Example 3.1. We will show how many numbers there are up to 1000 that are not divisible by three, five, or seven.

Solution. Before proceeding to the solution of the task, we will use divisibility relations to determine the number of all natural numbers smaller than 1000, each of which can be divided simultaneously by three, five, and seven.

First, we will generally show that if $3|a$, $5|a$, then $3 \cdot 5 = 15|a$, being valid if $3|a$, so $a = 3b$, if $5|a$, so $a = 5c$. The left sides are equal, so the right sides must be equal, too. Then

$$3b = 5c$$

Since $(3,5) = 1 \Rightarrow 3|\mathrm{c} \Rightarrow c = 3d$. Then $a = 5c = 15d \Rightarrow 15|a$.

Now, we will show that if $15|a$, $7|a$, then $15 \cdot 7 = 105|a$ is valid if $15|a$, so $a = 15e$, if $7|a$, so $a = 7f$. Since $a = a$, it holds true that

$$15e = 7f$$

From the relation $(15,7) = 1 \Rightarrow 15|\mathrm{f} \Rightarrow f = 15g$. Then $a = 7f = 105g \Rightarrow 105|a$.

We will do the division $\frac{1000}{105} = 9 + \frac{55}{105}$ and we see that there exist 9 numbers with the required property.

Let us get back to our basic task. There, we have $N = 1000$. Let a_1 be the property that "the number *n* is divisible by three", property a_2 stand for "the number *n* is divisible by five", property a_3 stand for "the number *n* is divisible by seven". At the same time, $N(0)$ is the number of searched numbers not divisible by any of the numbers 3, 5, 7.
Every third natural number is divisible by three since $1000 = 3 \cdot 333 + 1$. We have the number $N(a_1) = 333$, that is 333 numbers up to 1000 are divisible by three. By similar consideration, we determine $N(a_2) = 200,\ N(a_3) = 142$.

Based on the previous considerations, we determine the number $N(a_1a_2)$. It holds true that if a number is divisible by three and five, it is also divisible by its product, i.e. by the number 15 (inasmuch as the numbers 3 and 5 are relatively prime). Hence, $N(a_1a_2)$ equals the number of numbers up to 1000 divisible by 15 and $N(a_1a_2) = 66$. Similarly, we determine $N(a_2a_3) = 28$ and $N(a_1a_3) = 47$. For the number $N(a_1a_2a_3)$ it is valid that it will be equal to the number of numbers up to 1000 that are divisible by the product $3 \cdot 5 \cdot 7 = 105$, hence $N(a_1a_2a_3) = 9$.

Then, based on the inclusion and exclusion principle, we have in total

$$N(0) = 1000 - (333 + 200 + 142) + (66 + 28 + 47) - 9 = 457$$

Now we implement the given task into the Matlab computing environment to verify the result. First we create the function "count_the_divisors", which is the application of the inclusion and exclusion principle:

```
function cnt = count_the_divisors(N, a, b, c)
cnt_3 = floor(N / a); %counts of numbers divisible by a
cnt_5 = floor(N / b); %counts of numbers divisible by b
cnt_7 = floor(N / c); %counts of numbers divisible by c

cnt_3_5 = floor(N / (a * b)); %counts of numbers divisible by a and b
cnt_5_7 = floor(N / (b * c)); %counts of numbers divisible by b and c
cnt_3_7 = floor(N / (a * c)); %counts of numbers divisible by a and c

cnt_3_5_7 = floor(N / (a * b * c)); %counts of numbers divisible by a, b and c

%and now inclusion-exclusion principle applied
cnt = N - (cnt_3 + cnt_5 + cnt_7) + (cnt_3_5 + cnt_5_7 + cnt_3_7) - cnt_3_5_7;
```

We will call the function from the command line:

```
>> N = 1000;
>> count_the_divisors(N, 3, 5, 7)
```

```
ans =

    457
```

When creating functions or scripts solving various problems based on the inclusion and exclusion principle, it is possible to use various set operations (functions) built directly in Matlab without the need to create one's own structures. [8]

4 Conclusion

The principle of inclusion and exclusion is a "set problem" that falls within the field of discrete mathematics with different applications in combinatorics. However, this principle also plays a significant role in number theory when defining the so-called Euler's function or Fermat's theorem, or in clarifying and exploring the fundamental problems of number theory, such as expressing the distribution of prime numbers among natural numbers on the numerical axis and many other questions still open today.
The paper offered something different than just a set view of the inclusion and exclusion principle and its definition using number theory knowledge and the properties of combinatorial numbers. Our work is a guideline for solving selected practical tasks in which the involvement of the principle might not be expected at first sight. We also showed the possible application of ICT and the Matlab computing environment in solving computational problems in the field of number theory, which can be concurrently involved in mathematics teaching. In conclusion, the inclusion and exclusion principle has much more application than we allege in our short contribution and can be used to solve more difficult tasks, e.g. in algebra to solve specific systems of equations or to solve various problems in combination with the Dirichlet principle. Some research shows that the ability to solve problems also depends on the substitution thinking, which makes possible to use mathematical knowledge effectively in various areas of number theory [9].

References

[1] J. Sedláček. *Faktoriály a kombinační čísla*. Praha, Mladá fronta, 1964.

[2] A. Vrba. *Kombinatorika*. Praha: Mladá fronta, 1980.

[3] Š. Znám. *Teória čísel*. Bratislava, SPN, 1975.

[4] M.T. Keller, W.T. Trotter. *Applied Combinatorics*. American Institute of Mathematics, 2017.

[5] M. Škoviera. *Úvod do diskrétnej matematiky*. Bratislava, Katedra informatiky FMFI UK, 2007.

[6] K. H. Rosen. *Discrete mathematics and its applications*. 4th ed., WCB/McGraw Hill, Boston, 1999.

[7] M.J. Erickson. *Introduction to Combinatorics*. John Wiley & Sons, New York, ISBN: 0-471-15408-3, 1996.

[8] Mathwork. *Online documentation*. 2019. Available at: https://www.mathworks.com/help/matlab/set-operations.html, Accessed 15th of April 2019.

[9] D. Gonda: *The Elements of Substitution Thinking and Its Impact On the Level of Mathematical Thinking*. In: IEJME — MATHEMATICS EDUCATION, vol. 11, no. 7, p. 2402-2417, Look Academic Publishers, 2016.