Ratio Mathematica Volume 36, 2019, pp.89-95

The theorem of the complex exponentials

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Abstract

This paper describes a new theorem that relates the lengths of the legs of a right triangle with the ratio of three complex exponentials. The big novelty of the theorem consists in transforming two real measures of legs derived from Euclidean geometry into a combination of imaginary elements obtained from the complex analysis.

Keywords: complex analysis, complex geometry

2010 subject classification: 37K20.[†]

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[†] Received on December 10th, 2018. Accepted on April 24th, 2019. Published on June 30th, 2019. doi: 10.23755/rm.v36i1.472. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

It seems difficult, apparently, to imagine that the difference between the lengths of the legs of a right triangle may have some connection with the ratio of complex numbers, or vice-versa, that a ratio of complex numbers may be obtained from the difference of the legs of a right triangle, but this theorem, that we will call, precisely, *Theorem of the complex exponentials,* shows that it is possible.

Let's start with some historical and mathematical considerations, from which our research is inspired. In ancient Greece, the right triangles were basically solved by the *first and second theorem of Euclide* (IV-III century B.C.) and by the *theorem of Pythagoras* (about 575-495 B.C.). This happened because the Greek trigonometry, that was only applied to the study of astronomy, was based on the measurement of the ropes of a circle (subtended by a certain angle), rather than on that of sines and cosines. The functions sine and cosine, developed by the Indians in the IV-V century A.C., have been imported in the Arab world around the VIII century A.C., and then, to the West world, a few centuries later. From this moment, the triangles started to be solved by the relations that bind the lengths of the sides of the triangle with the values of the trigonometric functions of its angles. In particular, two fundamental trigonometric theorems were introduced, through which it has been possible to solve any problem related to the elements of a triangle: *the theorem of sines and the theorem of Carnot*. The first one states that *in any triangle the ratio between one side and the sine of the opposite angle is always constant and equal to the diameter of the circle circumscribed to the given triangle*; the second one states that *in any triangle the square of one side is equal to the sum of the squares of the other two, plus their product to the cosine of the angle included*. From the theorem of sines, applied to the right triangles, it descends the theorem according to which *in a right triangle a leg is equal to the product of the hypotenuse for the sine of the angle opposite to the leg*. Finally, we arrive at the XVIII century A.C., where, in another branch of mathematics completely different from the above one, is developed, in all its entirety, the theory of complex numbers of the form $x+iy$, with x and y real numbers and $i = \sqrt{-1}$ the imaginary unit. In particular, the studies of Abraham de Moivre (1667-1754) and Leonhard Euler (1707-1783) provided to the complex numbers a definitive and systematic structure from which descended the complex trigonometric functions and the complex exponential functions. De Moivre left us the famous formula (1739) that calculates the power of a complex number expressed in the form trigonometric $(\cos \alpha + \sin \alpha)^m = \cos(m\alpha) + i \sin(m\alpha)$, while Euler left us the equally famous formula (1748) that binds the trigonometric functions *sine* and *cosine* to the complex exponential function $e^{iw} = \cos w + i \sin w$.

Our theorem is inspired by a very specific motivation: considering that the sides of a right triangle may be expressed by a trigonometric function, and this one by a complex variable, we wanted to discover if the same sides may have a relationship with the elements of the complex analysis and, in case of positive response, in which way and form. With this purpose, we discovered two important results: the first one is that the ratio between the difference of the legs of a right triangle and the difference of their projections on the hypotenuse, multiplied by the cosine of half-difference of two angles opposite to the legs, is always constant; the second one is that this constant is given by the ratio between complex exponentials, or their powers, where the most important constants of the all mathematics are appearing: the constant of Napier **e** (or of Euler), introduced by John Napier on 1618 and used systematically by Euler (1736) for its exponentials; the imaginary unit **i**, officially introduced by Friedrich Gauss (1777-1855) in an essay of 1832; the constant of Archimedes (287-212 B.C.) $\boldsymbol{\pi}$, calculated with approximation by the greek mathematician in the III century B.C. and definitively calculated, with 35 decimal digits, by Ludolf van Ceulen on 1610. Both the above important results are set forth and proved in the following theorem.

2. The theorem of the complex exponentials

Statement: In a right triangle CAB (rectangle in A), where **a** is the hypotenuse, **c** and **b** the legs, **m** and **n** the projections of the respective legs **c** and **b** on the hypotenuse, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ the angles respectively opposed to **c** and **b**, it results:

$$\frac{c-b}{m-n}\cos\frac{\gamma-\beta}{2} = \frac{e^{i\pi/4}}{e^{2i\pi}+e^{i\pi/2}}$$

where **e**=2,71….is the Napier's constant, $\boldsymbol{\pi}$=3,14…..is the pi and **i**=$\sqrt{-1}$ is the imaginary unit.

Proof. Let us consider the right triangle CAB of Figure 1, rectangle in A (α=90°), having hypotenuse a, height h, minor leg b and major leg c, n and m the respective projections of b and c on the hypotenuse a, γ the angle opposed to c and β the angle opposed to b.

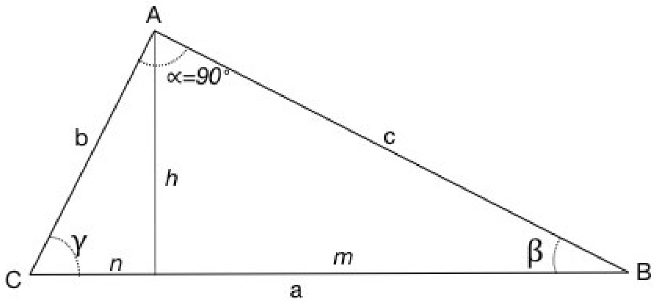

Figure 1

With reference to the right triangle of Figure 1, we know that the first Euclid's theorem asserts:

$$c^2 = a \cdot m$$
$$b^2 = a \cdot n \tag{1}$$

from which, subtracting member to member, it derives:

$$c^2 - b^2 = a\,(m - n) \tag{2}$$

namely:

$$(c + b) \cdot (c - b) = a \cdot (m - n) \tag{3}$$

Taking into account that it is: c=a sinγ and b=a sinβ, from (3) it derives:

$$\frac{c - b}{m - n} = \frac{a}{a(\sin\gamma + \sin\beta)} \tag{4}$$

Simplifying and applying the formulas Prosthaphaeresis to the denominator of second member (4), from (4) it's obtained:

$$\frac{c-b}{m-n} = \frac{1}{2\sin\frac{\gamma+\beta}{2}\cos\frac{\gamma-\beta}{2}} \tag{5}$$

But we know that $\gamma + \beta = 90°$, so in the denominator of (5) it's $2\sin\frac{\gamma+\beta}{2} = \sqrt{2}$, therefore from (5) it derives:

$$\frac{c-b}{m-n} = \frac{1}{\sqrt{2}\cos\frac{\gamma-\beta}{2}} \tag{6}$$

namely:

$$\frac{c-b}{m-n}\cos\frac{\gamma-\beta}{2} = \frac{1}{\sqrt{2}} \tag{7}$$

We know, from complex number's theory, that the trigonometric form of the complex number 1+i is:

$$1+i = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \tag{8}$$

For the formula of Euler it's:

$$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\pi/4} \tag{9}$$

Replacing (9) in (8), we obtain:

$$1+i = \sqrt{2}\cdot e^{i\pi/4} \tag{10}$$

namely:

$$\frac{1}{\sqrt{2}} = \frac{e^{i\pi/4}}{1+i} \tag{11}$$

Let us remind now that it is:

$$1 = e^{2i\pi} \quad \text{and} \quad i = e^{i\pi/2} \tag{12}$$

Replacing (12) in (11), we obtain:

$$\frac{1}{\sqrt{2}} = \frac{e^{i\pi/4}}{e^{2i\pi} + e^{i\pi/2}} \tag{13}$$

Finally, replacing the second member of (7) with the second member of (13), we obtain:

$$\frac{c-b}{m-n}\cos\frac{\gamma-\beta}{2} = \frac{e^{i\pi/4}}{e^{2i\pi} + e^{i\pi/2}}$$

And the theorem is thus proven.

Conclusions

We have shown a theorem born from the motivation to investigate and solve a problem: to link a geometric result of III century B.C., although it reworked by the trigonometric functions of XVI century, to the last theories of complex numbers of XVIII century, apparently irreconcilables with the Euclidean geometry. We think to have got two relevant teachings: on the one hand we have bound the elements of a right triangle (legs and angles) to a constant of complex analysis, given by the combination of three most important constants of mathematics; on the other hand we have notably pointed out a precise methodological procedure of the proof, based strictly on the deductive method, where, starting from a general axiom alleging geometric structure of the right triangles, we reached, through a series of rigorous logical concatenations, a particular result alleging new structure of complex analysis.

We finally think that from this article we also can draw another useful teaching: to discover this theorem allowed us to investigate on three completely different (among their) branches of mathematics (Euclidean geometry, trigonometric functions, complex analysis), born and developed in different ages, transmitted by several men separated by time and by different languages, cultures and religions, who, although not knowing themselves with each other, have always improved the ideas of their predecessors and transmitted it to the future generations. They have been united only by their love for mathematics, in addition to the desire to contribute to its development. We think that we all must pick up an example from this act of faith, that only mathematics, between all sciences, is able to provide.

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