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# Studies on the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen

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**Abstract**

In this paper, we continue the study initiated in preceding works of the argument by A. Einstein, B. Podolsky and N. Rosen according to which quantum mechanics could be "completed" into a broader theory recovering classical determinism. By using the previously achieved isotopic lifting of applied mathematics into isomathematics and that of of quantum mechanics into the isotopic branch of hadronic mechanics, we show that extended particles appear to progressively approach classical determinism in the interior of hadrons, nuclei and stars, and appear to recover classical determinism at the limit conditions in the interior of gravitational collapse.
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# 1. Introduction

## 1.1. The EPR argument

As it is well known, Albert Einstein did not consider quantum mechanical uncertainties to be final, for which reason he made his famous quote "God does not play dice with the universe."

More particularly, Einstein accepted quantum mechanics for atomic structures and other systems, but believed that quantum mechanics is an "incomplete theory," in the sense that it could be broadened into such a form to recover classical determinism at least under limit conditions.

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

Soon after the appearance of paper [1], N. N. Bohr published paper [2] expressing a negative judgment on the possibility of "completing" quantum mechanics along the lines of the EPR argument.

Bohr's paper was followed by a variety of papers essentially supporting Bohr's rejection of the EPR argument, among which we recall *Bell's inequality* [3] establishing that the $SU(2)$ spin algebra does not admit limit values with an identical classical counterpart.

We should also recall *von Neumann theorem* [4] achieving a rejection of the EPR argument via the uniqueness of the eigenvalues of quantum mechanical Hermitean operators under unitary transforms.

The field became known as *local realism* and was centered on the rejection of the EPR argument via additional claims that hidden variables [5] are not admitted by quantum axioms (see the review [6]).

## 1.2. The 1998 apparent proof of the EPR argument

In 1998, the author published paper [7] presenting an apparent proof of the EPR argument based on the following main steps that we here outline to render this paper minimally self-sufficient:

**Step 1:** The proof that Bell's inequality, von Neumann's theorem and other similar objections against the EPR argument [6] are indeed correct, but under the generally tacit assumptions of point-like particles moving in vacuum under sole potential/Hamiltonian interactions (*exterior dynamical systems*) when the systems are treated via quantum mechanics and its underlying 20th century mathematics, including Lie's theory and the

Newton-Leibnitz differential calculus;

**Step 2:** The proof that the above treatments are not applicable for extended, therefore deformable and hyperdense particles under conditions of mutual penetration or entanglement occurring in the structure of hadrons, nuclei, stars, and gravitational collapse such as for black holes, with novel non-linear, non-local, and non-potential/non-Hamiltonian interactions (*interior dynamical systems*);

**Step 3:** The treatment of interior systems via the axiom-preserving lifting of 20th century applied mathematics known as *isomathematics*, whose study was initiated by the author in the late 1970's when he was at Harvard University under DOE support, Refs. [8] to [12] and then continued by various mathematicians. Isomathematics is based on:

3-A) The axiom-preserving isotopy of the conventional associative product between generic quantities $a, b$ (numbers, functions, operators, etc.) first introduced in Eq. (5), p. 71 of Ref. [11]

$$ab \ \rightarrow a \star b = a\hat{T}b, \tag{1}$$

where $\hat{T}$ is a positive-definite quantity called the *isotopic element* providing a representation of the dimension, deformability and density of particles and physical media in which they are immersed via realizations of the type

$$\hat{T} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2})e^{-\Gamma}, \tag{2}$$

where: $n_4^2$ represents the density; $n_k^2, \ k = 1, 2, 3$ represents the deformable share of particles; $n_\mu^2, \mu = 1, 2, 3, 4$, and $\Gamma$ are solely restricted to be positive-definite but otherwise admit a functional dependence on any needed local variables, such as time $t$, coordinates $r$, momenta $p$, energy $E$, density $d$, temperature $\tau$, pressure $\pi$, wavefunctions $\psi$, their derivatives $\partial\psi$, etc.

$$n_\mu = n_\mu(t, r, p, E, d, \tau, \pi, \psi, \partial\psi, ....) > 0, \ \ \mu = 1, 2, 3, 4, \tag{3}$$

$$\Gamma = \Gamma(t, r, p, E, d, \tau, \pi, \psi, \partial\psi, ....) \gg 0. \tag{4}$$

$$e^{- \ \Gamma(t,r,p,E,d,\tau,\pi,\psi,\partial\psi,....)} \ll 1. \tag{5}$$

3-B) The formulation of isoassociative algebras on an *isofield* $\hat{F}(\hat{n}, \star, \hat{I})$ first introduced in Ref. [13] (see also independent work [14]), with *isounit*

$$\hat{I} = 1\hat{T}, \tag{6}$$

and *isoreal, isocomplex and isoquaternionic isonumbers* $\hat{n} = n\hat{I}$ under isoproduct (1), with ensuing isooperations such as the isosquare

$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n}. \tag{7}$$

Isofields also imply the lifting of functions into *isofunctions* [11] [20]

$$\hat{f}(\hat{r}) = [f(r\hat{I})]\hat{I}, \tag{8}$$

among which we quote the *isoexponentiation*

$$\hat{e}^X = (e^{X\hat{T}})\hat{I} = \hat{I}\ (e^{\hat{T}X}), \tag{9}$$

where $X$ is a Hermitean operator.

3-C) The ensuing axiom-preserving lifting of Lie's theory into a non-linear, non-local and non-Hamiltonian form first introduced in Ref. [11] (see also the recent paper [15] and independent work [16]), which theory is today known as the *Lie-Santilli isotheory,* with isobrackets at the foundation of Ref. [7]

$$[X\hat{,}Y] = X \star Y - Y \star X = X\hat{T}Y - Y\hat{T}X. \tag{10}$$

3-D) The isotopic lifting of the Newton-Leibnitz differential calculus, from its historical definition at isolated points, into a form defined on volumes, first introduced in Ref. [17] (see Refs. [18] for vast independent works) with *isodifferential*

$$\begin{aligned} \hat{d}\hat{r} &= \hat{T}(r,...)d\hat{r} = \\ &= \hat{T}(r,...)d[r\hat{I}(r,...)] = dr + r\hat{T}d\hat{I}(r,...), \end{aligned} \tag{11}$$

and corresponding *isoderivatives*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{12}$$

**Step 4:** The axiom-preserving lifting of quantum mechanics into the *isotopic branch of hadronic mechanics,* or *isomechanics* for short, whose study was initiated in Refs. [8] to [12] (see the 1995 monographs [19] [20] [21] with 2008 upgrade [22] and independent studies [23][24]).

Isomechanics is formulated on a *Hilbert-Myung-Santilli (HMS) isospace* [25] $\hat{\mathcal{H}}$ over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$, and it is based on the *iso-Heisenberg isoequations* for the time evolution of a Hermitean operator $\hat{Q}$ in the infinitesimal form

$$\hat{i} \star \frac{\hat{d}\hat{Q}}{\hat{d}\hat{t}} = [\hat{Q}\hat{,}\hat{H}] = \hat{Q} \star \hat{H} - \hat{H} \star \hat{Q} = \\ = \hat{Q}\hat{T}\hat{H} - \hat{H}\hat{T}\hat{Q}, \tag{13}$$

and the finite form

$$\hat{Q}(\hat{t}) = \hat{U}(\hat{t})^{\dagger} \star \hat{Q}(0) \star \hat{U}(\hat{t}) = \\ = \hat{e}^{\hat{H}\star\hat{t}\star\hat{i}} \star \hat{Q}(0) \star \hat{e}^{-\hat{i}\star\hat{t}\star\hat{H}} = \\ = e^{\hat{H}\hat{T}ti} Q(0) e^{-it\hat{T}\hat{H}}, \tag{14}$$

with the following rules for the basic *isounitary isotransforms*

$$\hat{U}(\hat{t})^{\dagger} \star \hat{U}(\hat{t}) = \hat{U}(\hat{t}) \star \hat{U}(\hat{t})^{\dagger} = \hat{I}, \tag{15}$$

where $\hat{t} = t\hat{I}_t$ is the *isotime* which is assumed hereon to coincide with conventional time, $\hat{I}_t = 1$. Dynamical equations (13) to (15) were first presented in Eq. (4.16.49), page 752 of Ref. [9] over conventional fields and reformulated via the full use of isomathematics in Ref. [17]).

Isomechanics is also based on the *iso-Schrödinger isorepresentation* characterized by the fundamental representation of the *isomomentum* permitted by the isodifferential isocalculus, Eq. (12),

$$\hat{p}\hat{|}\psi(\hat{t}, \hat{r}) >= -\hat{i} \star \hat{\partial}_{\hat{t},\hat{r}}\hat{|}\psi(\hat{t}, \hat{r}) >= \\ = -i\hat{I}\partial_{\hat{r}}\hat{|}\psi(\hat{t}, \hat{r}) >, \tag{16}$$

from which one can derive the *iso-Schrödinger isoequation,* [12] [17] [20]

$$\hat{i} \star \hat{\partial}_{\hat{t}}|\hat{\psi}(\hat{t}, \hat{r}) >= \hat{H} \star |\hat{\psi}(\hat{t}, \hat{r}) >= \\ = \hat{H}(r, p)\hat{T}(t, r, p, E, d, \tau, \pi, \psi, \partial\psi, ....)|\hat{\psi}(\hat{t}, \hat{r} >) = \\ = \hat{E} \star |\hat{\psi}(\hat{t}, \hat{r}) >= E|\hat{\psi}(\hat{t}, \hat{r}) > \tag{17}$$

and the *isocanonical isocommutation rules,*

$$[\hat{r}_i\hat{,}\hat{p}_j]|\hat{\psi} >= \hat{i} \star \hat{\delta}_{i.j} \star |\hat{\psi} >= i\delta_{ij}|\hat{\psi} > \tag{18}$$

$$[\hat{r}_i \hat{,} \hat{r}_j]|\hat{\psi}>= [\hat{p}_i \hat{,} \hat{p}_j]|\hat{\psi}>= 0. \tag{19}$$

Note that the characterization of extended particles at mutual distances smaller than their size requires the knowledge of *two quantities,* the conventional Hamiltonian $H$ for the representation of potential interactions, and the isotopic element $\hat{T}$ for the representation of dimension, shape, density as well as of non-linear, non-local and non-potential interactions.

**Step 5:** The proof in Ref. [7] that the isotopic $\hat{SU}(2)$-spin symmetry for extended particles immersed within a dense hadronic medium admits an explicit and concrete realization of *hidden variables* [5], e.g., of the type

$$\hat{T} = Diag.(\lambda, 1/\lambda), \quad Det\hat{T} = 1. \tag{20}$$

In particular, the isotopic $\hat{SU}(2)$-spin isosymmetry admits limit conditions with identical classical counterpart, Eq. (5.4) page 189 Ref. [7].

One aspect of isomathematics and isomechanics which is crucial for this paper is that *in all applications to date, the isotopic element $\hat{T}$ has values much smaller than* $1$, Eqs. (4) (5), as it has been the case for: the synthesis of the neutron from the hydrogen in the core of stars; the representation of nuclear magnetic moments and spin; new clean energies; and other applications [21].

It should be also noted that thanks to the new interactions represented by $\hat{T}$, isomathematics and isomechanics have permitted the first known identification of the *attractive force between identical valence electron pairs* in molecular structures [26]. A significant confirmation of values $|\hat{T}| \ll 1$ is provided by the fact that exact representations of binding energies for the hydrogen and water molecules have been achieved with *isoseries based on isoproduct (1) that are at least one thousand times faster than conventional quantum chemical series* [27] [28].

We should finally indicate that the numerical invariance of the isotopic element $\hat{T}$ and therefore, of the isounit $\hat{I} = 1/\hat{T}$, under isounitary time evolutions (14) (15) was proved in Ref. [29]. Detailed reviews and upgrades of isomathematics, isomechanics, and their applications to interior problems which are specifically written for the EPR argument should soon be available in Refs. [30] [31].

## 1.3. Aim of the paper

In this work, we shall attempt to complete the proof of the EPR argument of Ref. [7] by showing that extended particles in interior dynamical conditions appear to progressively recover classical determinism in interior dynamical conditions with the increase of the density and other characteristics, as indicated at the end of Ref. [7].

It should be stressed that a technical understanding of this work requires technical knowledge of hadronic mechanics, e.g., from Refs. [19] [20] [21] or from the forthcoming reviews and upgrades [30] [31].

We should indicate that the words "completion of quantum mechanics" is used in Einstein's sense for the intent of honoring his memory. For instance, the conventional associative product $ab$ of Eq. (1), which is at the foundation of quantum mechanics, admits a "completion" into the equally associative, yet more general isoproduct $a\hat{T}b$. Under no conditions Einstein's word "completed theory" should be confused with a 'final theory,' that is a theory admitting no additional Einstein's "completions." In fact, the time-reversal invariant, Lie-isotopic isomathematics and isomechanics studied in this work admit the "completion" into the covering, irreversible *Lie-admissible genomathematics and genomechanics* (in which $\hat{T}$ is no longer Hermitean) which, in turn, admit a covering via the most general mathematics and mechanics conceived by the human mind, the multi-valued *hypermathematics and hypermechanics* [32] [33], with additional "coverings" remaining possible in due time [19] [20] [21] .

The reader should be finally aware that the isotopic element $\hat{T}$ and isounit $\hat{I} = 1/\hat{T}$ are inverted in some of the early quoted literature not dealing with determinism without affecting their consistency. An important aim of this paper has been that of achieving the final selection of isotopic element and isounit which is compatible with studies on determinism.

# 2. Recovering of determinism in interior conditions?

## 2.1. Heisenberg uncertainty principle

Consider an electron in empty space represented with the $3$-dimensional Euclidean space $E(r, \delta, I)$, where $r$ represents coordinates, $\delta = Diag.(1, 1, 1)$ represents the Euclidean metric and $I = Dian(1, 1, 1, )$ is the space unit.

Let the operator representation of said electron be done in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ with states $\Psi(r)$ and familiar normalization

$$< \Psi(r)| \, |\Psi(r) >= \int_{-\infty}^{+\infty} \Psi(r)^\dagger \Psi(r) dr = 1. \tag{21}$$

As it is well known, the primary objections against the EPR argument

[2] [3] [4] were based on the *uncertainty principle* formulated by Werner Heisenberg in 1927, according to which *the position $r$ and the momentum $p$ of said electron cannot both be measured exactly at the same time.*

By introducing the *standard deviations* $\Delta r$ and $\Delta p$, the uncertainty principle is generally written in the form (see, e.g., [5])

$$\Delta r \Delta p \ \geq \ \frac{1}{2}\hbar, \tag{22}$$

easily derivable via the vacuum expectation value of the canonical commutation rule

$$\Delta r \Delta p \ \geq \ |\frac{1}{2i} < \Psi| \, [r,p] \, |\Psi > | = \frac{1}{2}\hbar. \tag{23}$$

The standard deviations have the known form [34] (with $\hbar = 1$)

$$\begin{aligned} \Delta r &= \sqrt{< \Psi(r)|[\, r - (< \Psi(r)| \, r \, |\Psi(r) >)]^2 |\Psi(r) >}, \\ \Delta p &= \sqrt{< \Psi(p)| \, [p - (< \Psi(p)| \, p \, |\Psi(p) >)]^2 |\Psi(p) >}, \end{aligned} \tag{24}$$

where $\Psi(r)$ and $\Psi(p)$ are the wavefunctions in coordinate and momentum spaces, respectively.

## 2.2. Particle in interior conditions

We consider now the electron, this time, in the core of a star classically represented with the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ [17] with basic isounit $\hat{I} = 1/\hat{T} > 0$, isocoordinates $\hat{r} = r\hat{I}$, isometric

$$\hat{\delta} = \hat{T}\delta, \tag{25}$$

and isotopic element of type (2) under conditions (3) to (5).

Besides being immersed in the core of a star, the electron has no Hamiltonian interactions. Consequently, we can represent the electron in the *HMS isospace* $\hat{\mathcal{H}}$ [25] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [13], and introduce the time independent *isoplanewave* [20]

$$\begin{aligned} \hat{\Psi}(\hat{r}) &= \hat{\psi}(\hat{r})\hat{I} = \\ &= \hat{N} \star (\hat{e}^{\hat{i} \star \hat{k} \star \hat{r}})\hat{I} = N(e^{ik\hat{T}\hat{r}})\hat{I}, \end{aligned} \tag{26}$$

where $\hat{N} = N\hat{I}$ is an *isonormalization isoscalar,* $\hat{k} = k\hat{I}$ is the *isowavenumber*, and the isoexponentiation is given by Eq. (9).

The corresponding representation in isomomentum isospace is given by
$$\hat{\Psi}(\hat{p}) = \hat{M} \star \hat{e}^{\hat{i} \star \hat{n} \star \hat{p}}, \tag{27}$$
where $\hat{M} = M\hat{I}$ is an isonormalization isoscalar and $\hat{n} = n\hat{I}$ is the isowavenumber in isomomentum isospace.

## 2.3. Isodeterministic isoprinciple

The *isopropability isofunction* is given by [20]
$$\begin{aligned} \hat{\mathcal{P}} &= \hat{<}| \star |\hat{>} =< \hat{\Psi}(\hat{r})| \, T \, |\hat{\Psi}(\hat{r}) > I = \\ &= [\textstyle\int_{-\infty}^{+\infty} \hat{\Psi}(\hat{r})^\dagger \star \hat{\Psi}(\hat{r}) \star \hat{d}\hat{r}]\hat{I} = \\ &= [\textstyle\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{\psi}(\hat{r}) \hat{d}\hat{r}]\hat{I}, \end{aligned} \tag{28}$$
where one should keep in mind that the isodifferential $\hat{d}\hat{r}$ is now given by Eqs. (11).

The *isoexpectation isovalues* of a Hermitean operator $\hat{Q}$ are then given by [20]
$$\begin{aligned} \hat{<}| \star \hat{Q} \star |\hat{>} &=< \hat{\Psi}(\hat{r})| \star \hat{Q} \star |\hat{\Psi}(\hat{r}) > \hat{I} = \\ &= [\textstyle\int_{-\infty}^{+\infty} \hat{\Psi}(\hat{r})^\dagger \star \hat{Q} \star \hat{\Psi}(\hat{r}) \hat{d}\hat{r}]\hat{I} = \\ &= [\textstyle\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{Q} \hat{\psi}(\hat{r}) \hat{d}\hat{r}]\hat{I}, \end{aligned} \tag{29}$$
with corresponding expressions for the isoexpectation isovalues in isomomentum isospace.

We now introduce, apparently for the first time in this paper, the *isotopic operator*
$$\hat{\mathcal{T}} = \hat{T}\hat{I} = I, \tag{30}$$
that, despite its seemingly irrelevant value, is indeed the correct operator formulation of the isotopic element for the transition of the isoproduct from its scalar form (1) into the isoscalar form
$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n} = \hat{n} \star \hat{\mathcal{T}} \star \hat{n} = \ n^2 \hat{I}. \tag{31}$$

Since the identity $I$ can be inserted anywhere in the expectation values of quantum mechanics without altering the results, realization (33) illustrates the central feature of the isotopies, namely, the property that the abstract axioms of quantum mechanics admit a "hidden" realization broader than that of the Copenhagen School whose degrees of freedom have been used in Ref.[7] for the proof of the EPR argument [1].

We now introduce the isoexpectation isovalue of the isotopic operator

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} =< \hat{\Psi}(\hat{r})| \star \hat{\mathcal{T}} \star |\hat{\Psi}(\hat{r}) > \hat{I} = \\ = [\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) d\hat{r}] \hat{I}, \tag{32}$$

and assume the isonormalization

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} = \\ = \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) d\hat{r} = \hat{T}. \tag{33}$$

We then introduce, in this paper apparently for the first time, the *isostandard isodeviation* for isocoordinates $\Delta\hat{r} = \Delta r \hat{I}$ and isomomenta $\Delta\hat{p} = \Delta p \hat{I}$, where $\Delta r$ and $\Delta p$ are the standard deviations in our space.

By using isocanonical isocommutation rules (18), we obtain the expression

$$\Delta\hat{r} \star \Delta\hat{p} = \Delta r \Delta p \hat{I} \approx \frac{1}{2} | < \hat{\Psi}(\hat{r})| \star [\hat{r} \hat{,} \hat{p}] \star \hat{\Psi}(\hat{r}) > | = \\ = \frac{1}{2} | < \hat{\Psi}(\hat{r}) | \hat{T} [\hat{r} \hat{,} \hat{p}] \hat{T} | \hat{\Psi}(\hat{r}) > . \tag{34}$$

By eliminating the common isounit $\hat{I}$, we then have the desired *isodeterministic isoprinciple* here proposed apparently for the first time

$$\Delta r \Delta p \approx \frac{1}{2} | < \hat{\Psi}(\hat{r})| \star [\hat{r} \hat{,} \hat{p}] \star |\hat{\Psi}(\hat{r}) >= \\ = \frac{1}{2} | < \hat{\Psi}(\hat{r}) | \hat{T} [\hat{r} \hat{,} \hat{p}] \hat{T} | \hat{\Psi}(\hat{r}) >= \\ \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) d\hat{r} = T \ll 1 \tag{35}$$

where the property $\Delta r \Delta p \ll 1$ follows from the fact that *the isotopic element $\hat{T}$ has always a value smaller than* 1 (Section 1.2).

It is now necessary to verify isoprinciple (35) by proving that the isostandard isodeviations tend to null values when $\hat{T} \to 0$.

For this purpose, we introduce the following simple isotopy of Eqs. (24) (where we ignore the common multiplication by the isounit)

$$\Delta r = \sqrt{< \hat{\Psi}(\hat{r}) | [\ \hat{r} - < \hat{\Psi}(\hat{r}) | \star \hat{r} \star | \hat{\Psi}(\hat{r}) >]^{\hat{2}} \hat{|} \Psi(\hat{r}) >}, \\ \Delta p = \sqrt{< \hat{\Psi}(\hat{p}) |\ [\hat{p} - < \hat{\Psi}(\hat{p}) | \star \hat{p} \star | \hat{\Psi}(\hat{p}) >]^{\hat{2}} | \hat{\Psi}(\hat{p}) >}, \tag{36}$$

where the differentiation between the isotopic elements for isocoordinates and isomomenta is ignored for simplicity.

It is then easy to see that the isosquare (7) implies the covering forms of the isostandard isodeviations

$$\Delta r = \sqrt{\hat{T} < \hat{\Psi}(\hat{r})|[\,\hat{r} - < \hat{\Psi}(\hat{r})| \star \hat{r} \star |\hat{\Psi}(\hat{r}) >]^2|\hat{\Psi}(\hat{r}) >},$$
$$\Delta p = \sqrt{\hat{T} < \hat{\Psi}(\hat{p})|\,[\hat{p} - < \hat{\Psi}(\hat{p})| \star \hat{p} \star |\hat{\Psi}(\hat{p}) >]^2|\hat{\Psi}(\hat{p}) >}, \quad (37)$$

that indeed approach null value under the limit conditions

$$Lim_{\hat{T}=0}\Delta r = 0,$$
$$Lim_{\hat{T}=0}\Delta p = 0, \quad (38)$$

thus confirming isodeterministic isoprinciple (35).

## 2.4. Particles under pressure

To illustrate the above expressions, we consider an electron in the center of a star, thus being under extreme pressures $\pi$ from the surrounding hadronic medium in all radial directions, while ignoring particle reactions in first approximation or under a sufficiently short period of time.

These conditions are here rudimentarily represented by assuming that the $\Gamma > 0$ function of the the isotopic element (2) is a constant linearly dependent on the pressure $\pi$, resulting in a realization of the isotopic element of the type

$$\hat{T} = e^{-w\pi} \ll 1, \;\; \hat{I} = e^{+w\pi} \gg 1, \quad (39)$$

where $w$ is a positive constant.

The isodeterministic isoprinciple for the considered particle is then given by

$$\Delta r \Delta p \approx \frac{1}{2} e^{-w\pi} \; \ll \; 1, \quad (40)$$

and tends to null values for diverging pressures.

The above example illustrates the consistency of isorenormalization (33) because, a constant isotopic element implies the consistent expression

$$\hat{<}\hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} =$$
$$T < \hat{\psi}(\hat{r})|\,|\hat{\psi}(\hat{r}) > \hat{I} = \quad (41)$$
$$< \hat{\psi}(\hat{r})|\,|\hat{\psi}(\hat{r}) >,$$

while, by contrast, the following alternative isonormalization

$$\hat{<}\hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} = \hat{I}, \quad (42)$$

would imply the expression

$$< \hat{\psi}(\hat{r})||\hat{\psi}(\hat{r}) > \hat{I} = \hat{I}, \quad (43)$$

which is manifestly inconsistent since $< \hat{\psi}(\hat{r})||\hat{\psi}(\hat{r}) >$ is an ordinary number while $\hat{I}$ is a matrix with integro-differential elements.

Note that we have considered a free particle immersed in a hadronic medium, rather than a bound state of extended particles in condition of mutual penetration. Consequently, in our view, isotopic element (2) represents a *subsidiary constraint* caused by the pressure of the hadronic medium encompassing the particle considered, by therefore restricting the values of the isostandard isodeviations for isocoordinates and isomomenta.

Illustrations of the isodeterministic isoprinciple in specific structure models of hadrons and related aspects have been studied in Ref. [21] and their interpretation in terms of the isodeterministic isoprinciple will be studied in future works.

## 2.5. Gravitational example

To provide a gravitational illustration, recall that isotopic element (2) contains as particular cases all possible symmetric metrics in $(3+1)$-dimensions, thus including the Riemannian metric [20].

We then consider the $3$-dimensional sub-case of isotopic element (2) and factorize the space component of the Schwartzchild metric $g_s(r)$ according to isotopic rule introduced in Refs. [35] [36]

$$g_s(r) = \hat{T}(r)\delta, \quad (44)$$

where $\delta$ is the Euclidean metric.

We reach in this way the following realization of the isotopic element

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}, \quad (45)$$

where $M$ is the gravitational mass of the body considered, with ensuing isodeterministic isoprinciple

$$\Delta\hat{r}\Delta\hat{p} \approx \hat{T} = \frac{r}{r - 2M} \Rightarrow_{r\rightarrow 0} = 0, \quad (46)$$

which confirms the statement in page 190 of Ref. [7], on the possible recovering of full classical determinism in the interior of gravitational collapse (see Ref. [37], Chapter 6 in particular, for a penetrating critical analysis of black holes).

It should perhaps be indicated that Refs. [35] [36] introduced the factorization of a full Riemannian metric $g(x),\ x = (r, t)$ in $(3+1)$-dimensions

$$g(x) = \hat{T}_{gr}(x)\eta, \tag{47}$$

where $\hat{T}_{gr}$ is the *gravitational isotopic element*, and $\eta$ is the Minkowski metric $\eta = Diag.(1, 1, 1, -1)$.

Refs. [35] [36] then reformulated the Riemannian geometry via the transition from a formulation over the field of real numbers $\mathcal{R}$ to that over the isofield of isoreal isonumbers $\hat{\mathcal{R}}$ where the *gravitational isounit* is evidently given by

$$\hat{I}_{gr}(x) = 1/\hat{T}_{gr}(x). \tag{48}$$

The above reformulation turns the Riemannian geometry into a new geometry called *iso-Minkowskian isogeometry*, which is locally isomorphic to the *Minkowskian* geometry, while maintaining the mathematical machinery of the Riemannian geometry (covariant derivative, connection, geodesics, etc.) us fully maintained, although reformulated in terms of the isodifferential isocalculus [38].

The apparent advantages of the *identical* iso-Minkowskian reformulation of Riemannian metrics and Einstein's field equations (see, e.g., Eqs. (2.9), page 390 of Ref. [38]) are:

1) The achievement of a consistent operator form gravity in terms of *relativistic hadronic mechanics* [39] whose axioms are those of quantum mechanics, only subjected to a broader realization;

2) The achievement of a universal *symmetry* of *all* non-singular Riemannian metrics, which symmetry is locally isomorphic to the Lorentz-Poincaré symmetry, today known as the *Lorentz-Poincaré-Santilli (LPS) isosymmetry* [40], and it is notoriously impossible on a conventional Riemannian space over the reals;

3) The achievement of clear compatibility of Einstein's field equation with 20th century sciences, such as a clear compatibility of general relativity with special relativity via the simple limit $\hat{I}_{gr} = I$ implying the transition from the universal LPS isosymmetry to the Poincaré symmetry of special relativity with ensuing recovering of conservation and other special relativity laws [41] [42]; the achievement of axiomatic compatibility of gravitation with electroweak interactions thanks to the replacement of curvature into the new notion of isoflatness with the ensuing, currently

impossible, foundations for a grand unification [43]; and other intriguing advances.

# 3. Concluding remarks

t In this paper, we have continued the study of the EPR argument [1] conducted in Ref. [7] and preceding works, with particular reference to the study of the uncertainties for extended particles immersed within hyperdense medias with ensuing linear and non-linear, local and non-local and Hamiltonian as well as non-Hamiltonian interactions.

This study has been conducted via the use of isomathematics and isomechanics characterized by the isotopic element $\hat{T}$ of Eq. (1) which represents the non-linear, non-local and non-Hamiltonian interactions of the particles with the medium [19] [20] [21].

The main result of this paper is that the standard deviations of coordinates and momenta for particles within hyperdense media are characterized by the isotopic element that, being always very small, $\hat{T} \ll 1$, reduces the uncertainties in a way inversely proportional to a non-linear increase of the density, pressure, temperature, and other characteristics of the medium, while admitting the value $\hat{T} = 0$ under extreme/limit conditions with ensuing recovering of full determinism as predicted by A. Einstein, B. Podolsky and N. Rosen [1].

We can, therefore, tentatively summarize the content of this paper with the following:

*ISODETERMINISTIC ISOPRINCIPLE: The product of isostandard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$, as well as the individual isodeviations, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei, and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.*

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