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Optimal Control Policy of a Production and Inventory System for multi-product in Segmented Market

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Abstract

In this paper, we use market segmentation approach in multi-product inventory - production system with deteriorating items. The objective is to make use of optimal control theory to solve the production inventory problem and develop an optimal production policy that maximize the total profit associated with inventory and production rate in segmented market. First, we consider a single production and inventory problem with multi-destination demand that vary from segment to segment. Further, we described a single source production multi destination inventory and demand problem under the assumption that firm may choose independently the inventory directed to each segment. This problem has been discussed using numerical example.

Key words: Market Segmentation, Production-Inventory System, Optimal Control Problem

MSC2010: 97U99.

1 Introduction

Market segmentation is an essential element of marketing in industrialized countries. Goods can no longer be produced and sold without considering customer needs and recognizing the heterogeneity of these needs [1]. Earlier in this century, industrial development in various sectors of economy induced strategies of mass production and marketing. Those strategies were manufacturing oriented, focusing on reduction of production costs rather than satisfaction of customers. But as production processes become more flexible, and customer's affluence led to the diversification of demand, firms that identified the specific needs of groups of customers were able to develop the right offer for one or more submarkets and thus obtained a competitive advantage. Segmentation has emerged as a key planning tool and the foundation for effective strategy formulation. Nevertheless, market segmentation is not well known in mathematical inventory-production models. Only a few papers on inventory-production models deal with market segmentation [2, 3]. Optimal control theory, a modern extension of the calculus of variations, is a mathematical optimization tool for deriving control policies. It has been used in inventory-production [4, 6] to derive the theoretical structure of optimal policies. Apart from inventory-production, it has been successfully applied to many areas of operational research such as Finance [7, 8], Economics [9, 10, 11], Marketing [12, 13, 14, 15], Maintenance [16] and the Consumption of Natural Resources [17, 18, 19] etc. The application of optimal control theory in inventory-production control analysis is possible due to its dynamic behaviour. Continuous optimal control models provide a powerful tool for understanding the behaviour of production-inventory system where dynamic aspect plays an important role. Several papers have been written on the application of optimal control theory in Production-Inventory system with deteriorating items [20, 21, 22, 23].

In this paper, we assume that firm has defined its target market in a segmented consumer population and that it develop a production-inventory plan to attack each segment with the objective of maximizing profit. In addition, we shed some light on the problem in the control of a single firm with a finite production capacity (producing a multi-product at a time) that serves as a supplier of a multi product to multiple market segments. Segmented customers place demand continuously over time with rates that vary from segment to segment. In response to segmented customer demand, the firm must decide on how much inventory to stock and when to replenish this stock by producing. We apply optimal control theory to solve the problem and find the optimal production and inventory policies. The rest of the paper is organized as follows. Following this introduction, all the notations and assumptions needed in the sequel is stated in Section 2. In section 3, we described the single source production-inventory problem with multi-destination demand that vary from segment to segment and developed the optimal control theory problem and its solution. In section 4 of this paper we introduce optimal control formulation of a single source production- multi destination demand and inventory problem and discussion of solution. Numerical illustration is presented in the section 5 and finally conclusions are drawn in section 6 with some future research directions.

2 Notations and Assumptions

Here we begin the analysis by stating the model with as few notations as possible. Let us consider a manufacturing firm producing m product in segmented market environment. We introduce the notation that is used in the development of the model:

Notations:

T	:	Length of planning period,
$P_j(t)$	:	Production rate for j^{th} product,
$I_j(t)$	:	Inventory level for j^{th} product,
$I_{ij}(t)$	:	Inventory level for j^{th} product in i^{th} segment,
$D_{ij}(t)$	:	Demand rate for j^{th} product in i^{th} segment,
$h_j(I_j(t))$	:	Holding cost rate for j^{th} product, (single source inventory)
$h_{ij}(I_{ij}(t))$	:	Holding cost rate for j^{th} product in i^{th} segment, (multi destination)
c_j	:	The unit production cost rate for j^{th} product,
$\theta_j(t, I_j(t))$	:	Deterioration rate for j^{th} product, (single source inventory)
$\theta_{ij}(t, I_{ij}(t))$	:	The deterioration rate for j^{th} product in i^{th} segment, (multi destination)
$K_j(P_j(t))$	:	cost rate corresponding to the production rate for j^{th} product,
r_{ij}	:	The revenue rate per unit sale for j^{th} product in i^{th} segment,
ρ	:	Constant non-negative discount rate.

The model is based on the following assumptions: We assume that the time horizon is finite. The model is developed for multi-product in segmented market. The production, demand, and deterioration rates are function of time. The holding cost rate is function of inventory level & production cost rate depends on the production rate. The functions $h_{ij}(I_{ij}(t))$ (in case of single source $h_j(I_j(t))$ and $\theta_{ij}(t, I_{ij}(t))$ (in case of single source $\theta_j(t, I_j(t))$) are convex. All functions are assumed to be non negative, continuous and differentiable functions. This allows us to derive the most general and robust conclusions. Further, we will consider more specific cases for which we obtain some important results.

3 Single Source Production and Inventory-Multi-Destination Demand Problem

Many manufacturing enterprises use a production-inventory system to manage fluctuations in consumers demand for the product. Such a system consists of a manufacturing plant and a finished goods warehouse to store those products which manufactured but not immediately sold. Here, we assume that once a product is made and put inventory into single warehouse, and demand for all products comes from each segment. Let there be m products and n segments. (i.e., $j = 1, \dots, m$ and $i = 1, \dots, n$).

Therefore, the inventory evolution in segmented market is described by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} I_j(t) = P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)), \ \forall j = 1, \dots, m. \tag{1}$$

So far, firm want to maximize the total Profit during planning period in segmented market. Therefore, the objective functional for all segments is defined as

$$\begin{aligned} \max_{P_j(t) \geq \sum_{i=1}^{m} D_{ij}(t) + \theta_j(t, I_j(t))} J = & \\ & \int_0^T \mathrm{e}^{-\rho t} \sum_{j=1}^{m} \left[\sum_{i=1}^{n} r_{ij} D_{ij}(t) - K_j(P_j(t)) - h_j(I_j(t)) \right] \mathrm{d}t \\ & + \int_0^T \mathrm{e}^{-\rho t} \sum_{j=1}^{m} \left[c_j \left(\sum_{i=1}^{n} D_{ij}(t) - P_j(t) \right) \right] \mathrm{d}t \end{aligned} \tag{2}$$

Subject to the equation (1).This is the optimal control problem with m-control variable (rate of production) with m-state variable (inventory states). Since total demand occurs at rate $\sum_{i=1}^{n} D_{ij}(t)$ and production occurs at controllable rate $P_j(t)$ for j^{th}, it follows that $I_j(t)$ evolves according to the above state equation (1). The constraints $P_j(t) \geq \sum_{i=1}^{m} D_{ij}(t) - \theta_j(t, I_j(t))$ and $I_j(0) = I_{j0} \geq 0$ ensure that shortage are not allowed.
Using the maximum principle [10], the necessary conditions for (P_j^*, I_j^*) to be an optimal solution of above problem are that there should exist a piecewise continuously differentiable function λ and piecewise continuous function μ ,

called the adjoint and Lagrange multiplier function, respectively such that

$$H(t, I^*, P^*, \lambda) \geq H(t, I^*, P, \lambda), \text{ for all } P_j(t) \geq \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t) \quad (3)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = -\frac{\partial}{\partial I_j} L(t, I_j, P_j, \lambda_j, \mu_j) \quad (4)$$

$$I_j(0) = I_{j0}, \ \lambda_j(T) = 0 \quad (5)$$

$$\frac{\partial}{\partial P_j} L(t, I_j, P_j, \lambda_j, \mu_j) = 0 \quad (6)$$

$$\begin{aligned} &P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) \geq 0, \ \mu_j(t) \geq 0, \\ &\mu_j(t)\left[P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t))\right] = 0 \end{aligned} \quad (7)$$

Where, $H(t, I, P, \lambda)$ and $L(t, I, P, \lambda, \mu)$ are Hamiltonian function and Lagrangian function respectively. In the present problem Hamiltonian function and Lagrangian function are defined as

$$\begin{aligned} H = &\sum_{j=1}^{m}\left[\sum_{i=1}^{n} r_{ij} D_{ij}(t) + c_j\left(\sum_{i=1}^{n} D_{ij}(t) - P_j(t)\right) - K_j(P_j(t)) - h_j(I_j(t))\right] \\ &+ \sum_{j=1}^{m}\left[\lambda_j(t)\left(P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t))\right)\right] \end{aligned} \quad (8)$$

$$\begin{aligned} L = &\sum_{j=1}^{m}\left[\sum_{i=1}^{n} r_{ij} D_{ij}(t) + c_j\left(\sum_{i=1}^{n} D_{ij}(t) - P_j(t)\right) - K_j(P_j(t)) - h_j(I_j(t))\right] \\ &+ \sum_{j=1}^{m}\left[(\lambda_j(t) + \mu_j(t))\left(P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t))\right)\right] \end{aligned} \quad (9)$$

A simple interpretation of the Hamiltonian is that it represents the overall profit of the various policy decisions with both the immediate and the future effects taken into account and the value of $\lambda_j(t)$ at time t describes the future effect on profits upon making a small change in $I_j(t)$. Let the Hamiltonian H for all segments is strictly concave in $P_j(t)$ and according to the Mangasarian sufficiency theorem [4, 10]; there exists a unique Production rate.

From equation (4) and (6), we have following equations respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = \rho\lambda_j(t) - \left\{-\frac{\partial h_j(I_j(t))}{\partial I_j} - (\lambda_j(t) + \mu_j(t))\frac{\partial \theta_j(t, I_j(t))}{\partial I_j}\right\}, \quad (10)$$

for all $j = 1, \cdots, m$

$$\lambda_j(t) + \mu_j(t) = c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t)). \quad (11)$$

Now, consider equation (7). Then for any t, we have either

$$P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) = 0 \text{ or}$$

$$P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) > 0 \;\forall\; j = 1, \cdots, m.$$

3.1 Case 1:

Let S is a subset of planning period $[0, T]$, when $P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) = 0$. Then $\frac{\mathrm{d}}{\mathrm{d}t}I_j(t) = 0$ on S, in this case $I^*(t)$ is obviously constant on S and the optimal production rate is given by the following equation

$$P_j^*(t) = \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j^*(t)), \text{ for all } t \in S \quad (12)$$

By equation (10) and (11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = \rho\lambda_j(t) - \left\{-\frac{\partial h_j(I_j(t))}{\partial I_j} - \left(c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t))\right)\frac{\partial \theta_j(t, I_j(t))}{\partial I_j}\right\} \quad (13)$$

After solving the above equation, we get a explicit from of the adjoint function $\lambda_j(t)$. From the equation (10)), we can obtain the value 0f Lagrange multiplier $\mu_j(t)$.

3.2 Case 2:

$P_j(t) - \sum_{i=1}^{n} D_{ij}(t) - \theta_j(t, I_j(t)) > 0$, for $t \in [0, T]\backslash S$. Then $\mu_j(t) = 0$ on $t \in [0, T]\backslash S$. In this case the equation (10) and (11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_j(t) = \rho\lambda_j(t) - \left\{-\frac{\partial h_j(I_j(t))}{\partial I_j} - \lambda_j(t)\frac{\partial \theta_j(t, I_j(t))}{\partial I_j}\right\}, \;\forall\; j = 1, \cdots, m \quad (14)$$

$$\lambda_j(t) = c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t)) \quad (15)$$

Cobining these equation with the state equation, we have the following second order differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_j(t)\frac{d^2}{dP_j^2}K_j(P_j) - \left[\rho + \frac{\partial\theta_j(t, I_j(t))}{\partial I_j}\right]\frac{\mathrm{d}}{dP_j}K_j(P_j) = c_j\left(\rho + \frac{\partial\theta_j(t, I_j(t))}{\partial I_j}\right) + \frac{\partial h_j(t, I_j(t))}{\partial I_j} \quad (16)$$

and $I_j(0) = I_{j0}$, $c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(T)) = 0$. For illustration purpose, let us assume the following forms the exogenous functions $K_j(P_j) = k_jP_j^2/2$, $h_j(t, I_j(t)) = h_jI_j(t)$ and $\theta_j(t, I_j(t)) = \theta_jI_j(t)$, where k_j h_j θ_j are positive constants for all $j = 1, \cdots, m$.

For these functions the necessary conditions for (P_j^*, I_j^*) to be optimal solution of problem (2) with equation (1) becomes

$$\frac{d^2}{\mathrm{d}t^2}I_j(t) - \rho\frac{\mathrm{d}}{\mathrm{d}t}I_j(t) - (\rho + \theta_j)\theta_{1j}I_j(t) = \eta_j(t) \quad (17)$$

with $I_j(0) = I_{j0}$, $c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(T)) = 0$.

Where, $\eta_j(t) = -\sum_{i=1}^n\left(\frac{\mathrm{d}}{\mathrm{d}t}D_{ij}(t)\right) + (\rho + \theta_{1j})\left(\sum_{i=1}^n D_{ij}(t)\right) + \frac{(c_j(\rho+\theta_{1j})+h_j)}{k_j}$.

This problem is a two point boundary value problem.

Proposition 3.1. *The optimal solution of (P_j^*, I_j^*) to the problem is given by*

$$I_j^*(t) = a_{1j}\mathrm{e}^{m_{1j}t} + a_{2j}\mathrm{e}^{m_{2j}t} + Q_j(t), \quad (18)$$

and the corresponding P_j^* is given by

$$P_j^*(t) = a_{1j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{1j}t} + a_{2j}(m_{2j} + \theta_j)\mathrm{e}^{m_{2j}t} + \frac{\mathrm{d}}{\mathrm{d}t}Q_j(t) + \theta_{1j}Q_j(t) + \sum_{i=1}^n D_{ij}. \quad (19)$$

The values of the constant a_{1j}, a_{2j}, m_{1j}, m_{2j} are given in the proof, and $Q_j(t)$ is a particular solution of the equation (17).

Proof. The solution of the two point boundary value problem (17) is given by standard method. Its characteristic equation $m_j^2 - \rho m_j - (\rho + \theta_j)\theta_{1j} = 0$, has two real roots of opposite sign, given by

$$m_{1j} = \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4(\rho + \theta_{1j})\theta_{1j}}\right) < 0,$$

$$m_{2j} = \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4(\rho + \theta_{1j})\theta_{1j}}\right) > 0,$$

and therefore $I_j^*(t)$ is given by (18), where $Q_j(t)$ is the particular solution. Then initial and terminal condition used to determineed the values of constant a_{1j} and a_{2j} as follows

$$
\begin{aligned}
& a_{1j} + a_{2j} + Q_j(0) = I_{j0}, \\
& a_{1j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{1j}T} + a_{2j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{2j}T} \\
& + \left(\frac{c_j}{k_j} + \frac{\mathrm{d}}{\mathrm{d}t} Q_j(T) + \theta_{1j} Q_j(T) + \sum_{i=1}^{n} D_{ij}(T) \right) = 0.
\end{aligned}
$$

By putting $b_{1j} = I_{j0} - Q_j(0)$ and $b_{2j} = -(\frac{c_j}{k_j} + \frac{\mathrm{d}}{\mathrm{d}t} Q_j(T) + \theta_{1j} Q_j(T) + \sum_{i=1}^{n} D_{ij}(T))$, we obtain the following system of two linear equation with two unknowns

$$
\begin{aligned}
& a_{1j} + a_{2j} = b_{1j} \\
& a_{1j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{1j}T} + a_{2j}(m_{1j} + \theta_{1j})\mathrm{e}^{m_{2j}T} = b_{2j}
\end{aligned} \tag{20}
$$

The value of P_j^* is deduced using the values of I_j^* and the state equation. □

4 Single Source Production- Multi Destination Demand and Inventory Problem

We assume the single source production and multi destination demand-inventory system. Hence, the inventory evolution in each segmented is described by the following differential equation:

$$
\frac{\mathrm{d}}{\mathrm{d}t} I_{ij}(t) = \gamma_{ij} P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t)), \ \forall\ j = 1, \cdots, m;\ i = 1, \cdots, n. \tag{21}
$$

Here, $\gamma_{ij} > 0$, $\sum_{i=1}^{n} \gamma_{ij} = 1$, $\forall\ j = 1, \cdots m$ with the conditions $I_{ij}(0) = I_{ij}^0$ and $\gamma_{ij} P_j(t) \geq D_{ij}(t) - \theta_{ij}(t, I_{ij}(t))$. We called $\gamma_{ij} > 0$ the segment production spectrum and $\gamma_{ij} P_j(t)$ define the relative segment production rate of j^{th} product towards i^{th} segment. We develop a marketing-production model in which firm seeks to maximize its all profit by properly choosing production and market segmentation. Therefore, we defined the profit maximization objective function as follows:

$$\max_{\gamma_{ij}P_j(t)\geq D_{ij}(t)-\theta_{ij}(t,I_{ij}(t))} J =$$
$$= \int_0^T \mathrm{e}^{-\rho t} \sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n} r_{ij}D_{ij}(t) + c_j\Bigg(\sum_{i=1}^{n}(D_{ij}(t) - \gamma_{ij}P_j(t))\Bigg)\Bigg]\mathrm{d}t$$
$$- \int_0^T \mathrm{e}^{-\rho t} \sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n} h_{ij}(I_{ij}(t)) - K_j(P_j(t))\Bigg]\mathrm{d}t \tag{22}$$

subject to the equation (21). This is the optimal control problem (production rate) with m control variable with mn state variable (stock of inventory). To solve the optimal control problem expressed in equation (21) and (22), the following Hamiltonian and Lagrangian are defined as

$$H = \sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n} r_{ij}D_{ij}(t) + c_j\Bigg(\sum_{i=1}^{n}(D_{ij}(t) - \gamma_{ij}P_j(t))\Bigg)\Bigg]$$
$$- \sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n} h_{ij}(I_{ij}(t)) + K_j(P_j(t))\Bigg]$$
$$+ \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_{ij}(t)[\gamma_i P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t))] \tag{23}$$

$$L = \sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n} r_{ij}D_{ij}(t) + c_j\Bigg(\sum_{i=1}^{n}(D_{ij}(t) - \gamma_{ij}P_j(t))\Bigg)\Bigg]$$
$$- \sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n} h_{ij}(I_{ij}(t)) + K_j(P_j(t))\Bigg]$$
$$+ \sum_{j=1}^{m}\sum_{i=1}^{n} (\lambda_{ij} + \mu_{ij}(t))[\gamma_i P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t))] \tag{24}$$

Equation (4), (6) and (21) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}(t) = \rho\lambda_{ij}(t) - \left\{-\frac{\partial h_{ij}(I_{ij}(t))}{\partial I_{ij}} - \lambda_{ij}(t)\frac{\partial \theta_{ij}(t, I_{ij}(t))}{\partial I_{ij}}\right\}, \tag{25}$$

for all $i = 1, \cdots, n,\ j = 1, \cdots, m$

$$\sum_{i=1}^{n}(\lambda_{ij}(t) + \mu_{ij}(t))\gamma_i = c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t)) \tag{26}$$

In the next section of the paper, we consider only case when

$$\gamma_{ij}P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t)) > 0,\ \forall\ i,\ j.$$

4.1 Case 2:

$\gamma_{ij}P_j(t) - D_{ij}(t) - \theta_{ij}(t, I_{ij}(t)) > 0 \;\forall\; i,\; j$, for $t \in [0,T]\backslash S$. Then $\mu_{ij}(t) = 0$ on $t \in [0,T]\backslash S$. In this case, the equation (25) and (26) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}(t) = \rho\lambda_{ij}(t) - \left\{ -\frac{\partial h_{ij}(I_{ij}(t))}{\partial I_{ij}} - \lambda_{ij}(t)\frac{\partial \theta_{ij}(t, I_{ij}(t))}{\partial I_{ij}} \right\} \tag{27}$$

$$\sum_{i=1}^{n} \gamma_{ij}\lambda_{ij}(t) = c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(t)) \tag{28}$$

Cobining these equation with the state equation, we have the following second order differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_j(t)\frac{d^2}{dP_j^2}K_j(P_j) - \frac{1}{n}\sum_{i=1}^{n}\left(\rho + \frac{\partial \theta_i(t, I_{ij}(t))}{\partial I_{ij}}\right)\frac{\mathrm{d}}{dP_j}K_j(P_j)$$
$$= \sum_{i=1}^{n} c_j\gamma_i\left(\rho + \frac{\partial \theta_{ij}(t, I_{ij}(t))}{\partial I_{ij}}\right) + \sum_{i=1}^{n}\gamma_i\frac{\partial h_{ij}(t, I_{ij}(t))}{\partial I_{ij}} \tag{29}$$

with $I_j(0) = I_{ij}^0$, $\sum_{i=1}^{n}\gamma_{ij}\lambda_{ij}(T) = 0 \;\; \rightarrow \lambda_{ij}(T) = 0 \;\forall\; i$ and j, $c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(T)) = 0$. For illustration, let us assume the following forms the exogenous functions $K_j(P_j) = k_jP_j^2/2$, $h_{ij}(t, I_{ij}(t)) = h_{ij}I_{ij}(t)$ and $\theta_{ij}(t, I_{ij}(t)) = \theta_{ij}I_{ij}(t)$, where k_j h_{ij} θ_{ij} are positive constants.
For these functions the necessary conditions for (P_j^*, I_{ij}^*) to be optimal solution of problem (19) with equation (18) becomes

$$\frac{d^2I_{ij}(t)}{\mathrm{d}t^2} + (\theta_{ij} - A)\frac{dI_{ij}(t)}{\mathrm{d}t} - A\theta_iI_{ij}(t) = \eta_{ij}(t) \tag{30}$$

with $I_{ij}(0) = I_{ij}^0$, $\lambda_{ij}(T) = 0 \;\forall\; i$, $c_j + \frac{\mathrm{d}}{dP_j}K_j(P_j(T)) = 0$.
Where, $\eta_{ij}(t) = -D_{ij}(t)A + \frac{\gamma_j}{k_j}\left[\sum_{i=1}^{n}\gamma_i(h_{ij} + c_j(\rho + \theta_{ij}))\right] + \frac{dD_{ij}(t)}{\mathrm{d}t}$, $A = \sum_{i=1}^{n}\frac{(\rho+\theta_{ij})}{n}$. This problem is a two point boundary value problem.
The above system of two point boundary value problem (29) is solved by same method that we used in to solve (17).

5 Numerical Illustration

In order to demonstrate the numerical results of the above problem, the discounted continuous optimal problem (2) is transferred into equivalent discrete problem [24] that is solved to present numerical solution. The discrete optimal control can be written as follows:

$$J = \sum_{k=1}^{T}\left(\sum_{j=1}^{m}\left[\sum_{i=1}^{n}(r_{ij}(k-1)D_{ij}(k-1))\right]\right)\left(\frac{1}{(1+\rho)^{k-2}}\right)$$
$$+\sum_{k=1}^{T}\left(\sum_{j=1}^{m}c_j\left(\sum_{i=1}^{n}D_{ij}(k-1)-P_j(k-1)\right)\right)\left(\frac{1}{(1+\rho)^{k-2}}\right)$$
$$-\sum_{k=1}^{T}\left(\sum_{j=1}^{m}[K_j(P_j(k-1)+h_j(I_j(k-1)))]\right)\left(\frac{1}{(1+\rho)^{k-2}}\right)$$

such that

$$I_j(k) = I_j(k-1)+p_j(k-1)-\sum_{i=1}^{n}D_{ij}(k-1)-\theta_j(k-1,I_j(k-1))$$

for all $j = 1,\cdots,m$.

Similar discrete optimal control problem can be written for single source production multi destination and inventory control problem. These discrete optimal control problems are solved by using Lingo11. We assume that the duration of all the time periods are equal and demand are equal from segment for each product. The number of market segmentsis 4 and the number of products is 3. The value of parameters are r_{i1} = 2.55, 2.53, 2.53, 2.54;

Table 1: The Optimal production and inventory rate in segment market

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
P_1	100	86	80	73	64	53	39	21	5	0
P_2	110	81	76	70	62	52	38	21	5	0
P_3	140	79	75	69	61	51	38	21	5	0
I_1	20	98	154	199	232	254	262	255	231	193
I_2	20	107	156	194	222	238	241	231	205	166
I_3	20	137	179	211	233	244	244	231	203	161

r_{i2} = 2.52, 2.53, 2.54, 2.53; r_{i3} = 2.51, 2.54, 2.54, 2.52 for segments $i = 1$ to 4; $c_j = 1$; $k_j = 2$; $\theta_j = 0.10,\ 0.12,\ 0.13$; $h_j = 1$; for all the three products. The optimal production rate and inventory for every product for each segment is shown in Table 1 and their corresponding total profit is $177402.70.

The optimal trajectories of production and inventory rate for every product for each segment are shown in Fig1, Fig2 and Fig3 respectively (Appendix). In case of single source production-multi destination demand and inventory, the number of market segments M is 4 and the number of products is 3. The values of additional parameters are each segment is shown in Table 2.

Table 2: The values of parameter of deteriorating rate and holding cost rate constant

Segment	θ_{i1}	θ_{i2}	θ_{i3}	h_{i1}	h_{i2}	h_{i3}
$M1$	0.10	0.11	0.11	1.0	1.1	1.0
$M2$	0.11	0.12	0.12	1.1	1.2	1.1
$M3$	0.13	0.11	0.11	1.2	1.1	1.2
$M4$	0.11	0.13	0.11	1.1	1.0	1.3

Table 3: Values of the parameter for single source production-multi destination demand and inventory problem in each segment

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10
P_1	100	85	79	73	65	54	41	23	5	0
P_2	110	82	77	70	62	52	38	21	5	0
P_3	140	83	77	71	63	52	38	21	5	0
I_{11}	20	98	153	197	231	254	263	258	236	197
I_{12}	20	97	152	195	227	247	255	248	225	185
I_{13}	20	97	150	192	223	242	247	239	214	173
I_{14}	20	97	149	190	218	236	240	230	204	162
I_{21}	20	108	158	198	227	245	250	242	217	178
I_{22}	20	107	157	195	222	238	242	232	206	167
I_{23}	20	108	158	198	227	245	250	242	217	178
I_{24}	20	107	156	192	218	232	235	223	196	156
I_{31}	20	138	186	223	250	265	268	258	231	191
I_{32}	20	138	184	220	244	258	260	248	219	178
I_{33}	20	138	185	223	250	265	268	258	231	191
I_{34}	20	138	186	223	250	265	268	258	231	191

The optimal production rate and inventory for every product for each segment is shown in Table 3 with production spectrum $\gamma_{11} = 0.10$, $\gamma_{12} = 0.10$, $\gamma_{13} = 0.77$, $\gamma_{14} = 0.03$; $\gamma_{21} = 0.12$, $\gamma_{22} = 0.12$, $\gamma_{23} = 0.75$, $\gamma_{24} = 0.01$; $\gamma_{31} = 0.14$, $\gamma_{32} = 0.14$, $\gamma_{33} = 0.72$, $\gamma_{34} = 0.04$. The optimal value of total profit for all products is \$185876.90. In case of single source production-multi destination demand and inventory, The optimal trajectories of production and inventory rate for every product for each segment are shown in Fig4, Fig5, Fig6 and Fig7 respectively (Appendix).

6 Conclusion

In this paper, we have introduced market segmentation concept in the production inventory system for multi product and its optimal control formulation. We have used maximum principle to determine the optimal production rate policy that maximizes the total profit associated with inventory and production rate. The resulting analytical solution yield good insight on how production planning task can be carried out in segmented market environment. In order to show the numerical results of the above problem, the discounted continuous optimal problem is transferred into equivalent discrete problem [24] that is solved using Lingo 11 to present numerical solution. In the present paper, we have assumption that the segmented demand for each product is a function of time only. A natural extension to the analysis developed here is the consideration of segmented demand that is a general functional of time and amount of onhand stock (inventory).

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Appendix

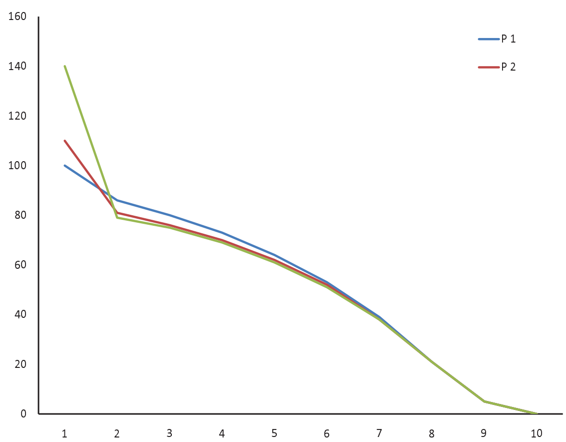

Fig-1

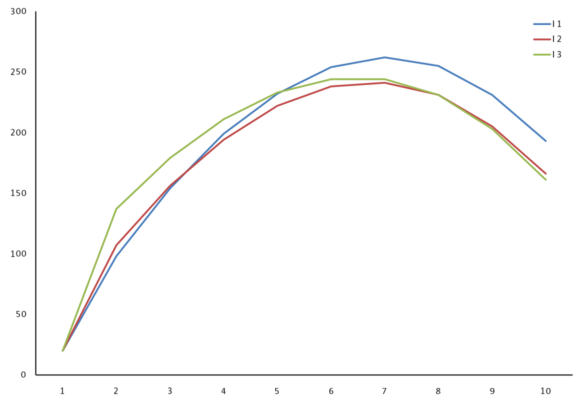

Fig-2

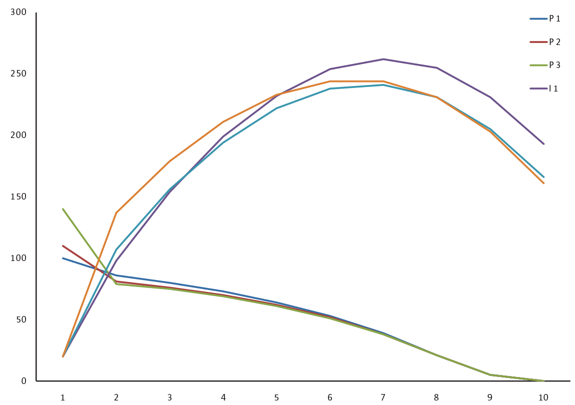

Fig-3

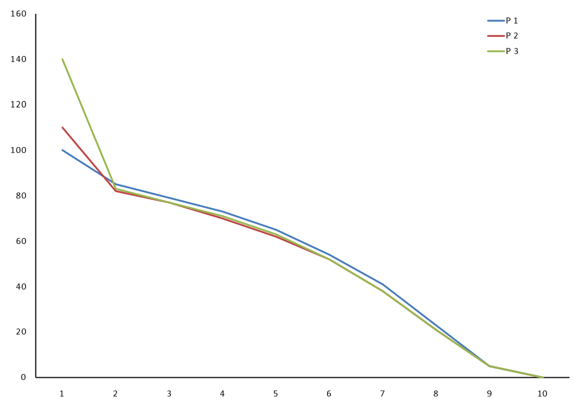

Fig-4

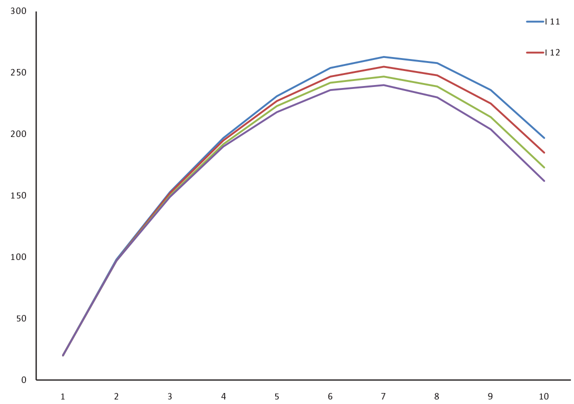

Fig-5

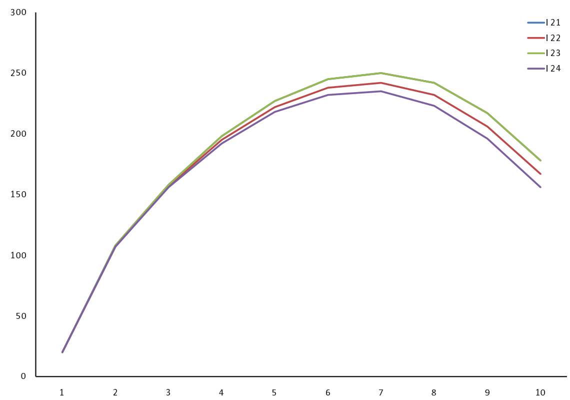

Fig-6

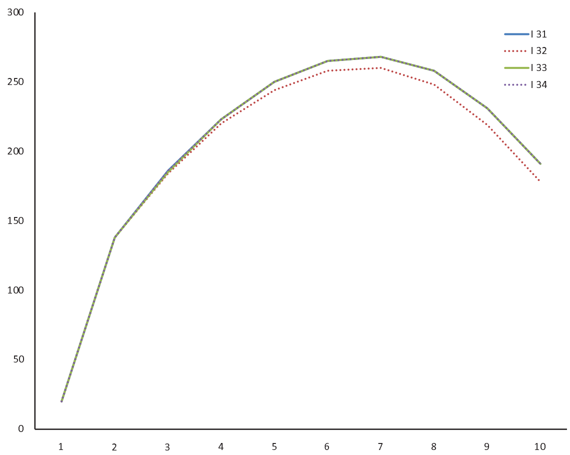

Fig-7