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The LV-hyperstructures

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Abstract

The largest class of hyperstructures is the one which satisfy the weak properties and they are called H_v-structures introduced in 1990. The H_v-structures have a partial order (poset) on which gradations can be defined. We introduce the LV-construction based on the Levels Variable.

Key words: hyperstructures, H_v-structures, hopes, weak hopes.

MSC2010: 20N20.

1 Fundamental Definitions

In a set H is called *hyperoperation* (abbreviation *hyperoperation=hope*) in a set H, is called any map $\cdot : H \times H \to \mathcal{P}(H) - \{\emptyset\}$.

Definition 1.1 (Marty 1934). A hyperstructure $(H, \cdot)$ is a *hypergroup* if $(\cdot)$ is an associative hyperoperation for which the reproduction axiom: $hH = Hh = H, \forall x \in H$, is valid.

Definition 1.2 (Vougiouklis 1990). In a set H with a hope we abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \emptyset, \forall x, y \in H$. The hyperstructure $(H, \cdot)$ is called H_v*-semigroup* if it is *WASS,* it is called H_v*-group* if it is reproductive H_v-semigroup, i.e. $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called H_v*-ring* if both $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to

$$(+) : x(y+z) \cap (xy+xz) \neq \emptyset, (x+y)z \cap (xz+yz) \neq \emptyset, \forall x, y, z \in R$$

.

Definition 1.3 (Santilly-Vougiouklis)**.** A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hyperoperation, is called *e-hyperfield* if the following axioms are valid:

1. $(F,+)$ is an abelian group with the additive unit 0,
2. $(\cdot)$ is WASS,
3. $(\cdot)$ is weak distributive with respect to $(+)$,
4. 0 is absorbing element: $0\cdot x = x\cdot 0 = 0, \forall x \in F$,
5. there exists a multiplicative scalar unit 1, i.e. $1\cdot x = x\cdot 1 = x, \forall x \in F$,
6. for every $x \in F$ there exists a unique inverse x^{-1}, such that
$$1 \in x\cdot x^{-1} \cap x^{-1}\cdot x.$$

The elements of an e-hyperfield are called *e-hypernumbers.* In the case that the relation: $1 = x\cdot x^{-1} = x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Construction 1.4. *The Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hyperoperations $(\otimes)$ as follows:

$$x \otimes y = \{xy, g_1, g_2, \ldots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \ldots \in G - \{e\}$$

$g_1, g_2, \ldots$ are not necessarily the same for each pair (x, y). Then $(G,\otimes)$ becomes an H_v-group, in fact is H_b-group which contains the $(G,\cdot)$. The H_v-group $(G,\otimes)$ is an e-hypergroup. Moreover, if for each x, y such that $xy = e$, so we have $x \otimes y = xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

For more definitions and applications on H_v-structures, see the books and papers [1-20].

The main tool to study hyperstructures are the *fundamental relations* β^*, γ^* and ε^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp. Fundamental relations are used for general definitions. Thus, an H_v-ring $(R,+,\cdot)$ is called *H_v-field* if R/γ^* is a field.

Definition 1.5. Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups defined on the same set H. Then $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an $f \in Aut(H,*)$ such that $xy \subset f(x * y), \forall x, y \in H$. Then we write $\cdot \leq *$ and we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called *H_b-structure.*

Theorem 1.6 (The Little Theorem). *Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

This Theorem leads to a partial order on H_v-structures, thus we have posets. The determination of all H_v-groups and H_v-rings is very interesting. To compare classes we can see the small sets. The problem of enumeration of classes of H_v-structures was started very early but recently we have results by using computers. The partial order in H_v-structures restricts the problem in finding the minimals.

2 Enumeration Theorems

Theorem 2.1 (Chung-Choi). *There exists up to isomorphism,* 13 *minimal H_v-groups of order* 3 *with scalar unit, i.e. minimal e-hyperstructures of order* 3.

Theorem 2.2 (Bayon-Lygeros).

- *There exist, up to isomorphism,* 20 *H_v-groups of order* 2.
- *There exist, up to isomorphism,* 292 *H_v-groups of order* 3 *with scalar unit, i.e. e-hyperstructures of order* 3.
- *There exist, up to isomorphism,* 6494 *minimal H_v-groups of order* 3.
- *There exist, up to isomorphism,* 1026462 *H_v-groups of order* 3.

Theorem 2.3 (Bayon-Lygeros).

- *There exist, up to isomorphism,* 631609 *H_v-groups of order* 4 *with scalar unit, i.e. e-hyperstructures of order* 4.
- *There exist, up to isomorphism,* 8.028.299.905 *abelian Hv-groups of order* 4.

Theorem 2.4 (Bayon-Lygeros).

- *The number of abelian H_v-groups of order* 4 *with scalar unit (i.e. abelian e-hyperstructures) in respect with their automorphism group are the following*

$\lvert \mathrm{Aut}\,(H_v) \rvert$	1	2	3	4	6	8	12	24
	—	—	—	32	—	46	5510	626021

- *There are* 63 *isomorphism classes of hyperrings of order* 2.
- *There are* 875 *isomorphism classes of H_v-rings of order* 2.
- *There are* 33277642 *isomorphism classes of hyperrings of order* 3.

In all the above results we construct the poset of hyperstructures of order 2 and 3 in the sense of inclusion for hyperproducts. We compute the Betti numbers of the poset of Hv-groups of order 2 and we have the following results: $(1,5),(2,4),(3,6),(4,4),(5,1)$. We also compute the Betti numbers of the poset of hypergroups of order 3 and we have the following results: $(1,59),(2,168),(3,294),(4,438),(5,568),(6,585),$ $(7,536),(8,480),$ $(9,358),$ $(10,245),(11,160),(12,66),(13,29),(14,10),$ $(15,2),(16,1)$.

We explicitly compute the Cayley subtables of the minimal e-hyperstructures with $H = \{e,a,b\}$ and we have for the products (aa, ab, ba, bb) the following results: (b; e; e; a), (eb; a; a; e), (e; ab; ab; e), (a; eb; eb; a), (ab; ea; ea; e), (H; eb; a; ea), (H; a; eb; ea), (a; H; H; e), (b; H; H; e), (a; H; H; b), (H; b; a; H), (H; a; b; H), (H; e; ab; H).

3 Construction Theorems

There are several ways to organize such posets using hyperstructure theory. We present now a new construction on posets and we name this LV-construction since it is based on gradations where the Levels are used as Variable. Thus LV means Level Variable.

Theorem 3.1. *The LV-Construction I*

Consider the set $\mathbf{P}_n$ of all H_v-groups defined on a set of n *elements. Take the following gradation on $\mathbf{P}_n$ based on posets:*

Level 0 *(or grade* 0*), denoted by $\mathbf{g}_0$, is the set of all minimals of $\mathbf{P}_n$. Level (grade)* 1*, denoted by $\mathbf{g}_1$, is the set of all H_v-groups obtained from minimals by adding one only element to anyone of the results of the products of two elements on the minimals of $\mathbf{P}_n$, i.e. of $\mathbf{g}_0$. Level* 2 *(or grade* 2*), denoted by $\mathbf{g}_2$, is the set of all H_v-groups obtained from minimals by adding only two elements to anyone of the results of the products of two elements of the minimals $\mathbf{g}_0$. Then inductively the Level k is defined, denoted by $\mathbf{g}_k$. In the* *case that an H_v-group is obtained by adding k_1 elements of one minimal and by adding k_2 elements of another minimal then we consider that it belongs to the Level* $\min(k_1, k_2)$.

Denote by r the cardinality of the minimals, $|\mathbf{g}_0| = r$, and by s the number of levels. Take any H_v-group with r elements corresponding to the r elements of $\mathbf{g}_0$, so we have an H_v-group $(\mathbf{g}_0, *)$. Then we define a hope on

$$\mathbf{P}_n = \mathbf{g}_0 \cup \mathbf{g}_1 \cup, \ldots, \cup \mathbf{g}_{s-1},$$

as follows

$$x \otimes y = \begin{cases} x * y, & \forall x, y \in \mathbf{g}_0 \\ \mathbf{g}_{\kappa+\lambda}, & \forall x \in \mathbf{g}_\kappa, y \in \mathbf{g}_\lambda, \text{ where } (\kappa, \lambda) \neq (0,0) \end{cases}$$

Then the hyperstructure $(\mathbf{P}_n, \otimes)$ is an H_v-group where its fundamental group is isomorphic to $\mathbf{Z}_s$, thus we have

$$\mathbf{P}_n/\beta^* \approx \mathbf{Z}_s.$$

Proof. Let us correspond, numbered, the levels with the elements of $\mathbf{Z}_s$: $\mathbf{g}_i \to \underline{i}, i = 0, \ldots, s-1$.

From the definition of $(\otimes)$ any hyperproduct of elements from several levels, apart of $\mathbf{g}_0$, equals to only one special set of H_v-groups that constitute one level. Moreover we have

$$x \otimes y = \mathbf{g}_0, \forall x \in \mathbf{g}_\kappa, y \in \mathbf{g}_{-\kappa}, \text{ for any } \kappa \neq 0.$$

That means that the elements of $\mathbf{g}_0$are β^*-equivalent. Therefore all elements of each level are β^*-equivalent and there are no β^*-equivalent elements from different levels. That proves that

$$\mathbf{P}_n/\beta^* \approx \mathbf{Z}_s. \qquad \square$$

The above is a construction similar to the one from the book [15, p.27]

A generalization of the above construction is the following:

Theorem 3.2. *The LV-Construction II*

Consider a graded finite poset with n elements: $\mathbf{P}_n = \mathbf{g}_0 \cup \mathbf{g}_1 \cup, \ldots, \cup \mathbf{g}_{s-1}$, with s levels (grades) $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{s-1}$, such that

$$\sum_{i=0}^{s-1} |\mathbf{g_i}| = n.$$

Denoting $|\mathbf{g}_0| = r$, *we consider two* H_v*-groups* $(\mathbf{E}, \cdot)$ *and* $(\mathbf{S}, *)$ *such that* $|\mathbf{E}| = r$, $|\mathbf{S}| = s$ *and moreover* $\mathbf{S}$ *has a unit single element* e. *Then we take 1:1 maps from* $\mathbf{E}$ *onto* $\mathbf{g}_0$ *and from* $\mathbf{S}$ *onto* $\{\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{s-1}\}$, *so we obtain two* H_v*-groups:* $(\mathbf{g}_0, \cdot)$ *and* $\big(\mathbf{G} = \{\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{s-1}\}, *\big)$ *where* $\mathbf{E} = \mathbf{g}_0$ *corresponds to the single element* e. *We define a hope on* $\mathbf{P}_n$ *as follows:*

$$x \otimes y = \begin{cases} x \cdot y, & \forall x, y \in \mathbf{g}_0 \\ \mathbf{g}_\kappa * \mathbf{g}_\lambda, & \forall \mathbf{g}_\kappa, \mathbf{g}_\lambda \in \mathbf{G}, \text{ where } (\kappa, \lambda) \neq (0,0) \end{cases}$$

Then the hyperstructure $(\mathbf{P}_n, \otimes)$ *is an* H_v*-group where its fundamental group is isomorphic to the fundamental group of* $(\mathbf{S}, *)$, *therefore we have*

$$(\mathbf{P}_n, \otimes)/\beta^* \approx (\mathbf{S}, *)/\beta^*.$$

Proof. From the reproductivity of $(\mathbf{G}, *)$, for each $\mathbf{g}_\kappa, \kappa \neq 0$, there exists a $\mathbf{g}_\lambda$ such that $\mathbf{g}_0 \in \mathbf{g}_\kappa * \mathbf{g}_\lambda$. But $\mathbf{g}_0$ is a single element of $(\mathbf{S}, *)$, therefore we have $\mathbf{g}_0 = \mathbf{g}_\kappa * \mathbf{g}_\lambda$. Then, by the definition, for any $x \in \mathbf{g}_\kappa$, $y \in \mathbf{g}_\lambda$ we have, $x \otimes y = \mathbf{g}_0$. Therefore, all the elements of $\mathbf{g}_0$ are β^*-equivalent. On the other side, from the definition, all elements of each level are β^*-equivalent and they are β^*-equivalent elements with different levels if and only if they are β^*-equivalent in $(\mathbf{G}, *)$. In other wards they follow exactly the β^*-equivalence of $(\mathbf{G}, *)$.

That proves that

$$(\mathbf{P}_n, \otimes)/\beta^* \approx (\mathbf{S}, *)/\beta^*. \qquad \square$$

With this LV-construction we can define the poset for H_v-groups of order 2. So we get a non-connected poset with Betti numbers for the two subposets (1,4), (2,4), (3,1) and (1,1), (2, 4), (3,6).

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