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Some characterizations of fuzzy comultisets and quotient fuzzy multigroups

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Abstract

The idea of fuzzy multisets has been applied to some group theoretic notions. Nonetheless, the notions of cosets and quotient groups have not been substantiated in fuzzy multigroup environment. The aim of this paper is to present the concepts of cosets and quotient groups in fuzzy multigroup context with some related results. To start with, the connection between fuzzy comultisets of fuzzy multigroups and the cosets of groups is established. Some characterizations of fuzzy comultisets are outlined with proofs. In addition, quotient fuzzy multigroup is proposed and some of its properties are explored. It is proven that a normal fuzzy submultigroup, $\tilde{H}$ of a fuzzy multigroup, $\tilde{G}$ is commutative if and only if the quotient fuzzy multigroup, $\frac{\tilde{G}}{\tilde{H}}$ of $\tilde{G}$ by $\tilde{H}$ is commutative. Finally, group theoretic isomorphism theorems are established in fuzzy multigroup setting.

Keywords: fuzzy multiset; fuzzy multigroup; fuzzy comultiset; quotient/factor fuzzy multigroup.

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1 Introduction

Fuzzy set theory proposed in 1965 by Zadeh (1965), although with vehement opposition as at then, has been extensively researched with applicative expressions ranging from engineering and computer science to medical diagnosis and social behavior, etc. In a way of extending the application of fuzzy sets to group theory, Rosenfeld (1971) proposed the notion of fuzzy groups as an extension of group theory and some number of results were obtained. Several studies have been carried out on some group theoretic notions in fuzzy group setting (see Ajmal and Prajapati, 1992; Bhattacharya and Mukherjee, 1987; Ejegwa and Otuwe, 2019; Mukherjee and Bhattacharya, 1986, 1984; Onasanya and Ilori, 2013).

With the interest derived from fuzzy sets and multisets (see Blizard, 1989), the idea of fuzzy multisets or fuzzy bags was proposed in (Yager, 1986) as a generalization of fuzzy sets in multiset framework. Myriad of works have been carried out on the fundamentals and properties of fuzzy multisets (see Biswas, 1999; Ejegwa, 2014, 2019a; Miyamoto, 1996; Miyamoto and Mizutani, 2004; Onasanya and Sholabomi, 2019). In recent times, the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory (Shinoj et al., 2015). The ideas of abelian fuzzy multigroups and order of fuzzy multigroups have been studied with some results (Baby et al., 2015; Ejegwa, 2018b), and the notions of center and centralizer in fuzzy multigroup context were established (Ejegwa, 2018b). In the same vein, the concept of fuzzy multigroupoids was introduced and the idea of fuzzy submultigroups was explored with a number of results (Ejegwa, 2018d). The concepts of normal subgroups, characteristic subgroups and Frattini subgroups have been established in fuzzy multigroup setting with some results (Ejegwa, 2018a; Ejegwa et al., 2020; Ejegwa, 2020c). In (Ejegwa, 2018c), the idea of homomorphism in the environment of fuzzy multigroups was defined and some homomorphic properties of fuzzy multigroups were elaborated. Subsequently, the idea of direct product of fuzzy multigroups was proposed and a number of results were established (Ejegwa, 2019b). The idea of alpha-cuts of fuzzy multigroups and its homomorphic properties have been studied (Ejegwa, 2020b,a). The concept of fuzzy multigroups was redefined by Rasuli (2020) as an extension of the work in (Anthony and Sherwood, 1979).

The present paper is a further study of fuzzy multigroups in group theoretic analogs. Motivated by the researches done in fuzzy multigroups so far, it is expedient to investigate the notions of cosets and quotient groups in the light of fuzzy multigroups to strengthen the theory of fuzzy multigroups. Establishing the ideas of fuzzy comultisets and quotient/factor multigroups shall enhance the plausibility of studying nilpotency and solvability in fuzzy multigroup setting. Actually, this work is an application of fuzzy multisets to cosets and factor groups. In so doing, this paper assay to introduce fuzzy comultisets and quotient fuzzy multigroups with some analog results. The relationship between fuzzy comultisets of fuzzy multigroups and that of cosets of groups is examined, and the isomorphism theorems are duly established. By organization, the paper is thus presented: Section 2 provides some preliminaries on fuzzy multisets and fuzzy multigroups. In Section 3, the idea of fuzzy comultisets is proposed and some of its properties are discussed. Section 4 discusses the concept of quotient or factor fuzzy multigroups with some results. Finally, Section 5 concludes the paper and provides direction for future studies.

2 Preliminaries

In this section, we present some existing definitions and results to be used in the sequel.

2.1 Fuzzy multisets

Definition 2.1. (Yager, 1986) Assume X is a set of elements. Then, a fuzzy bag/multiset, $\tilde{G}$ drawn from X can be characterized by a count membership function, $CM_{\tilde{G}}$ such that

$$CM_{\tilde{G}} : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval, $I = [0, 1]$.

A fuzzy multiset, $\tilde{G}$ can be characterized by a function

$$CM_{\tilde{G}} : X \to N^I \text{ or } CM_{\tilde{G}} : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By Miyamoto and Mizutani (2004), it implies that $CM_{\tilde{G}}(x)$ for $x \in X$ is given as

$$CM_{\tilde{G}}(x) = \{\mu^1_{\tilde{G}}(x), \mu^2_{\tilde{G}}(x), ..., \mu^n_{\tilde{G}}(x), ...\},$$

where $\mu^1_{\tilde{G}}(x), \mu^2_{\tilde{G}}(x), ..., \mu^n_{\tilde{G}}(x), ... \in [0, 1]$ such that $\mu^1_{\tilde{G}}(x) \geq \mu^2_{\tilde{G}}(x) \geq ... \geq \mu^n_{\tilde{G}}(x) \geq ...$, whereas in a finite case, we write

$$CM_{\tilde{G}}(x) = \{\mu^1_{\tilde{G}}(x), \mu^2_{\tilde{G}}(x), ..., \mu^n_{\tilde{G}}(x)\},$$

for $\mu^1_{\tilde{G}}(x) \geq \mu^2_{\tilde{G}}(x) \geq ... \geq \mu^n_{\tilde{G}}(x)$.

A fuzzy multiset, $\tilde{G}$ can be represented in the form $\tilde{G} = \{\frac{\langle CM_{\tilde{G}}(x)\rangle}{x} \mid x \in X\}$.

We denote the set of all fuzzy multisets by $FMS(X)$.

Definition 2.2. (Yager, 1986) Let $\tilde{G}, \tilde{H} \in FMS(X)$. Then, $\tilde{H}$ is called a fuzzy submultiset of $\tilde{G}$ written as $\tilde{H} \subseteq \tilde{G}$ if $CM_{\tilde{H}}(x) \leq CM_{\tilde{G}}(x)$, $\forall x \in X$. Also, if $\tilde{H} \subseteq \tilde{G}$ and $\tilde{H} \neq \tilde{G}$, then $\tilde{H}$ is called a proper fuzzy submultiset of $\tilde{G}$ and denoted as $\tilde{H} \subset \tilde{G}$.

Definition 2.3. (Yager, 1986) Let $\tilde{G}, \tilde{H} \in FMS(X)$. Then, $\tilde{G}$ and $\tilde{H}$ are comparable to each other if and only if $\tilde{H} \subseteq \tilde{G}$ or $\tilde{G} \subseteq \tilde{H}$, and $\tilde{G} = \tilde{H}$ if and only if $CM_{\tilde{G}}(x) = CM_{\tilde{H}}(x)$, $\forall x \in X$.

Definition 2.4. (Miyamoto, 1996) Let $\tilde{G}, \tilde{H} \in FMS(X)$. Then, the intersection and union of $\tilde{G}$ and $\tilde{H}$, denoted by $\tilde{G} \cap \tilde{H}$ and $\tilde{G} \cup \tilde{H}$ are defined by the rules that for any object $x \in X$,

(i) $CM_{\tilde{G}\cap\tilde{H}}(x) = CM_{\tilde{G}}(x) \wedge CM_{\tilde{H}}(x)$,

(ii) $CM_{\tilde{G}\cup\tilde{H}}(x) = CM_{\tilde{G}}(x) \vee CM_{\tilde{H}}(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum operations, respectively.

Before finding the intersection and union of $\tilde{G}$ and $\tilde{H}$, the membership sequences of $\tilde{G}$ and $\tilde{H}$ should be equal. If not, it could be completed by affixing zero(s).

2.2 Fuzzy multigroups

We denote group by X and assume that all fuzzy multigroups are drawn from $FMG(X)$, which is the set of all fuzzy multigroups of X.

Definition 2.5. (Shinoj et al., 2015) A fuzzy multiset $\tilde{G}$ of X is said to be a fuzzy multigroup of X if it satisfies the following two conditions:

(i) $CM_{\tilde{G}}(xy) \geq CM_{\tilde{G}}(x) \wedge CM_{\tilde{G}}(y)$, $\forall x, y \in X$,

(ii) $CM_{\tilde{G}}(x^{-1}) = CM_{\tilde{G}}(x)$, $\forall x \in X$.

It can be easily verified that if $\tilde{G}$ is a fuzzy multigroup of X, then

$$CM_{\tilde{G}}(e) = \bigvee_{x\in X} CM_{\tilde{G}}(x),$$

that is, $CM_{\tilde{G}}(e)$ is the tip of $\tilde{G}$.

Remark 2.1. (Shinoj et al., 2015) We notice that a fuzzy multiset, $\tilde{G}$ of a group X is a fuzzy multigroup if $\forall x, y \in X$,

$$CM_{\tilde{G}}(xy^{-1}) \geq CM_{\tilde{G}}(x) \wedge CM_{\tilde{G}}(y)$$

holds.

Definition 2.6. (Shinoj et al., 2015) Let $\tilde{G}$ be a fuzzy multigroup of a group X. Then $\tilde{G}^{-1}$ is defined by $CM_{\tilde{G}^{-1}}(x) = CM_{\tilde{G}}(x^{-1}), \forall x \in X$.

By Definition 2.5, we get $CM_{\tilde{G}^{-1}}(x) = CM_{\tilde{G}}(x^{-1}) = CM_{\tilde{G}}(x)$. That is, $\tilde{G}^{-1} = \tilde{G}$. Thus, $\tilde{G} \in FMG(X) \Leftrightarrow \tilde{G}^{-1} \in FMG(X)$.

Proposition 2.1. (Shinoj et al., 2015) *Let $\tilde{G}, \tilde{H} \in FMG(X)$. Then, $\tilde{G} \cap \tilde{H} \in FMG(X)$.*

Definition 2.7. (Ejegwa, 2018d) Let $\{\tilde{G}_i\}_{i\in I}, I = 1, ..., n$ be an arbitrary family of fuzzy multigroups of X. Then,

$$CM_{\bigcap_{i\in I}\tilde{G}_i}(x) = \bigwedge_{i\in I} CM_{\tilde{G}_i}(x), \ \forall x \in X$$

and

$$CM_{\bigcup_{i\in I}\tilde{G}_i}(x) = \bigvee_{i\in I} CM_{\tilde{G}_i}(x), \ \forall x \in X.$$

The family of fuzzy multigroups $\{\tilde{G}_i\}_{i\in I}$ of X is said to have inf or sup assuming chain if either $\tilde{G}_1 \subseteq \tilde{G}_2 \subseteq ... \subseteq \tilde{G}_n$ or $\tilde{G}_1 \supseteq \tilde{G}_2 \supseteq ... \supseteq \tilde{G}_n$, respectively.

Definition 2.8. (Baby et al., 2015) Let $\tilde{G} \in FMG(X)$. Then, $\tilde{G}$ is said to be commutative if for all $x, y \in X$,

$$CM_{\tilde{G}}(xy) = CM_{\tilde{G}}(yx).$$

Definition 2.9. (Ejegwa, 2018a) Let $\tilde{G}, \tilde{H} \in FMG(X)$. Then, the product, $\tilde{G} \circ \tilde{H}$ of $\tilde{G}$ and $\tilde{H}$ is defined to be a fuzzy multiset of X as follows:

$$CM_{\tilde{G}\circ\tilde{H}}(x) = \begin{cases} \bigvee_{x=yz}[CM_{\tilde{G}}(y) \wedge CM_{\tilde{H}}(z)], & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.10. (Ejegwa, 2018d) Let $\tilde{G} \in FMG(X)$. A fuzzy submultiset, $\tilde{H}$ of $\tilde{G}$ is called a fuzzy submultigroup of $\tilde{G}$ denoted by $\tilde{H} \subseteq \tilde{G}$ if $\tilde{H}$ is a fuzzy multigroup. A fuzzy submultigroup, $\tilde{H}$ of $\tilde{G}$ is a proper fuzzy submultigroup denoted by $\tilde{H} \subset \tilde{G}$, if $\tilde{H} \subseteq \tilde{G}$ and $\tilde{H} \neq \tilde{G}$.

Definition 2.11. (Ejegwa, 2018a) Let $\tilde{H}, \tilde{G} \in FMG(X)$ such that $\tilde{H} \subseteq \tilde{G}$. Then, $\tilde{H}$ is called a normal fuzzy submultigroup of $\tilde{G}$ if

$$CM_{\tilde{H}}(xyx^{-1}) = CM_{\tilde{H}}(y), \ \forall x, y \in X.$$

Definition 2.12. (Ejegwa, 2018a) Let $\tilde{H}$ be a fuzzy submultiset of $\tilde{G} \in FMG(X)$. Then, the normalizer of $\tilde{H}$ in $\tilde{G}$ is the set given by

$$N(\tilde{H}) = \{g \in X \mid CM_{\tilde{H}}(gy) = CM_{\tilde{H}}(yg), \ \forall y \in X\}.$$

Theorem 2.1. (Ejegwa, 2018a) *Let X be a finite group and $\tilde{H}$ be a fuzzy submultigroup of $\tilde{G} \in FMG(X)$. Define*

$$\begin{aligned} H &= \{g \in X \mid CM_{\tilde{H}}(g) = CM_{\tilde{H}}(e)\}, \\ K &= \{x \in X \mid CM_{\tilde{H}x}(y) = CM_{\tilde{H}e}(y)\}, \end{aligned}$$

where e denotes the identity element of X. Then H and K are subgroups of X. Again, $H = K$.

Definition 2.13. (Ejegwa, 2018d; Shinoj et al., 2015) Let $\tilde{G} \in FMG(X)$. Then, the set $\tilde{G}_*$ defined by

$$\tilde{G}_* = \{x \in X \mid CM_{\tilde{G}}(x) > 0\}$$

is the level set or support of $\tilde{G}$. It follows that $\tilde{G}_*$ is a subgroup of X.

Also, the set $\tilde{G}^*$ defined by

$$\tilde{G}^* = \{x \in X \mid CM_{\tilde{G}}(x) = CM_{\tilde{G}}(e)\}$$

is a subgroup of X.

Definition 2.14. (Ejegwa, 2018c) Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose $\tilde{G}$ and $\tilde{H}$ are fuzzy multigroups of X and Y, respectively. Then, f induces a homomorphism from $\tilde{G}$ to $\tilde{H}$ which satisfies

(i) $CM_{\tilde{G}}(f^{-1}(y_1y_2)) \geq CM_{\tilde{G}}(f^{-1}(y_1)) \wedge CM_{\tilde{G}}(f^{-1}(y_2))$, $\forall y_1, y_2 \in Y$,

(ii) $CM_{\tilde{H}}(f(x_1x_2)) \geq CM_{\tilde{H}}(f(x_1)) \wedge CM_{\tilde{H}}(f(x_2))$, $\forall x_1, x_2 \in X$,

where

(i) the image of $\tilde{G}$ under f, denoted by $f(\tilde{G})$, is a fuzzy multiset over Y defined by

$$CM_{f(\tilde{G})}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_{\tilde{G}}(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of $\tilde{H}$ under f, denoted by $f^{-1}(\tilde{H})$, is a fuzzy multiset over X defined by

$$CM_{f^{-1}(\tilde{H})}(x) = CM_{\tilde{H}}(f(x)),\ \forall x \in X.$$

Proposition 2.2. (Ejegwa, 2018c) *Let $f : X \to Y$ be a homomorphism and $\tilde{G} \in FMG(X)$. If f is injective, then $f^{-1}(f(\tilde{G})) = \tilde{G}$.*

Theorem 2.2. (Ejegwa, 2018c) *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then the following statements hold.*

(i) $\tilde{G} \in FMG(X)$ if and only if $f(\tilde{G}) \in FMG(Y)$.

(ii) $\tilde{H} \in FMG(Y)$ if and only if $f^{-1}(\tilde{H}) \in FMG(X)$.

3 Fuzzy comultiset and some of its properties

In this section, we define fuzzy comultiset and characterize some of its properties.

Definition 3.1. Suppose $\tilde{H}$ is a fuzzy submultigroup of a fuzzy multigroup $\tilde{G}$ of X. Then, the fuzzy submultiset, $y\tilde{H}$ of $\tilde{G}$ for $y \in X$ defined by

$$CM_{y\tilde{H}}(x) = CM_{\tilde{H}}(y^{-1}x),\ \forall x \in X$$

is called the left fuzzy comultiset of $\tilde{H}$. Similarly, the fuzzy submultiset, $\tilde{H}y$ of $\tilde{G}$ for $y \in X$ defined by

$$CM_{\tilde{H}y}(x) = CM_{\tilde{H}}(xy^{-1}),\ \forall x \in X$$

is called the right fuzzy comultiset of $\tilde{H}$.

The following result proves that the right and left fuzzy comultisets of a fuzzy submultigroup in a fuzzy multigroup are equal.

Proposition 3.1. *If $\tilde{H}$ is a fuzzy submultigroup of $\tilde{G} \in FMG(X)$, then the right and left fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$ are identical.*

Proof. Let $x, y, \in X$. Assume $\tilde{H}$ is a fuzzy submultigroup of $\tilde{G}$. Then, we have

$$\begin{array}{rcl} CM_{y\tilde{H}}(x) = CM_{\tilde{H}}(y^{-1}x) & \geq & CM_{\tilde{H}}(y) \wedge CM_{\tilde{H}}(x) \\ & = & CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y) \\ & = & CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y^{-1}). \end{array}$$

Suppose by hypothesis, $CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y^{-1}) = CM_{\tilde{H}}(xy^{-1})$. Then, we have

$$CM_{y\tilde{H}}(x) \geq CM_{\tilde{H}y}(x).$$

Again,

$$\begin{array}{rcl} CM_{\tilde{H}y}(x) = CM_{\tilde{H}}(xy^{-1}) & \geq & CM_{\tilde{H}}(x) \wedge CM_{\tilde{H}}(y) \\ & = & CM_{\tilde{H}}(y) \wedge CM_{\tilde{H}}(x) \\ & = & CM_{\tilde{H}}(y^{-1}) \wedge CM_{\tilde{H}}(x). \end{array}$$

By the same hypothesis, we get

$$CM_{\tilde{H}y}(x) \geq CM_{y\tilde{H}}(x).$$

Hence, $CM_{y\tilde{H}}(x) = CM_{\tilde{H}y}(x) \Rightarrow y\tilde{H} = \tilde{H}y$. □

Remark 3.1. Let $\tilde{H}$ be a fuzzy submultigroup of $\tilde{G} \in FMG(X)$. We notice that

(i) the right and left fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$ are fuzzy submultigroups of $\tilde{G}$.

(ii) $x\tilde{H} = y\tilde{H} = z\tilde{H} = \tilde{H}$, $\forall x, y, z \in X$. This is not applicable in the conventional case.

(iii) there is a one-to-one correspondence between the set of right fuzzy comultisets and the set of left fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$.

(iv) the number of fuzzy comultisets of $\tilde{H}$ in $\tilde{G}$ equals the cardinality of $\tilde{H}_*$.

(v) $x\tilde{H} \cap y\tilde{H} \cap z\tilde{H} = \tilde{H} = x\tilde{H} \cup y\tilde{H} \cup z\tilde{H}, \forall x, y, z \in X$.

Theorem 3.1. *Let $\tilde{H}$ be a fuzzy submultigroup of $\tilde{G} \in FMG(X)$. Then, $g\tilde{H} = h\tilde{H}$ for $g, h \in X$ if and only if*

$$CM_{\tilde{H}}(g^{-1}h) = CM_{\tilde{H}}(h^{-1}g) = CM_{\tilde{H}}(e).$$

Proof. Let $g\tilde{H} = h\tilde{H}$. Then, $CM_{g\tilde{H}}(g) = CM_{h\tilde{H}}(g)$ and $CM_{g\tilde{H}}(h) = CM_{h\tilde{H}}(h)$ $\forall g, h \in X$. Hence,

$$CM_{\tilde{H}}(g^{-1}h) = CM_{\tilde{H}}(h^{-1}g) = CM_{\tilde{H}}(e).$$

Conversely, let $CM_{\tilde{H}}(g^{-1}h) = CM_{\tilde{H}}(h^{-1}g)$ $\forall g, h \in X$. For every $x \in X$, we have

$$\begin{aligned} CM_{g\tilde{H}}(x) = CM_{\tilde{H}}(g^{-1}x) &= CM_{\tilde{H}}(g^{-1}hh^{-1}x) \\ &\geq CM_{\tilde{H}}(g^{-1}h) \wedge CM_{\tilde{H}}(h^{-1}x) \\ &= CM_{\tilde{H}}(h^{-1}x) \\ &= CM_{h\tilde{H}}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} CM_{h\tilde{H}}(x) = CM_{\tilde{H}}(h^{-1}x) &= CM_{\tilde{H}}(h^{-1}gg^{-1}x) \\ &\geq CM_{\tilde{H}}(h^{-1}g) \wedge CM_{\tilde{H}}(g^{-1}x) \\ &= CM_{\tilde{H}}(g^{-1}x) \\ &= CM_{g\tilde{H}}(x). \end{aligned}$$

Hence, $CM_{g\tilde{H}}(x) = CM_{h\tilde{H}}(x) \Rightarrow g\tilde{H} = h\tilde{H}$. □

Corolary 3.1. *Let $\tilde{H}$ be a fuzzy submultigroup of $\tilde{G} \in FMG(X)$. Then $\tilde{H}g = \tilde{H}h$ for $g, h \in X$ if and only if*

$$CM_{\tilde{H}}(gh^{-1}) = CM_{\tilde{H}}(hg^{-1}) = CM_{\tilde{H}}(e).$$

Proof. Straightforward from Theorem 3.1. □

Proposition 3.2. *Let $\tilde{H}, \tilde{G} \in FMG(X)$ such that $\tilde{H} \subseteq \tilde{G}$. If $g\tilde{H} = h\tilde{H}$, then $CM_{\tilde{H}}(g) = CM_{\tilde{H}}(h)$, $\forall g, h \in X$.*

Proof. Let $g, h \in X$. Suppose $g\tilde{H} = h\tilde{H}$, then we have

$$\begin{aligned} CM_{g\tilde{H}}(g) = CM_{h\tilde{H}}(g) &\Rightarrow CM_{\tilde{H}}(g^{-1}g) = CM_{\tilde{H}}(h^{-1}g) \\ &\Rightarrow CM_{\tilde{H}}(e) = CM_{\tilde{H}}(h^{-1}g) \end{aligned}$$

$\forall g, h \in X$. The fact that,

$$CM_{\tilde{H}}(e) = CM_{\tilde{H}}(h^{-1}g) \Rightarrow CM_{\tilde{H}}(h) = CM_{\tilde{H}}(g),$$

the result follows.

Alternatively, suppose $z \in X$, we get

$$\begin{aligned} CM_{g\tilde{H}}(z) = CM_{h\tilde{H}}(z) &\Rightarrow CM_{\tilde{H}}(g^{-1}z) = CM_{\tilde{H}}(h^{-1}z) \\ &\Rightarrow CM_{\tilde{H}z^{-1}}(g) = CM_{\tilde{H}z^{-1}}(h) \\ &\Rightarrow CM_{\tilde{H}}(g) = CM_{\tilde{H}}(h), \end{aligned}$$

because $\tilde{H}z^{-1} = \tilde{H}$. □

Theorem 3.2. *Let $\tilde{G} \in FMG(X)$. Any fuzzy submultigroup $\tilde{H}$ of $\tilde{G}$ and for any $z \in X$, the fuzzy submultiset, $z\tilde{H}z^{-1}$, where $CM_{z\tilde{H}z^{-1}}(x) = CM_{\tilde{H}}(z^{-1}xz)$ for each $x \in X$ is a fuzzy submultigroup of $\tilde{G}$.*

Proof. Let $x, y \in X$ and $\tilde{H} \subseteq \tilde{G}$. We prove that $z\tilde{H}z^{-1}$ is a fuzzy submultigroup of $\tilde{G}$. Now

$$\begin{aligned} CM_{z\tilde{H}z^{-1}}(xy^{-1}) &= CM_{\tilde{H}}(z^{-1}xy^{-1}z) \\ &= CM_{\tilde{H}}(z^{-1}xzz^{-1}y^{-1}z) \\ &\geq CM_{\tilde{H}}(z^{-1}xz) \wedge CM_{\tilde{H}}(z^{-1}y^{-1}z) \\ &= CM_{z\tilde{H}z^{-1}}(x) \wedge CM_{z\tilde{H}z^{-1}}(y^{-1}) \\ &= CM_{z\tilde{H}z^{-1}}(x) \wedge CM_{z\tilde{H}z^{-1}}(y), \ \forall z \in X. \end{aligned}$$

Hence, $z\tilde{H}z^{-1}$ is a fuzzy submultigroup of $\tilde{G}$. □

Corolary 3.2. *Let* $\{\tilde{H}_i\}_{i\in I} \in FMG(X)$*, then*

(i) $\bigcap_{i\in I} z\tilde{H}_i z^{-1} \in FMG(X),\ \forall z \in X,$

(ii) $\bigcup_{i\in I} z\tilde{H}_i z^{-1} \in FMG(X),\ \forall z \in X$ *provided* $\{\tilde{H}_i\}_{i\in I}$ *have sup/inf assuming chain.*

Proof. The results follow from Theorem 3.2. □

The following results are the application of product of fuzzy multigroups to the idea of fuzzy comultisets.

Proposition 3.3. *Suppose* $\tilde{H}$ *is a fuzzy submultigroup of* $\tilde{H} \in FMG(X)$*. Then;*

(i) $\tilde{H}g \circ \tilde{H}g = \tilde{H}g$.

(ii) $\tilde{H}g \circ \tilde{H}h = \tilde{H}h \circ \tilde{H}g$.

(iii) $(\tilde{H}g \circ \tilde{H}h)^{-1} = (\tilde{H}h)^{-1} \circ (\tilde{H}g)^{-1}$.

(iv) $(\tilde{H}g \circ \tilde{H}h)^{-1} = \tilde{H}g \circ \tilde{H}h$.

Proof. Using Definition 2.9, the results follow. □

Remark 3.2. Proposition 3.3 also holds for left fuzzy comultisets.

Proposition 3.4. *If* $\tilde{H}$ *is a fuzzy submultigroup of a commutative fuzzy multigroup* $\tilde{G}$ *of* X*, then*

(i) $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz,\ \forall y, z \in X,$

(ii) $y\tilde{H} \circ z\tilde{H} = yz\tilde{H},\ \forall y, z \in X.$

Proof. Let $\tilde{H} \in FMG(X)$ and $x, y, z \in X$, then we have

(i)

$$\begin{aligned}
CM_{\tilde{H}y\circ\tilde{H}z}(x) &= \bigvee_{x=zy}[CM_{\tilde{H}y}(z) \wedge CM_{\tilde{H}z}(y)],\ \forall y, z \in X \\
&= \bigvee_{x=zy}[CM_{\tilde{H}}(zy^{-1}) \wedge CM_{\tilde{H}}(yz^{-1})],\ \forall y, z \in X \\
&= \bigvee_{x=zy}[CM_{\tilde{H}\cap\tilde{H}}((zy^{-1})(yz^{-1}))],\ \forall y, z \in X \\
&= CM_{\tilde{H}}(xz^{-1}y^{-1}) \\
&= CM_{\tilde{H}yz}(x).
\end{aligned}$$

Hence, $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz$.

(ii) Similar to (i).

□

Corolary 3.3. *Suppose $\tilde{H}$ is a fuzzy submultigroup of a commutative fuzzy multigroup $\tilde{G}$ of X. Then, the following statements are equivalent.*

(i) $(\tilde{H}y \circ \tilde{H}z)^{-1} = \tilde{H}y \circ \tilde{H}z$.

(ii) $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz$.

Proof. The result is easy to see by combining Definition 2.9 and Proposition 3.4. □

Remark 3.3. If $(\tilde{H}y \circ \tilde{H}y)^{-1} = \tilde{H}y \circ \tilde{H}z$ and $\tilde{H}y \circ \tilde{H}z = \tilde{H}yz$, then $(\tilde{H}y \circ \tilde{H}z)^{-1} = \tilde{H}yz$.

Theorem 3.3. *If $\tilde{H}$ is a fuzzy submultigroup of $\tilde{G} \in FMG(X)$ such that $\tilde{G}$ is commutative, then $\tilde{H}g \circ \tilde{H}h = \tilde{H}gh$ if and only if $g\tilde{H} \circ h\tilde{H} = gh\tilde{H}, \forall\, g, h \in X$. Consequently, $\tilde{H}gh = gh\tilde{H}$.*

Proof. Suppose $\tilde{H}g \circ \tilde{H}h = \tilde{H}gh$. By Definition 2.9, we get

$$\begin{aligned}
CM_{\tilde{H}gh}(x) &= CM_{\tilde{H}g\circ\tilde{H}h}(x)\\
&= \bigvee_{y\in X}[CM_{\tilde{H}g}(y) \wedge CM_{\tilde{H}h}(y^{-1}x)]\\
&= \bigvee_{y\in X}[CM_{\tilde{H}}(yg^{-1}) \wedge CM_{\tilde{H}}(y^{-1}xh^{-1})]\\
&= \bigvee_{y\in X}[CM_{\tilde{H}}(g^{-1}y) \wedge CM_{\tilde{H}}(h^{-1}y^{-1}x)]\\
&= \bigvee_{y\in X}[CM_{g\tilde{H}}(y) \wedge CM_{h\tilde{H}}(y^{-1}x)]\\
&= CM_{g\tilde{H}\circ h\tilde{H}}(x)\\
&= CM_{gh\tilde{H}}(x)
\end{aligned}$$

$\Rightarrow g\tilde{H} \circ h\tilde{H} = gh\tilde{H}$.

Conversely, assuming $g\tilde{H} \circ h\tilde{H} = gh\tilde{H}$. Then

$$\begin{aligned}
CM_{gh\tilde{H}}(x) &= CM_{g\tilde{H}\circ h\tilde{H}}(x) \\
&= \bigvee_{y\in X} [CM_{g\tilde{H}}(y) \wedge CM_{h\tilde{H}}(y^{-1}x)] \\
&= \bigvee_{y\in X} [CM_{\tilde{H}}(g^{-1}y) \wedge CM_{\tilde{H}}(h^{-1}y^{-1}x)] \\
&= \bigvee_{y\in X} [CM_{\tilde{H}}(yg^{-1}) \wedge CM_{\tilde{H}}(y^{-1}xh^{-1})] \\
&= \bigvee_{y\in X} [CM_{\tilde{H}g}(y) \wedge CM_{\tilde{H}h}(y^{-1}x)] \\
&= CM_{\tilde{H}g\circ\tilde{H}h}(x) \\
&= CM_{\tilde{H}gh}(x)
\end{aligned}$$

$\Rightarrow \tilde{H}g \circ \tilde{H}h = \tilde{H}gh$. Hence, the result follow. □

Theorem 3.4. *Suppose $\tilde{G} \in FMG(X)$ and $\tilde{H}$ a fuzzy submultigroup of $\tilde{G}$. Define*

$$H = \{g \in X \mid CM_{\tilde{H}}(g) = CM_{\tilde{H}}(e)\}.$$

Then $Hx = Hy \Leftrightarrow \tilde{H}x = \tilde{H}y, \forall x, y \in X$. Similarly, $xH = yH \Leftrightarrow x\tilde{H} = y\tilde{H}$.

Proof. This result gives a relationship between fuzzy comultisets of a fuzzy submultigroup of a fuzzy multigroup and the cosets of a subgroup of a given group.

By Theorem 2.1, we know that H is a subgroup of X and

$$H = \{x \in X \mid CM_{\tilde{H}x}(z) = CM_{\tilde{H}e}(z)\}.$$

Now, suppose that $Hx = Hy$. Then $xy^{-1} \in H$. Thus

$$CM_{\tilde{H}xy^{-1}}(z) = CM_{\tilde{H}e}(z)\,\forall z \in X \text{ and so } CM_{\tilde{H}}(zyx^{-1}) = CM_{\tilde{H}}(z).$$

Put $z = zy^{-1}$, we get

$$\begin{aligned}
CM_{\tilde{H}}(zy^{-1}yx^{-1}) = CM_{\tilde{H}}(zy^{-1}) &\Rightarrow CM_{\tilde{H}}(zx^{-1}) = CM_{\tilde{H}}(zy^{-1}) \\
&\Rightarrow CM_{\tilde{H}x}(z) = CM_{\tilde{H}y}(z)
\end{aligned}$$

and so, $\tilde{H}x = \tilde{H}y$.

Conversely, suppose that $\tilde{H}x = \tilde{H}y$, that is $CM_{\tilde{H}x}(z) = CM_{\tilde{H}y}(z)$, $\forall z \in X$. This implies that

$$CM_{\tilde{H}}(zx^{-1}) = CM_{\tilde{H}}(zy^{-1}).$$

Put $z = y$, we get

$$CM_{\tilde{H}}(yx^{-1}) = CM_{\tilde{H}}(e).$$

So, $yx^{-1} \in H$. Thus, $Hx = Hy$.

The proof of $xH = yH \Leftrightarrow x\tilde{H} = y\tilde{H}$ is similar. □

4 Quotient fuzzy multigroups

In this section, we present the notion of quotient groups in fuzzy multigroup setting and establish the isomorphism theorems.

Definition 4.1. Suppose $\tilde{G}$ is a fuzzy multigroup of X and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then, the union of the set of left/right fuzzy comultisets of $\tilde{H}$ such that the fuzzy comultisets satisfy

$$x\tilde{H} \circ y\tilde{H} = xy\tilde{H},\ \forall x, y \in X$$

is called quotient or factor fuzzy multigroup of $\tilde{G}$ by $\tilde{H}$, denoted by $\frac{\tilde{G}}{\tilde{H}}$.

Remark 4.1. Suppose $\frac{\tilde{G}}{\tilde{H}}$ is a factor fuzzy multigroup of $\tilde{G}$ by $\tilde{H}$, it implies that $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$ and $\frac{\tilde{G}}{\tilde{H}} = e\tilde{H} = \tilde{H}$. This property is not applicable in classical case.

Remark 4.2. Suppose $\tilde{G}$ is a fuzzy multigroup of X, and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then

(i) if $\tilde{I}$ is a fuzzy submultigroup of $\tilde{G}$ such that $\tilde{H} \subseteq \tilde{I} \subseteq \tilde{G}$, then $\frac{\tilde{I}}{\tilde{H}}$ is a fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$.

(ii) every fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ is of the form $\frac{\tilde{I}}{\tilde{H}}$, for some fuzzy submultigroup $\tilde{I}$ of $\tilde{G}$ such that $\tilde{H} \subseteq \tilde{I} \subseteq \tilde{G}$.

Theorem 4.1. *If $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G} \in FMG(X)$. Then $\tilde{H}$ is commutative if and only if $\frac{\tilde{G}}{\tilde{H}}$ is commutative.*

Proof. Let $x, y \in X$. Suppose $\tilde{H}$ is commutative, then

$$CM_{\tilde{H}}(xyx^{-1}y^{-1}) = CM_{\tilde{H}}(e),$$

and hence,

$$CM_{\tilde{H}}(xy) = CM_{\tilde{H}}(yx).$$

Consequently, $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$ by Definition 2.11.
Thus, since

$$CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(xyx^{-1}y^{-1}) = CM_{\tilde{H}}(e),$$

we have

$$\begin{aligned} CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(e) &\Rightarrow CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(xy(xy)^{-1}) \\ &\Rightarrow CM_{\tilde{H}yx}(xy) = CM_{\tilde{H}xy}(xy). \end{aligned}$$

Thus, $\tilde{H}xy = \tilde{H}yx$. It follows that, $\tilde{H}x \circ \tilde{H}y = \tilde{H}y \circ \tilde{H}x$ since $\tilde{H}x \circ \tilde{H}y = \tilde{H}xy$ and $\tilde{H}y \circ \tilde{H}x = \tilde{H}yx$ by Proposition 3.4. Hence, $\frac{\tilde{G}}{\tilde{H}}$ is commutative.

Conversely, assume $\frac{\tilde{G}}{\tilde{H}}$ is commutative, then

$$\tilde{H}x \circ \tilde{H}y = \tilde{H}y \circ \tilde{H}x \Rightarrow \tilde{H}xy = \tilde{H}yx.$$

Thus,

$$CM_{\tilde{H}}(xy(yx)^{-1}) = CM_{\tilde{H}}(e) \Rightarrow CM_{\tilde{H}}(xy) = CM_{\tilde{H}}(yx),$$

completes the proof. □

Theorem 4.2. *Suppose $\tilde{G} \in FMG(X)$ and $\tilde{H}, \tilde{I}$ are normal fuzzy submultigroups of $\tilde{G}$ and $\tilde{H} \subseteq \tilde{I}$, then $\frac{\tilde{I}}{\tilde{H}}$ is a normal fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$.*

Proof. Let $x \in X$. Then $CM_{\frac{\tilde{I}}{\tilde{H}}}(x) \leq CM_{\frac{\tilde{G}}{\tilde{H}}}(x)$ since $\tilde{H} \subseteq \tilde{I}$ and, $\tilde{H}$ and $\tilde{I}$ are normal fuzzy submultigroups of $\tilde{G}$. So, $\frac{\tilde{I}}{\tilde{H}}$ is a fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$. Subsequently,

$$CM_{\frac{\tilde{I}}{\tilde{H}}}(yxy^{-1}) = CM_{\frac{\tilde{I}}{\tilde{H}}}(x)\, \forall x, y \in X.$$

Hence, $\frac{\tilde{I}}{\tilde{H}}$ is a normal fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ by Definition 2.11. □

Remark 4.3. Let $\tilde{G}$ be a fuzzy multigroup of X, and $\tilde{I}$ a normal fuzzy submultigroup of $\tilde{G}$. Then, every normal fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ is of the form $\frac{\tilde{I}}{\tilde{H}}$, for some normal fuzzy submultigroup $\tilde{H}$ of $\tilde{G}$ such that $\tilde{H} \subseteq \tilde{I} \subseteq \tilde{G}$.

Theorem 4.3. *Suppose $\tilde{G}, \tilde{H} \in FMG(X)$ and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then $\frac{\tilde{H}\cap\tilde{G}}{\tilde{H}_*}$ is a normal fuzzy submultigroup of $\tilde{H}$.*

Proof. By Definition 2.13, $\tilde{H}_*$ is a subgroup of X and $\tilde{H} \cap \tilde{G} \in FMG(X)$ by Proposition 2.1. So, $\frac{\tilde{H}\cap\tilde{G}}{\tilde{H}_*}$ is a fuzzy multigroup of X. Since $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$, then $\tilde{H}\cap\tilde{G}$ is a fuzzy submultigroup of $\tilde{G}$ and $\frac{\tilde{H}\cap\tilde{G}}{\tilde{H}_*}$ is a fuzzy submultigroup of $\tilde{H}$. We show that $\frac{\tilde{H}\cap\tilde{G}}{\tilde{H}_*}$ is a normal fuzzy submultigroup of $\tilde{H}$. Let $x, y \in \tilde{H}_*$. Then $xyx^{-1} \in \tilde{H}_*$ since

$$CM_{\tilde{H}}(xyx^{-1}) = CM_{\tilde{H}}(y) > 0$$

by definition of $\tilde{H}_*$. This proves that $\tilde{H}_*$ is a normal subgroup of X.

It is easy to see that $\tilde{H} \cap \tilde{G}$ is normal since

$$\begin{aligned} CM_{\tilde{H}\cap\tilde{G}}(xyx^{-1}) &= CM_{\tilde{H}}(xyx^{-1}) \wedge CM_{\tilde{G}}(xyx^{-1}) \\ &= CM_{\tilde{H}}(y) \wedge CM_{\tilde{G}}(y) \\ &= CM_{\tilde{H}\cap\tilde{G}}(y). \end{aligned}$$

In fact, $\tilde{H} \cap \tilde{G}$ is a normal fuzzy submultigroup since $\tilde{H} \cap \tilde{G} = \tilde{H}$, and $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$. Hence, $\frac{\tilde{H} \cap \tilde{G}}{\tilde{H}_*}$ is a normal fuzzy submultigroup of $\tilde{H}$. □

Theorem 4.4. *Suppose $\tilde{G}$ is a fuzzy multigroup of X, $\tilde{H} \subseteq \tilde{G}$ and $N(\tilde{H})$ is a normalizer. Then $N(\tilde{H})$ is a subgroup of X and $\frac{\tilde{H}}{N(\tilde{H})}$ is a normal fuzzy submultigroup of $\tilde{G}$.*

Proof. Clearly, $e \in N(\tilde{H})$. Let $x, y \in N(\tilde{H})$. Then for any $z \in X$, we have

$$\begin{aligned} CM_{\tilde{H}}((xy^{-1})z) &= CM_{\tilde{H}}(x(y^{-1}z)) \\ &= CM_{\tilde{H}}((y^{-1}z)x) \\ &= CM_{\tilde{H}}(y^{-1}(zx)) \\ &= CM_{\tilde{H}}(y(zx)^{-1}) \\ &= CM_{\tilde{H}}(y(x^{-1}z^{-1})) \\ &= CM_{\tilde{H}}(z(xy^{-1})). \end{aligned}$$

Hence, $xy^{-1} \in N(\tilde{H})$. Therefore, $N(\tilde{H})$ is a subgroup of X. By Definition 4.1, it follows that $\frac{\tilde{H}}{N(\tilde{H})} \in FMG(N(\tilde{H}))$ and clearly, $\frac{\tilde{H}}{N(\tilde{H})}$ is a fuzzy submultigroup of $\tilde{H}$. Since, $CM_{\frac{\tilde{H}}{N(\tilde{H})}}(xyx^{-1}) = CM_{\frac{\tilde{H}}{N(\tilde{H})}}(y)$, $\forall x, y \in X$, it implies that $\frac{\tilde{H}}{N(\tilde{H})}$ is a normal fuzzy submultigroup of $\tilde{H}$. □

Theorem 4.5. *Suppose $\tilde{G}$ is a commutative fuzzy multigroup of X and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then, there exists a natural homomorphism $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ defined by $CM_{f(\tilde{G})}(y) = CM_{\tilde{H}}(x^{-1}y)$, $\forall\, x, y \in X$.*

Proof. Let $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ be a mapping defined by

$$CM_{f(\tilde{G})}(y) = CM_{\tilde{H}}(x^{-1}y),\ \forall\, x, y \in X.$$

That is, $CM_{f(\tilde{G})}(y) = CM_{x\tilde{H}}(y) \Rightarrow f(\tilde{G}) = x\tilde{H}$ (consequently, $f(\tilde{G}_*) = x\tilde{H}_*$). Since $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ is derived from $f : \tilde{G}_* \to \frac{\tilde{G}_*}{\tilde{H}_*}$ such that $\tilde{H}_*$ is a normal subgroup of $\tilde{G}_*$, then to prove that f is a homomorphism, we show that

$$CM_{xy\tilde{H}}(z) = CM_{x\tilde{H} \circ y\tilde{H}}(z),\ \forall z \in X \Rightarrow f(xy) = f(x)f(y).$$

Since $\tilde{H}$ is commutative, then

$$CM_{\tilde{H}}(xz) = CM_{\tilde{H}}(zx) \Rightarrow CM_{\tilde{H}}(z^{-1}xz) = CM_{\tilde{H}}(x),\ \forall z \in X.$$

It is certain that,

$$CM_{x\tilde{H}}(z) = CM_{\tilde{H}}(x^{-1}z) \text{ and } CM_{y\tilde{H}}(z) = CM_{\tilde{H}}(y^{-1}z).$$

Then

$$CM_{xy\tilde{H}}(z) = CM_{\tilde{H}}((xy)^{-1}z).$$

Now,

$$\begin{aligned} CM_{x\tilde{H}\circ y\tilde{H}}(z) &= \bigvee_{z=rs} [CM_{x\tilde{H}}(r) \wedge CM_{y\tilde{H}}(s)] \\ &= \bigvee_{z=rs} [CM_{\tilde{H}}(x^{-1}r) \wedge CM_{\tilde{H}}(y^{-1}s)]. \end{aligned}$$

Similarly,

$$\begin{aligned} CM_{xy\tilde{H}}(z) = CM_{\tilde{H}}((xy)^{-1}z) &= CM_{\tilde{H}}(y^{-1}x^{-1}z) \\ &\geq \bigvee_{z=rs} [CM_{\tilde{H}}(x^{-1}r) \wedge CM_{\tilde{H}}(y^{-1}s)]. \end{aligned}$$

Suppose by hypothesis,

$$CM_{\tilde{H}}(y^{-1}x^{-1}z) = \bigvee_{z=rs} [CM_{\tilde{H}}(x^{-1}r) \wedge CM_{\tilde{H}}(y^{-1}s)],$$

then it follows that $CM_{xy\tilde{H}}(z) = CM_{x\tilde{H}\circ y\tilde{H}}(z)$, $\forall z \in X$. Consequently, we have $f(xy) = f(x)f(y)$, $\forall x, y \in X$. Hence, f is a homomorphism. □

Corolary 4.1. *Let* $\tilde{G}, \tilde{H} \in FMG(X)$ *such that* $CM_{\tilde{G}}(x) = CM_{\tilde{G}}(y)$, $\forall x, y \in X$ *and* $CM_{\tilde{G}}(e) \geq CM_{\tilde{H}}(x)$, $\forall x \in X$. *If* $f : \tilde{G} \to \frac{\tilde{G}}{\tilde{H}}$ *is a natural homomorphism defined by* $CM_{f(\tilde{G})}(y) = CM_{\tilde{H}}(x^{-1}y)$, $\forall\, x, y \in X$, *then* $f^{-1}(f(\tilde{H})) = \tilde{G} \circ \tilde{H}$.

Proof. Let $x \in X$. To proof the result, we assume that $f(x) = f(y)$, $\forall x, y \in X$. Thus,

$$\begin{aligned} CM_{f^{-1}(f(\tilde{H}))}(x) &= \bigvee_{x\in X} [CM_{f(\tilde{H})}(f(x))] \\ &= \bigvee_{x\in X} [CM_{\tilde{H}}(f^{-1}(f(y))] \\ &= CM_{\tilde{H}}(y). \end{aligned}$$

Again,

$$\begin{aligned} CM_{\tilde{G}\circ\tilde{H}}(x) &= \bigvee_{x=zy}[CM_{\tilde{G}}(z)\wedge CM_{\tilde{H}}(y)] \\ &= \bigvee_{x\in X}[CM_{\tilde{G}}(xy^{-1})\wedge CM_{\tilde{H}}(y)] \\ &= \bigvee_{x\in X}[CM_{\tilde{G}}(e)\wedge CM_{\tilde{H}}(y)] \\ &= CM_{\tilde{H}}(y). \end{aligned}$$

Hence, the proof follows. □

Remark 4.4. Assuming there is a bijective correspondence between every (normal) fuzzy submultigroup of $\tilde{G}$ that contains $\tilde{H}$ and the (normal) fuzzy submultigroups of $\frac{\tilde{G}}{\tilde{H}}$. That is, if $\tilde{I}$ is a (normal) fuzzy submultigroup of $\tilde{G}$ containing $\tilde{H}$, then the corresponding (normal) fuzzy submultigroup of $\frac{\tilde{G}}{\tilde{H}}$ is $f(\tilde{I})$.

Theorem 4.6. *Let X and Y be groups, $f : X \to Y$ be an isomorphism and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}\in FMG(X)$ such that $CM_{\tilde{G}}(x) = CM_{\tilde{G}}(y)\forall x, y \in X$ with $ker f = \{e\}$. Then $\frac{\tilde{G}}{\tilde{H}} \cong \frac{f(\tilde{G})}{f(\tilde{H})}$.*

Proof. By Theorem 2.2 and Definition 4.1, $\frac{\tilde{G}}{\tilde{H}}$ and $\frac{f(\tilde{G})}{f(\tilde{H})}$ are fuzzy multigroups, respectively.

Let $h : \frac{\tilde{G}}{\tilde{H}} \to \frac{f(\tilde{G})}{f(\tilde{H})}$ be defined by $h(\tilde{H}x) = f(\tilde{H})(f(x))$, $\forall x \in X$. If $\tilde{H}x = \tilde{H}y$, then $CM_{\tilde{H}}(xy^{-1}) = CM_{\tilde{H}}(e)$. Since $ker f = \{e\}$ meaning $ker f \subseteq A^*$, then $f^{-1}(f(\tilde{H})) = \tilde{H}$ by Proposition 2.2. Thus,

$$CM_{f^{-1}(f(\tilde{H}))}(xy^{-1}) = CM_{f^{-1}(f(\tilde{H}))}(e)$$

$\Rightarrow$

$$CM_{f(\tilde{H})}(f(xy^{-1})) = CM_{f(\tilde{H})}(f(e))$$

$\Rightarrow$

$$CM_{f(\tilde{H})}(f(x)(f(y))^{-1}) = CM_{f(\tilde{H})}(f(e))$$

$\Rightarrow$

$$CM_{f(\tilde{H})}(f(x)) = CM_{f(\tilde{H})}(f(y)e') \text{ (where } f(e) = e').$$

Hence,

$$CM_{f(\tilde{H})}(f(x)) = CM_{f(\tilde{H})}(f(y)) \Rightarrow f(\tilde{H})(f(x)) = f(\tilde{H})(f(y)).$$

Hence, h is well-defined. It is also a homomorphism because

$$\begin{aligned} h(\tilde{H}x\tilde{H}y) = h(\tilde{H}xy) &= f(\tilde{H})(f(xy)) \\ &= f(\tilde{H})(f(x)f(y)) \\ &= f(\tilde{H})(f(x))f(\tilde{H})(f(y)) \\ &= h(\tilde{H}x)h(\tilde{H}y). \end{aligned}$$

Suppose f is an epimorphism, then $\exists\, x \in X$ such that $f(x) = y$. Thus,

$$h(\tilde{H}x) = f(\tilde{H})(f(x)) = f(\tilde{H})(y).$$

Moreover,

$$f(\tilde{H})(f(x)) = f(\tilde{H})(f(y)) \Rightarrow CM_{f(\tilde{H})}(f(x)(f(y))^{-1}) = CM_{f(\tilde{H})}(e') \Rightarrow$$

$$CM_{f(\tilde{H})}(f(xy^{-1})) = CM_{f(\tilde{H})}(f(e)) \Rightarrow CM_{f^{-1}(f(\tilde{H}))}(xy^{-1}) = CM_{f^{-1}(f(\tilde{H}))}(e)$$

implies $CM_{\tilde{H}}(xy^{-1}) = CM_{\tilde{H}}(e) \Rightarrow \tilde{H}x = \tilde{H}y$, which proves that h is an isomorphism. Hence, the result follows. □

Corolary 4.2. *Let $f : X \to Y$ be an isomorphism and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G} \in FMG(Y)$ such that $CM_{\tilde{G}}(x) = CM_{\tilde{G}}(y)$, $\forall x, y \in Y$. Then $\frac{f(\tilde{G})}{f(\tilde{H})} \simeq \frac{\tilde{G}}{\tilde{H}}$.*

Proof. By Theorem 2.2, $f(\tilde{G}), f(\tilde{H}) \in FMG(X)$ and $\frac{f(\tilde{G})}{f(\tilde{H})}$ and $\frac{f(\tilde{G})}{f(\tilde{H})}$ are fuzzy multigroups by Definition 4.1.

Again, since $\tilde{H} \in FMG(Y)$, then

$$f(f^{-1}(\tilde{H})) = \tilde{H}.$$

If $x \in kerf$, then $f(x) = e' = f(e)$, and so

$$CM_{\tilde{H}}(f(x)) = CM_{\tilde{H}}(f(e)) \Rightarrow CM_{f^{-1}(\tilde{H})}(x) = CM_{f^{-1}(\tilde{H})}(e).$$

Hence, $kerf \subseteq f^{-1}(\tilde{H}^*)$. The proof is completed following the same logic as in Theorem 4.6. □

Theorem 4.7. *Let $\tilde{H}, \tilde{G} \in FMG(X)$ and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then $\frac{\tilde{G}}{\tilde{G}_*} \approx \frac{\tilde{G}}{\tilde{H}}$.*

Proof. Let f be a natural homomorphism from $\tilde{G}_*$ onto $\frac{\tilde{G}_*}{\tilde{H}_*}$ defined by $f(x\tilde{H}_*) = x\tilde{H}_*\ \forall x \in \tilde{G}_*$. Then, we have

$$CM_{f(\frac{\tilde{G}}{\tilde{G}_*})}(x\tilde{H}_*) = \vee[CM_{\frac{\tilde{G}}{\tilde{G}_*}}(z)],\ \forall z \in \tilde{G}_*, f(z) = x\tilde{H}_*.$$

Since $\frac{\tilde{G}}{\tilde{G}_*}$ and $\tilde{G}$ are bijective correspondence to each other (by Remark 4.4) and $z = f^{-1}(x\tilde{H}_*) = x\tilde{H}_*$, it follows that

$$\begin{aligned} CM_{f(\frac{\tilde{G}}{\tilde{G}_*})}(x\tilde{H}_*) &= \vee[CM_{\frac{\tilde{G}}{\tilde{G}_*}}(z)],\ \forall z \in \tilde{G}_*, f(z) = x\tilde{H}_* \\ &= \vee[CM_{\tilde{G}}(y)],\ \forall y \in x\tilde{H}_* \\ &= CM_{\frac{\tilde{G}}{\tilde{H}}}(x\tilde{H}_*),\ \forall x \in \tilde{G}_*, \end{aligned}$$

because $\frac{\tilde{G}}{\tilde{H}}$ and $\tilde{G}$ are bijective correspondence to each other. Hence, it follows that $\frac{\tilde{G}}{\tilde{G}_*} \approx \frac{\tilde{G}}{\tilde{H}}$. □

Lemma 4.1. *Suppose $f : X \to Y$ and $\tilde{G} \in FMG(X)$, then $(f(\tilde{G}))_* = f(\tilde{G}_*)$.*

Proof. Straightforward. □

Theorem 4.8. *Let $\tilde{G} \in FMG(X)$. Suppose Y is a group and $\tilde{I} \in FMG(Y)$ such that $\tilde{G} \approx \tilde{I}$. Then, there exists a normal fuzzy submultigroup $\tilde{H}$ of $\tilde{G}$ such that $\frac{\tilde{G}}{\tilde{H}} \cong \frac{\tilde{I}}{\tilde{I}_*}$.*

Proof. Since $\tilde{G} \approx \tilde{I}, \exists$ an epimorphism f of X onto Y such that $f(\tilde{G}) = \tilde{I}$. Define $\tilde{H} \in FMG(X)$ as follows: $\forall\, x \in X$,

$$CM_{\tilde{H}}(x) = \begin{cases} CM_{\tilde{G}}(x) & \text{if } x \in ker f \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $\tilde{H} \subseteq \tilde{G}$. If $x \in ker f$, then $yxy^{-1} \in ker f,\ \forall y \in X$, and so

$$CM_{\tilde{H}}(yxy^{-1}) = CM_{\tilde{G}}(yxy^{-1}) = CM_{\tilde{G}}(x) = CM_{\tilde{H}}(x),\ \forall y \in X.$$

If $x \notin ker f$, then $CM_{\tilde{H}}(x) = 0$ and so

$$CM_{\tilde{H}}(yxy^{-1}) = CM_{\tilde{H}}(x) = 0,\ \forall y \in X.$$

Hence, $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{G}$. Also, $\tilde{G} \approx \tilde{I} \Rightarrow f(\tilde{G}) = \tilde{I}$ which further implies $(f(\tilde{G}))_* = \tilde{I}_*$ and $f(\tilde{G}_*) = \tilde{I}_*$ by Lemma 4.1. Let $f = g$. Then g is a homomorphism of $\tilde{G}_*$ onto $\tilde{I}_*$ and $ker\, g = \tilde{H}_*$. Thus, there exists an isomorphism h of $\frac{\tilde{G}_*}{\tilde{H}_*}$ onto $\tilde{I}_*$ such that $h(x\tilde{H}_*) = g(x) = f(x),\ \forall x \in \tilde{G}_*$. For such an h, we have

$$\begin{aligned} CM_{h(\frac{\tilde{G}}{\tilde{H}})}(z) &= \vee[CM_{\frac{\tilde{G}}{\tilde{H}}}(x\tilde{H}_*)],\ \forall x \in \tilde{G}_*, h(x\tilde{H}_*) = z \\ &= \vee(\vee[CM_{\tilde{G}}(y)],\ \forall y \in x\tilde{H}_*), \forall x \in \tilde{G}_*, g(x) = z \\ &= \vee[CM_{\tilde{G}}(y)],\ \forall y \in \tilde{G}_*, g(y) = z \\ &= \vee[CM_{\tilde{G}}(y)],\ \forall y \in X, f(y) = z \\ &= CM_{\tilde{G}}(f^{-1}(z)) = CM_{f(\tilde{G})}(z) = CM_{\tilde{I}}(z),\ \forall z \in \tilde{I}_*. \end{aligned}$$

Therefore, $\frac{\tilde{G}}{\tilde{H}} \cong \frac{\tilde{I}}{\tilde{I}_*}$. □

Theorem 4.9. *Suppose $\tilde{G}$ is a fuzzy multigroup of X and $\tilde{H}$ a normal fuzzy submultigroup of $\tilde{G}$. Then $\frac{\tilde{G}}{(\tilde{H}\cap\tilde{G})} \simeq \frac{(\tilde{H}\circ\tilde{G})}{\tilde{H}}$.*

Proof. From Definition 2.13, it is easy to infer that $\tilde{H}_*$ is also a normal subgroup of X. By the Second Isomorphism Theorem for groups, we deduce

$$\frac{\tilde{G}_*}{\tilde{H}_* \cap \tilde{G}_*} \cong \frac{\tilde{H}_*\tilde{G}_*}{\tilde{H}_*}.$$

Assume that

$$(\tilde{H} \cap \tilde{G})_* = \tilde{H}_* \cap \tilde{G}_* \text{ and } (\tilde{H} \circ \tilde{G})_* = \tilde{H}_*\tilde{G}_*.$$

Consequently, we have

$$\frac{\tilde{G}_*}{(\tilde{H} \cap \tilde{G})_*} \cong \frac{(\tilde{H} \circ \tilde{G})_*}{\tilde{H}_*},$$

where f is given by

$$f(x(\tilde{H} \cap \tilde{G})_*) = x\tilde{H}_*,\ \forall x \in \tilde{G}_*.$$

Thus,

$$\begin{aligned} CM_{f(\frac{\tilde{G}}{\tilde{H}\cap\tilde{G}})}(y\tilde{H}_*) &= C_{\frac{\tilde{G}}{\tilde{H}\cap\tilde{G}}}(y(\tilde{H} \cap \tilde{G})_*) \\ &= \vee[CM_{\tilde{G}}(z)],\ \forall z \in y(\tilde{H} \cap \tilde{G})_* \\ &\leq \vee[CM_{\tilde{H}\circ\tilde{G}}(z)],\ \forall z \in y(\tilde{H}_* \cap \tilde{G}_*) \\ &\leq \vee[CM_{\tilde{H}\circ\tilde{G}}(z)],\ \forall z \in y\tilde{H}_* \\ &= CM_{\frac{\tilde{H}\circ\tilde{G}}{\tilde{H}}}(y\tilde{H}_*),\ \forall y \in \tilde{G}_*. \end{aligned}$$

Hence, $f(\frac{\tilde{G}}{(\tilde{H}\cap\tilde{G})}) \subseteq \frac{\tilde{H}\circ\tilde{G}}{\tilde{H}}$. Therefore, $\frac{\tilde{G}}{(\tilde{H}\cap\tilde{G})} \simeq \frac{(\tilde{H}\circ\tilde{G})}{\tilde{H}}$. □

Theorem 4.10. *Let $\tilde{H}, \tilde{I}, \tilde{G} \in FMG(X)$ such that $\tilde{H} \subseteq \tilde{I}$, and $\tilde{H}$ and $\tilde{I}$ are normal fuzzy submultigroups of $\tilde{G}$. Then $(\frac{\tilde{G}}{\tilde{H}})/(\frac{\tilde{I}}{\tilde{H}}) \cong \frac{\tilde{G}}{\tilde{I}}$.*

Proof. If $\tilde{H}, \tilde{I} \in FMG(X)$ and $\tilde{H}$ is a normal fuzzy submultigroup of $\tilde{I}$, then $\tilde{H}_*$ is a normal subgroup of $\tilde{I}_*$ and both $\tilde{H}_*$ and $\tilde{I}_*$ are normal subgroups of $\tilde{G}_*$. From the principle of Third Isomorphism Theorem for groups, it follows that

$$(\frac{\tilde{G}_*}{\tilde{H}_*})/(\frac{\tilde{I}_*}{\tilde{H}_*}) \cong (\frac{\tilde{G}_*}{\tilde{I}_*}),$$

where f is given by

$$f(x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*})) = x\tilde{I}_*,\ \forall x \in \tilde{G}_*.$$

Then

$$
\begin{aligned}
CM_{f((\frac{\tilde{G}}{\tilde{H}})/(\frac{\tilde{I}}{\tilde{H}}))}(x\tilde{I}_*) &= CM_{(\frac{\tilde{G}}{\tilde{H}})/(\frac{\tilde{I}}{\tilde{H}})}(x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*})) \\
&= \vee[CM_{\frac{\tilde{G}}{\tilde{H}}}(y\tilde{H}_*)],\ \forall y \in \tilde{G}_*, y\tilde{H}_* \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[\vee(CM_{\tilde{G}}(z)), \forall z \in y\tilde{H}_*],\ \forall y \in \tilde{G}_*,\ y\tilde{H}_* \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in \tilde{G}_*, z\tilde{H}_* \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in x\tilde{H}_*(\frac{\tilde{I}_*}{\tilde{H}_*}) \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in \tilde{G}_*, f(z) \in x\tilde{I}_* \\
&= \vee[CM_{\tilde{G}}(z)],\ \forall z \in \tilde{G}_*, f(z) = z \\
&= CM_{\frac{\tilde{G}}{\tilde{I}}}(x\tilde{I}_*),
\end{aligned}
$$

$\forall x \in \tilde{G}_*$, where the equalities hold since f is one-to-one. Hence, the result follows. $\square$

5 Conclusions

In this paper, the ideas of fuzzy comultisets and quotient fuzzy multigroups have been proposed in an attempt to further strengthen the theory of fuzzy multigroups. A number of some related results were duly discussed in details. The connection between fuzzy comultisets of fuzzy multigroups and the cosets of groups has been proven. Some characterizations of fuzzy comultisets were delineated with verifications. In addition, quotient fuzzy multigroup was explicated and some of its properties were explored. Finally, group theoretic isomorphism theorems were substantiated in fuzzy multigroup setting. Nevertheless, some group analog concepts could still be exploited in fuzzy multigroup context.

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