

Some properties of certain Subhypergroups

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Abstract

The structure of the hypergroup is much more complicated than that of the group. Thus there exist various kinds of subhypergroups. This paper deals with some of these subhypergroups and presents certain properties of the closed, invertible and ultra-closed subhypergroups.

Key words: Hypergroup, Subhypergroup.

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1 Introduction

In 1934 F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [11]. An *operation* or *composition* in a non-void set H is a function from $H \times H$ to H , while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset $P(H)$ of H . An algebraic structure that satisfies the axioms:

- i. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in H$ (associative axiom) and
- ii. $a \cdot H = H \cdot a = H$ for every $a \in H$ (reproductive axiom).

is called *group* if \cdot is a composition [16] and *hypergroup* if \cdot is a hypercomposition [11]. When there is no likelihood of confusion \cdot can be omitted. If A and B are subsets of H , then AB signifies the union $\bigcup_{(a,b) \in A \times B} ab$, in particular if $A = \emptyset$ or $B = \emptyset$ then $AB = \emptyset$. Ab and aB have the same meaning as $A\{b\}$ and $\{a\}B$. In general, the singleton $\{a\}$ is identified with its member a .

Proposition 1.1. *If a non-void set H is endowed with a composition which satisfies the associative and the reproductive axioms, then H has a bilateral neutral element and any element in H has a bilateral symmetric.*

Proof. Let $x \in H$. Per reproductive axiom $x \in xH$. Therefore there exists $e \in H$ such that $xe = x$. Next, let y be an arbitrary element in H . Per reproductive axiom there exists $z \in H$ such that $y = zx$. Consequently $ye = (zx)e = z(xe) = zx = y$. Hence e is a right neutral element. In an analogous way there exists a left neutral element e' . Then the equality $e = e'e = e'$ is valid. Therefore e is the bilateral neutral element of H . Now, per reproductive axiom $e \in xH$. Thus there exists $x' \in H$, such that $e = xx'$. Hence any element in H has a right symmetric. Similarly any element in H has a left symmetric and it is easy to prove that these two symmetric elements coincide. \square

Remark 1.2. An analogous Proposition to Proposition 1.1 is not valid when H is endowed with a hypercomposition. In hypergroups there exist different types of neutral elements [15] (e.g. scalar [4], strong [8,17] ect). There also exist special types of hypergroups which have a neutral element and each one of their elements has one or more symmetric. Such hypergroups are for example the canonical hypergroups [21], the quasicanonical hypergroups [12], the fortified join hypergroups [17], the fortified transposition hypergroups [8], the transposition polysymmetrical hypergroups [19], the canonical polysymmetrical hypergroups [14], etc.

Proposition 1.3. *If H is a hypergroup, then $ab \neq \emptyset$ is valid for all the elements a, b of H .*

Proof. Suppose that $ab = \emptyset$ for some $a, b \in H$. Per reproductive axiom, $aH = H$ and $bH = H$. Hence, $H = aH = a(bH) = (ab)H = \emptyset H = \emptyset$, which is absurd. \square

In [11], F. Marty also defined the two induced hypercompositions (right and left division) that result from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|b} = \{x \in H | a \in xb\} \text{ and } \frac{a}{|b} = \{x \in H | a \in bx\}.$$

It is obvious that the two induced hypercompositions coincide, if the hypergroup is commutative. For the sake of notational simplicity, a/b or $a : b$ is used for right division and $b \backslash a$ or $a..b$ for left division [7, 13].

Proposition 1.4. *If H is a hypergroup, then $a/b \neq \emptyset$ and $b \backslash a \neq \emptyset$ for all the elements a, b of H .*

Proof. Per reproductive axiom, $Hb = H$ for all $b \in H$. Hence, for every $a \in H$ there exists $x \in H$, such that $a \in xb$. Thus, $x \in a/b$ and, therefore, $a/b \neq \emptyset$. Dually, $b \setminus a \neq \emptyset$. \square

In Proposition 2.3 of [13] the following properties were proved for any hypergroup H (see also Proposition 1 in [7])

Proposition 1.5. i) $(a/b)/c = a/(cb)$ and $c \setminus (b \setminus a) = (bc) \setminus a$, for all $a, b, c \in H$.

ii) $b \in (a/b) \setminus a$ and $b \in a / (b \setminus a)$, for all $a, b \in H$.

In [7] and then in [8] a principle of duality is established in the theory of hypergroups and in the theory of transposition hypergroups as follows:

Given a theorem, the dual statement which results from the interchanging of the order of the hypercomposition . (and necessarily interchanging of the left and the right division), is also a theorem.

This principle is used throughout this paper.

2 Closed, invertible and ultra-closed subhypergroups

The structure of the hypergroup is much more complicated than that of the group. There are various kinds of subhypergroups. In particular a non-empty subset K of H is called semi-subhypergroup when it is stable under the hypercomposition, i.e. it has the property $xy \subseteq K$ for all $x, y \in K$. K is a subhypergroup of H if it satisfies the reproductive axiom, i.e. if the equality $xK = Kx = K$ is valid for all $x \in K$ (for the fuzzy case see e.g [3]). This means that when K is a subhypergroup and $a, b \in K$, the relations $a \in bx$ and $a \in yb$ always have solutions in K . Although the non-void intersection of two subhypergroups is stable under the hypercomposition, it usually is not a subhypergroup since the reproductive axiom fails to be valid for it. This led, from the very early steps of hypergroup theory, to the consideration of more special types of subhypergroups. One of them is the *closed subhypergroup* (e.g. see [5], [9]). A subhypergroup K of H is called *left closed* with respect to H if for any two elements a and b in K , all the solutions of the relation $a \in yb$ lie in K . This means that K is left closed if and only if $a/b \subseteq K$, for all $a, b \in K$ (see [13]). Similarly K is *right closed* when all the solutions of the relation $a \in bx$ lie in K or equivalently if $b \setminus a \subseteq K$, for all $a, b \in K$ [13]. Finally K is *closed* when it is both right and left closed. In the case of the closed subhypergroups, the non-void intersection of any family of closed

subhypergroups is a closed subhypergroup. It must be mentioned though that a hypergroup may have subhypergroups, but no proper closed ones. For example if Q is a quasi-order hypergroup [6], a^2 is a subhypergroup of Q , for each $a \in Q$, but $a/a = a \setminus a = Q$ for all $a \in Q$. Also fortified transposition hypergroups [8, 17] consisting only of attractive elements have no proper closed subhypergroups [18].

Proposition 2.1. *If K is a subset of a hypergroup H such that $a/b \subseteq K$ and $b \setminus a \subseteq K$, for all $a, b \in K$, then K is a subhypergroup of H .*

Proof. Let a be an element of K . It must be shown that $aK = Ka = K$. Suppose that $x \in K$. Then $a \setminus x \subseteq K$, therefore $x \in aK$, hence $K \subseteq aK$. For the reverse inclusion now suppose that $y \in aK$. Then $K/y \subseteq K/aK$. So $K \cap (K/aK)y \neq \emptyset$. Thus, $y \in (K/aK) \setminus K$. Per Proposition 1.4 (i) the equality $K/aK = (K/K)/a$ is valid. Thus $(K/aK) \setminus K = ((K/K)/a) \setminus K \subseteq (K/a) \setminus K \subseteq (K/K) \setminus K \subseteq K \setminus K \subseteq K$. Hence $y \in K$ and so $aK \subseteq K$. Therefore $aK = K$. The equality $Ka = K$ follows by duality. \square

In [13] it is also proved that the equalities

$$K = K/a = a/K = a \setminus K = K \setminus a$$

are valid for every element a of a closed subhypergroup K .

Next some properties of these subhypergroups will be presented.

Proposition 2.2. *If K is a subhypergroup of H , then $H - K \subseteq (H - K)s$ and $H - K \subseteq s(H - K)$, for all $s \in K$.*

Proof. Let r be an element in $H - K$ which does not belong to $(H - K)s$. Because of the reproductive axiom, $r \in Hs$ and since $r \notin (H - K)s$, r must be a member of Ks . Thus, $r \in Ks \subseteq KK = K$. This contradicts the assumption and so $H - K \subseteq (H - K)s$. The second inclusion follows by duality. \square

Proposition 2.3. (i) *A subhypergroup K of H is left closed in H , if and only if $(H - K)s = H - K$ for all $s \in K$.*

(ii) *A subhypergroup K of H is right closed in H , if and only if $s(H - K) = H - K$ for all $s \in K$.*

(iii) *A subhypergroup K of H is closed in H , if and only if $s(H - K) = (H - K)s = H - K$ for all $s \in K$.*

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Proof. (i) Let K be left closed in H . Suppose that z lies in $H - K$ and assume that $zs \cap K \neq \emptyset$. Then, there exists an element y in K such that $y \in zs$, or equivalently, $z \in y/s$. Therefore $z \in K$, which is absurd. Hence $(H - K)s \subseteq H - K$. Next, because of Proposition 1, $H - K \subseteq (H - K)s$ and therefore $H - K = (H - K)s$. Conversely now. Suppose that $(H - K)s = H - K$ for all $s \in K$. Then $(H - K)s \cap K = \emptyset$ for all $s \in K$. Hence $x \notin rs$ and so $r \notin x/s$ for all $x, s \in K$ and $r \in H - K$. Therefore $x/s \cap (H - K) = \emptyset$ which implies that $x/s \subseteq K$. Thus K is closed in H . (ii) follows by duality and (iii) is an obvious consequence of (i) and (ii). \square

Corollary 2.4. (i) If K is a left closed subhypergroup in H , then $xK \cap K = \emptyset$, for all $x \in H - K$.

(ii) If K is a right closed subhypergroup in H , then $Kx \cap K = \emptyset$, for all $x \in H - K$.

(iii) If K is a closed subhypergroup in H , then $xK \cap K = \emptyset$ and $Kx \cap K = \emptyset$, for all $x \in H - K$.

Proposition 2.5. If K is a subhypergroup of H , $A \subseteq K$ and $B \subseteq H$, then (i) $A(B \cap K) \subseteq AB \cap K$ and (ii) $(B \cap K)A \subseteq BA \cap K$.

Proof. Let $t \in A(B \cap K)$. Then $t \in ax$, with $a \in A$ and $x \in B \cap K$. Since x lies in $B \cap K$, it derives that $x \in B$ and $x \in K$. Hence $ax \subseteq aB$ and $ax \subseteq aK = K$. Thus $ax \subseteq AB \cap K$ and therefore $t \in AB \cap K$. Duality gives (ii) and so the Proposition. \square

Proposition 2.6. (i) If K is a left closed subhypergroup in H , $A \subseteq K$ and $B \subseteq H$, then $(B \cap K)A = BA \cap K$.

(ii) If K is a right closed subhypergroup in H , $A \subseteq K$ and $B \subseteq H$, then $A(B \cap K) = AB \cap K$.

Proof. (i) Let $t \in BA \cap K$. Since K is right closed, for any element y in $B - K$, it is valid that $yA \cap K \subseteq yK \cap K = \emptyset$. Hence $t \in (B \cap K)A \cap K$. But $(B \cap K)A \subseteq KK = K$. Thus $t \in (B \cap K)A$. Therefore $BA \cap K \subseteq (B \cap K)A$. Next the inclusion becomes equality because of Proposition 2.5. (ii) derives from the duality. \square

Proposition 2.7. (i) If K is a left closed subhypergroup in H , $A \subseteq K$ and $B \subseteq H$, then $(B \cap K)/A = (B/A) \cap K$.

(ii) If K is a right closed subhypergroup in H , $A \subseteq K$ and $B \subseteq H$, then $(B \cap K) \setminus A = B \setminus A \cap K$.

Proof. (i) Since $B \cap K \subseteq B$, it derives that $(B \cap K)/A \subseteq B/A$. Moreover $A \subseteq K$ and $B \cap K \subseteq K$, thus $(B \cap K)/A \subseteq K$. Hence $(B \cap K)/A \subseteq$

$(B/A) \cap K$. For the reverse inclusion now suppose that $x \in (B/A) \cap K$. Then, there exist $a \in A$, $b \in B$ such that $x \in b/a$ or equivalently $b \in ax$. Since $ax \subseteq K$ it derives that $b \in K$ and so $b \in B \cap K$. Therefore $b/a \subseteq (B \cap K)/A$. Thus $x \in (B \cap K)/A$. Hence $(B/A) \cap K \subseteq (B \cap K)/A$, QED. Duality gives (ii) and so the Proposition. \square

Krasner generalized the notion of the closed subhypergroups, considering closed subhypergroups in other subhypergroups [9]. Let us define the restriction of the right and left division in subset A of a hypergroup H as follows:

$$a/_A b = \{x \in A | a \in xb\} \text{ and } b \backslash_A a = \{x \in A | a \in bx\}$$

Thus, if K is a subhypergroup of H and $K \subseteq A$, then K is right closed in A , if $b \backslash_A a \subseteq K$ for all $a, b \in K$ and K is left closed in A , if $a/_A b \subseteq K$ for all $a, b \in K$.

Proposition 2.8. *Let K, M be two subhypergroups of a hypergroup H , such that $K \subseteq M$. If K is left (or right) closed in M and M is left (or right) closed in H , then K is left (or right) closed in H .*

Proof. Since K is left closed in M , the inclusion $a/_M b \subseteq K$ is valid, for all $a, b \in K$. This means that if x is an element of M such that $a \in xb$, then $x \in K$. Next if there exists $y \in H - M$ such that $a \in yb$, then a/b will not be a subset of M . Hence M will not be left closed in H . This contradicts the assumption, and so the Proposition. \square

Corolary 2.9. *Let K, M be two subhypergroups of a hypergroup H , such that $K \subseteq M$. If K is closed in M and M is closed in H , then K is closed in H .*

Proposition 2.10. *Let K, M be two subhypergroups of a hypergroup H and suppose that K is left (or right) closed in H . Then $K \cap M$ is left (or right) closed in M .*

Proof. Let $a, b \in K \cap M$. Then $a/b = \{x \in H | a \in xb\} \subseteq K$. Hence $\{x \in M | a \in xb\} \subseteq K \cap M$. Therefore $a/_M b \subseteq K \cap M$. Thus $K \cap M$ is left closed in M . \square

Corolary 2.11. *Let K, M be two subhypergroups of a hypergroup H and suppose that K is closed in H . Then $K \cap M$ is closed in M .*

Proposition 2.12. *If two subhypergroups K, M of a hypergroup H are left (or right) closed in H and their intersection is not void, then $K \cap M$ is left (or right) closed in M .*

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Proof. Let $a, b \in K \cap M$. Since K, M are left closed in H , $a/b = \{x \in H \mid a \in xb\}$ is a subset of both K and M . Hence $a/b \subseteq K \cap M$ and so the Proposition. \square

Corolary 2.13. *The non-void intersection of two closed subhypergroups is a closed subhypergroup.*

The next type of hypergroups was introduced by Drescher and Ore in [5] and immediately after that, M. Krasner used them in [9]. In both [5] and [9] they are named reversible subhypergroups. In our days these subhypergroups are called invertible. The Definition that follows was given by Jantosciak in [7].

Definition 2.14. A subhypergroup K of a hypergroup H is **right invertible** if $a/b \cap K \neq \emptyset$, implies that $b/a \cap K \neq \emptyset$, $a, b \in H$. K is **left invertible** if $b \setminus a \cap K \neq \emptyset$, implies that $a \setminus b \cap K \neq \emptyset$, $a, b \in H$. If K is both right and left invertible, then it is called **invertible**.

Theorem 4 in [1] gives an interesting example of an invertible subhypergroup in a join hypergroup of partial differential operators. Moreover the closed subhypergroups of the quasicanonical or of the canonical hypergroups are invertible [21].

Direct consequences of the above definition are the following propositions:

Proposition 2.15. (i) *K is right invertible in H , if and only if the following implication is valid: $b \in Ka \Rightarrow a \in Kb$, $a, b \in H$.*

(ii) *K is left invertible in H , if and only if the following implication is valid: $b \in aK \Rightarrow a \in bK$, $a, b \in H$.*

Proposition 2.16. (i) *K is right invertible in H , if and only if the following implication is valid: $Ka \neq Kb \Rightarrow Ka \cap Kb = \emptyset$, $a, b \in H$.*

(ii) *K is left invertible in H , if and only if the following implication is valid: $aK \neq bK \Rightarrow aK \cap bK = \emptyset$, $a, b \in H$.*

Proposition 2.17. *If K is right (left) invertible in H , then K is right (left) closed in H .*

In [2] one can find examples of closed hypergroups that are not invertible.

Definition 2.18. A subhypergroup K of a hypergroup H is **right ultra-closed** if it is right closed and $a/a \subseteq K$ for each $a \in H$. K is **left ultra-closed** if it is left closed and $a \setminus a \subseteq K$ for each $a \in H$. If K is both right and left ultra-closed, then it is called **ultra-closed**.

Proposition 2.19. (i) *If K is right ultra-closed in H , then either $a/b \subseteq K$ or $a/b \cap K = \emptyset$, for all $a, b \in H$. Moreover if $a/b \subseteq K$, then $b/a \subseteq K$.*

(ii) *If K is left ultra-closed in H , then either $b \setminus a \subseteq K$ or $b \setminus a \cap K = \emptyset$, for all $a, b \in H$. Moreover if $b \setminus a \subseteq K$, then $a \setminus b \subseteq K$.*

Proof. Suppose that $a/b \cap K \neq \emptyset$, $a, b \in H$. Then $a \in kb$, for some $k \in K$. Next assume that $b/a \cap (H - K) \neq \emptyset$. Then $b \in ra$, $r \in H - K$. Thus $a \in k(ra) = (kr)a$. Since K is right closed, per Proposition 2.3, $kr \subseteq H - K$. So $a \in va$, for some $v \in H - K$. Therefore $a/a \cap (H - K) \neq \emptyset$, which is absurd. Hence $b/a \subseteq K$. Now let there be x in K such that $b \in xa$. If $a/b \cap (H - K) \neq \emptyset$, there exists $y \in H - K$ such that $a \in yb$. Therefore $b \in x(yb) = (xy)b$. Since K is right closed, per Proposition 2.3, $xy \subseteq H - K$. So $b \in zb$, for some $z \in H - K$. Therefore $b/b \cap (H - K) \neq \emptyset$, which is absurd. Hence $a/b \subseteq K$. Duality gives (ii). \square

Corolary 2.20. *If K is right (left) ultra-closed in H , then K is right (left) invertible in H .*

Ultra-closed subhypergroups were introduced by Y. Sureau [22] (see also [2, 20]). The following Proposition proves that the above given definition is equivalent to the definition used by Sureau:

Proposition 2.21. (i) *K is right ultra-closed in H , if and only if*

$$Ka \cap (H - K)a = \emptyset \text{ for all } a \in H.$$

(ii) *K is left ultra-closed in H , if and only if $aK \cap a(H - K) = \emptyset$ for all $a \in H$.*

Proof. Suppose that K is right ultra-closed in H . Then $a/a \subseteq K$ for all $a \in H$. Since K is right closed, $(a/a)/k \subseteq K$ is valid, or equivalently $a/(ak) \subseteq K$ for all $k \in K$. Proposition 2.19 yields $(ak)/a \subseteq K$ for all $k \in K$. If $Ka \cap (H - K)a \neq \emptyset$, then there exist $k \in K$ and $v \in H - K$, such that $ka \cap va \neq \emptyset$, which implies that $v \in ak/a$. But $(ak)/a \subseteq K$, hence $v \in K$ which is absurd. Conversely now: Let $Ka \cap (H - K)a = \emptyset$ for all $a \in H$. If $a \in K$, then $K \cap (H - K)a = \emptyset$. Therefore $k \notin ra$, for each $k \in K$ and $r \in H - K$. Equivalently $k/a \cap (H - K) = \emptyset$, for all $k \in K$. Hence $k/a \subseteq K$ for all $k \in K$ and $a \in K$. So K is right closed. Next suppose that $a/a \cap (H - K) \neq \emptyset$ for some $a \in H$. Then $a \in (H - K)a$, or $Ka \subseteq K(H - K)a$. Since K is closed, per Proposition 2.3, $K(H - K) \subseteq H - K$ is valid. Thus $Ka \subseteq (H - K)a$, which contradicts the assumption. Duality gives (ii). \square

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