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Hypermatrix Based on Krasner Hypervector Spaces

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Abstract

In this paper we extend a very specific class of hypervector spaces called Krasner hypervector spaces in order to obtain a hypermatrix. For reaching to this goal, we will define dependent and independent vectors in this kind of hypervector space and define basis and dimension for it. Also, by using multivalued linear transformations, we examine the possibility of existing a free object here. Finally, we study the fundamental relation on Krasner hypervector spaces and we define a functor.

Key words: Hypermatrix, Hypervector spaces, Basis of a hypervector space, Multivalued linear transformations.

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1 Introduction

The notion of a hypergroup was introduced by F. Marty in 1934 [5]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more details see [2],[3]). Using hyperstructures

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theory, mathematicians have defined and studied variety of algebraic structures. Among them the notion of hypervector spaces has been studied mainly by Vougiuklis [8, 9], Tallini [6, 7] and Krasner [3].(see also [1]). There are differences mainley about operation or a hyperoperation in these three type of hypervector spaces. Vougiuklis has studied H_V vector spaces which deals with a very weak condition regarding intersections. Tallini defined a hypervector spaces considering a crisp sum and using a hyperexternal operation which assigns to the production every element of a field and every element of the abelian geroup $(V, +)$, a non empty subset of V, while Krasner in the definition of a hypervector space used a hypersum to make a canonical hypergroup and by using a singlevalued operation he defined the Krasner hypervector space with some definitions.
In this paper we have chosen the definition of Krasner and we defined the generalized subset of it. Also, to make a correct logical relation between definitions we had to define the notion of a multivalued linear transformation and by using this notion we could talk about basis and dimension of a Krasner hypervector space. In the sequel, considering the multivalued functions, we have constructed a kind of matrix whith hyperarrays with coefficients taken from the hyperfield of Krasner and elements of the basis. Also, we studied the notion of singular and nonsingular transformations. Finally, we studied the category of Krasner hypervector spaces and defines the fundamental relation on it. In the last part we have defines a functor.

2 Preliminaries

In this section we present definitions and properties of hypervector spaces and subsets, that we need for developing our paper.

A mapping $\circ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation* (or a join operation), where $P^*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B\}$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively. Generally, the singleton $\{a\}$ is identified by its element a.

A hypergroupoid $(H, \circ)$, which is associative, i. e, $x \circ (y \circ z) = (x \circ y) \circ z$, $\forall x, y, z \in H$ is called a semihypergroup. A hypergroup is a semihypergroup such that for all $x \in H$, we have $x \circ H = H = H \circ x$, which is called reproduction axiom.

Definition 2.1. [3] A semihypergroup $(H, +)$ is called a canonical hypergroup if the following conditions are satisfied:
$(i) x + y = y + x, \forall x, y \in R$;
$(ii) \exists 0 \in R$(unique) such that for every $x \in R, x \in 0 + x = x$;
(iii) for every $x \in R$, there exists a unique element, say $\acute{x}$ such that $0 \in x + \acute{x}$.(we denote $\acute{x}$ by -x);
(iv) for every $x, y, z \in R$, $z \in x + y \Longleftrightarrow x \in z - y \Longleftrightarrow y \in z - x$.
from the definition it can be easily verified that $-(-x) = x$ and $-(x + y) = -x - y$.

Definition 2.2. [3] A Krasner hyperring is a hyperstructure $(R, \oplus, \star)$ where
(i) $(A, \oplus)$ is a canonical hypergroup;
(ii) $(A, \star)$ is a semigroup endowed with a two-sided absorbing element 0;
(iii) the product distributes from both sides over the sum.

A hyperfield is a Krasner hyperring $(K, \oplus, \star)$, such that $(K - \{0\}, \star)$ is a group.

Definition 2.3. [3] Let $(K, \oplus, \star)$ be a hyperskewfield and $(V, \oplus)$ be a canonical hypergroup. We define a *Krasner hypervector* space over K to be the quadrupled $(V, \oplus, \cdot, K)$, where " $\cdot$ " is a single-valued operation

$$\cdot : K \times V \quad \longrightarrow \quad V,$$

such that for all $a \in K$ and $x \in V$ we have $a \cdot x \in V$, and for all $a, b \in K$ and $x, y \in V$ the following conditions hold:
(H_1) $a \cdot (x \oplus y) = a \cdot x \oplus a \cdot y$;
(H_2) $(a \oplus b) \cdot x = a \cdot x \oplus b \cdot x$;
(H_3) $a \cdot (b \cdot x) = (a \star b) \cdot x$;
(H_4) $0 \cdot x = 0$;
(H_5) $1 \cdot x = x$.
We say that $(V, \oplus, \cdot, K)$ is *anti-left distributive* if for all $a, b \in K$, $x \in V$, $(a + b) \cdot x \supseteq a \cdot x + b \cdot x$, and *strongly left distributive*, if for all $a, b \in K$, $x \in V$, $(a \oplus b) \cdot x = a \cdot x \oplus b \cdot x$,
In a similar way we define the *anti-right distributive and* strongly right distributive hypervector spaces, respectively. V is called strongly distributive if it is both strongly left and strongly right distributive.

In the sequel by a hypervector space we mean a Krasner hypervector space.

3 Krasner Subhypervector Space

Here we study some basic results of Krasner hypervector spaces and after defining the category of Krasner hypervector spaces, we continue to find a free object in the category of Krasner hypervector spaces.

Definition 3.1. A nonempty subset S of V is a *subhyperspace* if $(S, \oplus)$ is a canonical subhypergroup of V and for all $a \in K$, $x \in S$, we have $a \cdot x \in S$.

Here we present example of a Krasner hypervector spaces.

Example 3.2. Let F be a field , V be a vector space and F^* be a multiplicative subgroup of F. For all $x, y \in V$ we define the equivalence relation $\sim$ on V as follows:

$$x \sim y \Longleftrightarrow x = ty \qquad t \in F^*$$

Now, let $\bar{V}$ be the set of all classes of V modulo $\sim$. $\bar{V}$ together with the hypersum $\oplus$, construct a canonical hypergroup:

$$\bar{x} \oplus \bar{y} = \{\bar{v} \in \bar{V} | \ \bar{v} \subseteq \bar{x} \oplus \bar{y}\}$$

Here we consider the external composition

$$\cdot : \bar{f} \times \bar{V} \longrightarrow \bar{V}$$
$$\bar{a} \cdot \bar{v} \longmapsto \bar{av}$$

Now, $(\bar{V}, \oplus, \cdot, F)$ is a hypervector space.

Lemma 3.3. *Let V_i be a hypervector space, for all $i \in I$, then $\bigcap V_i$ is also a hypervector space.*

Definition 3.4. Let V be a hypervector spaces and S a nonempty subset of it, then the smallest subhypervector space of V containing S is called linear space generated by S and is denoted by $< S >$. Moreover, $< S >= \bigcap_{S \subseteq W \leq V} W$.

Theorem 3.5. *Let V be a hypervector space and S a nonempty subset of it, then*

$$< S > = \{t \in V | t \in \sum_{i=1}^{n} a_i \cdot s_i, a_i \in K, s_i \in S, n \in N\} =$$
$$= \{t_1 \oplus ... \oplus t_n | t_i = a_i \cdot s_i\}.$$

Proof. Let $A = \{t \in V | t \in \sum_{i=1}^{n} a_i \cdot s_i, a_i \in K, s_i \in S, n \in N\}$. We claim that $(A, \oplus, \cdot, K)$ is the smallest hypervector space generated by S.
First we show that $(A, \oplus)$ is a canonical hypergroup. Commutativity is obvious.
For all $x \in A$, we have $x \in \sum_{i=1}^{n} a_i \cdot s_i$. Suppose there exists a scalar identity

$0_A \in A$ such that $0_A \in \sum_{i=1}^{n} b_i \cdot r_i$, for $b_i \in K$ and $r_i \in S$, we should have

$$x \oplus 0_A = \sum_{i=1}^{n} a_i \cdot s_i \oplus \sum_{i=1}^{n} b_i \cdot r_i = \sum_{i=1}^{n} a_i \cdot s_i \ni x.$$

Since for all $s_i \in A$, we have $s_i \in S \subseteq V$, and $(V, \oplus)$ is a canonical hypergroup, then there exists a scalar identity in V called 0_V such that $s_i \oplus 0_V = s_i$. Hence in the above equation it is enough to choose $b_i = a_i$ and $r_i = 0_V$, we obtain

$$x \oplus 0_A = \sum_{i=1}^{n} a_i \cdot s_i \oplus \sum_{i=1}^{n} a_i \cdot 0_V = \sum_{i=1}^{n} a_i \cdot (s_i \oplus 0_V) = \sum_{i=1}^{n} a_i \cdot s_i \ni x.$$

Now for all $x \in A$ we define $-x = \sum_{i=1}^{n} a_i \cdot (-s_i)$, then we have

$$0_S = \sum_{i=1}^{n} a_i \cdot 0_V \in \sum_{i=1}^{n} a_i \cdot s_i \oplus \sum_{i=1}^{n} a_i \cdot (-s)_i = \sum_{i=1}^{n} a_i \cdot (s_i \oplus (-s_i))$$

Hence every element in $(A, \oplus)$ has a unique identity. Moreover, every element in $(A, \oplus)$ is reversible, because suppose for all $x, y, z \in A$, we have $x = \sum_{i=1}^{n} a_i \cdot s_i,\ y = \sum_{i=1}^{n} \acute{a}_i \cdot \acute{s}_i,\ z = \sum_{i=1}^{n} \acute{\acute{a}}_i \cdot \acute{\acute{s}}_i$. Since for $s_i, \acute{s}_i, \acute{\acute{s}}_i \in S \subseteq V$, if $\acute{\acute{s}}_i \in s_i \oplus \acute{s}_i$ we have $\acute{s}_i \in \acute{\acute{s}}_i \oplus (-s_i)$, then it is sufficient to choose $\acute{\acute{a}}_i = \acute{a}_i = a_i$. Therefore $(A, \oplus)$ is a canonical subhypergroup.

Now for all $t \in A$, $k \in K$, we have

$$k \cdot t \subseteq k \cdot \sum_{i=1}^{n} a_i \cdot s_i = \sum_{i=1}^{n} (k \star a_i) \cdot s_i \subseteq A.$$

Then $(A, \oplus, \cdot, K)$ is a subhypervector space of V.

Let W be another subhypervector space of V containg S. let $t \in A$, then $t \in \sum_{i=1}^{n} a_i \cdot s_i$, for $a_i \in K, s_i \in S, n \in N$. Since W is a subhypervector space of V containing S, then $\sum_{i=1}^{n} a_i \cdot s_i \subseteq W$ and $A \subseteq W$. So, A is the smallest subhypervector space of V. Also, for all $s \in S$, we have $s = 1 \cdot s$, then $s \in A$, therefore $S \subseteq A$. □

Definition 3.6. Let $(V, \oplus, \cdot), (W, \oplus, \cdot)$ be two hypervector spaces over a hyperskewfield K, then the mapping $T : V \longrightarrow P^*(W)$ is called

(i) *multivalued linear transformation* if

$$T(x \oplus y) \subseteq T(x) \oplus T(y) \qquad \text{and} \qquad T(a \cdot x) = a \cdot T(x).$$

(ii) *multivalued good linear transformation* if

$$T(x \oplus y) = T(x) \oplus T(y) \qquad \text{and} \qquad T(a \cdot x) = a \cdot T(x).$$

where, $P^*(W)$ is the nonempty power set of W.

From now on, by *mv- linear transformation* we mean a multivalued linear transformation.

Remark 3.7. We define $T(0_V) = 0_W$.

Definition 3.8. Let V, W be two hypervector spaces over a hyperskewfield K, and $T : V \longrightarrow P(W)$ be a mv-linear transformation. Then the kernel of T is denoted by $kerT$ and defined by

$$KerT = \{x \in V \mid 0_W \in T(x)\}$$

Theorem 3.9. *Let V, W be two hypervector spaces on a hyperskewfield K and $T : V \longrightarrow W$ be a linear transformation. Then $KerT$ is a subhypervector space of V.*

Proof. By Remark 4.18, we have $T(0_V) = 0_W$ which means that $0_V \in KerT$ and $KerT \neq \varnothing$, then we have $x \in x \oplus 0_V = x$, for all $x \in KerT$. The other properties of a canonical subhypervector space will inherit from V. □

Theorem 3.10. *Let V, U be two hypervector spaces and $T : V \longrightarrow P^*(U)$ be a mv-linear transformation :*

(i) if W is a subhypervector space of V, then $T(W)$ is also a subhypervector space of U.

(ii) if L is a subhypervector space of U, then $T^{-1}(L)$ is also a subhypervector space of V containing $kerT$.

Proof. (i) Let $a \in K$ and $\acute{x}, \acute{y} \in T(W)$, such that $\acute{x} = T(x), \acute{y} = T(y)$ for some $x, y \in W$. Then $\acute{x} \oplus \acute{y} = T(x) \oplus T(y) = T(y) \oplus T(x) = \acute{y} \oplus \acute{x}$, hence commutativity holds.

For all $x \in V$ we have $x = x \oplus 0_V$, then we obtain

$$T(x) = T(x \oplus 0_V) \subseteq T(x) \oplus T(0_V) = T(x) \oplus 0_U.$$

Also, for all $x \in V$, there exists $\acute{x} = -x \in V$ such that $0_V \in x \oplus (-x)$. By Remark 4.18 we have

$$0_U = T(0_V) \in T(x \oplus (-x)) \subseteq T(x) \oplus T(-x) = \acute{x} \oplus \overline{\acute{x}}.$$

where $\overline{\acute{x}} = T(-x)$ is the unique inverse of $\acute{x}$.

Now suppose for all $x, y, z \in V$ we have

$$x \in y \oplus z \Longrightarrow y \in x \oplus (-z)$$

This is equivalent to

$$T(x) \in T(y \oplus z) \subseteq T(y) \oplus T(z) \Longrightarrow T(y) \in T(x) \oplus T(-z).$$

So, $(T(W), \oplus)$ is a canonical hypergroup. Now for $a \in K$ and $\acute{x} \in T(W)$, we have

$$a \cdot \acute{x} = a \cdot T(x) = T(a \cdot x) \subseteq T(W).$$

Hence, $(T(W), \oplus, \cdot)$, is a subhypervector space of V.

(ii) let $a \in K$ and $x, y \in T^{-1}(L)$. Suppose $\acute{x} = T(x), \acute{y} = T(y)$, for $\acute{x}, \acute{y} \in L$. Since $(U, \oplus)$ is a canonical hypergroup, then we have

$$x \oplus y = T^{-1}(\acute{x}) \oplus T^{-1}(\acute{y}) = T^{-1}(\acute{y}) \oplus T^{-1}(\acute{x}) = y \oplus x.$$

Also, we have

$$x \oplus 0_V = T^{-1}(\acute{x}) \oplus T^{-1}(0_U) \supseteq T^{-1}(\acute{x} \oplus 0_U) \supseteq T^{-1}(\acute{x}) = x.$$

for all $\acute{x} \in V$, there exists $\acute{-x}$ such that $0_U \in \acute{x} \oplus (\acute{-x})$, hence for $x \in T^{-1}(\acute{x})$, there exists $T^{-1}(-\acute{x}) \in T^{-1}(L)$ such that

$$x \oplus (-x) = T^{-1}(\acute{x}) \oplus T^{-1}(-\acute{x}) = T^{-1}(x \oplus (-\acute{x})) = T^{-1}(0_U) = 0_V.$$

Now for all $\acute{x}, \acute{y}, \acute{z} \in L$, we have

$$\acute{x} \in \acute{y} \oplus \acute{z} \Longrightarrow \acute{y} \in \acute{x} \oplus (-\acute{z})$$

Suppose $x, y, z \in T^{-1}(L)$. The above relation is equivalent to

$$y \oplus z = T^{-1}(\acute{y}) \oplus T^{-1}(\acute{z}) \supseteq T^{-1}(\acute{y} \oplus \acute{z}) \supseteq T^{-1}(\acute{x}) = x$$
$$\Longrightarrow x \oplus (-z) = T(\acute{x}) \oplus T(-\acute{z}) \supseteq T(\acute{x} \oplus (-\acute{z})) \supseteq T(\acute{y}) = y.$$

which means that $x \in y \oplus z \Longrightarrow y \in x \oplus (-z)$.
Moreover, $a \cdot x = a \cdot T^{-1}(\acute{x}) = T^{-1}(a \cdot \acute{x}) \subseteq T^{-1}(L)$. Hence $(T^{-1}(L), \oplus, \cdot)$ is a subhypervector space of V.
Now for $x \in KerT$ we have $T(x) = 0_U \in L$, then we obtain $x \in T^{-1}(L)$, hence $KerT \subseteq T^{-1}(L)$. □

Theorem 3.11. *Let U, V be two hypervector spaces on a hyperskewfield K and $T : V \longrightarrow P^*(U)$ be a good linear transformation. Then there is a one to one correspondence between sunhypervector spaces of V containing $KerT$ and subhypervector spaces of U.*

Proof. Suppose $A = \{W | W \leq V,\ W \supseteq KerT\}$ and $B = \{L | L \leq U\}$. We show that the following map is one to one and onto:

$$\phi : A \longrightarrow B$$
$$W \longrightarrow T(W)$$

By Theorem 3.10, $T(W)$ belongs to B, for all $W \in A$. Now let W_1, W_2 be two elements of A such that $W_1 \neq W_2$, then there exists $w_1 \in W_1 - W_2$ or $w_2 \in W_2 - W_1$. Let $w_1 \in W_1 - W_2$, then $T(w_1) \in T(W_1) - T(W_2)$ and hence $T(W_1) \neq T(W_2)$. If $w_2 \in W_2 - W_1$, then $T(W_1) \neq T(W_2)$, too. So, ϕ is well defined and one to one. Now for $L \in B$, put $W = T^{-1}(L)$, then by Theorem 3.9 we have $W \in A$ and $T(W) = L$. Therefore, ϕ is onto, hence the result. □

4 Construction of a hypermatrix

Now, we will talk about the basis of a hypervector space and verify that considering a multivalued linear transformation will imply some conditions to this definition. Finally, with the elements of hyperfield and basis we will construct a hypermatrix.

Definition 4.1. A subset S of V is called linearly independent if for every vectors $v_1, ..., v_n \in S$, and $c_1, ..., c_n \in K$, if we have $0_V \in c_1 \cdot v_1 \oplus ... \oplus c_n \cdot v_n$, implies that $c_1 = ... = c_n = 0_K$. Otherwise S is called linearly dependent.

Theorem 4.2. *Let V be a hypervector space and $v_1, ..., v_n$ be independent in V. Then every element in the linear space $< v_1, ..., v_n >$ belongs to a unique sum of the form $\sum_{i=1}^{n} a_i \cdot v_i$ where $a_i \in K$.*

Proof. Every element of $< v_1, ..., v_n >$ belongs to a set of the form $\sum_{i=1}^{n} a_i \cdot v_i$ where $a_i \in K$. We will show that this form is unique. Let $u \in V$ such that $u \subseteq \sum_{i=1}^{n} a_i \cdot v_i$ and $u \subseteq \sum_{i=1}^{n} b_i \cdot v_i$, where $a_i, b_i \in K$. Since V is a hypervector space we have :

$$0_V \in u - u \subseteq \sum_{i=1}^{n} a_i \cdot v_i - \sum_{i=1}^{n} b_i \cdot v_i = \sum_{i=1}^{n} a_i \cdot v_i \oplus \sum_{i=1}^{n} (-b)_i \cdot v_i.$$

Therefore, $0_V \subseteq \sum_{i=1}^{n} (a_i \oplus (-b_i)) \cdot v_i$. And since $v_1, ...v_n$ are independent we have $a_i \oplus (-b_i) = 0,\ \forall i$, then $a_i = -(-b_i) = b_i$. □

Theorem 4.3. *Let V be a hypervector space. Then vectors $v_1, ...v_n \in V$ are independent or v_j for some $1 \leq j \leq r$, belongs to the linear combination of the other vectors.*

Proof. Let $v_1, ..., v_n$be dependent and let $0_V \subseteq \sum_{i=1}^{n} a_i \cdot v_i$ such that at least one of the scalars such as a_j is not zero. Then there exists $t_i, (i = 1, ..., n)$ such that

$$0_V \in t_1 \oplus t_2 \oplus ... \oplus t_n,$$

where $t_i = a_i \cdot v_i$, which means that

$$\begin{aligned} & t_j \in 0 \oplus (-(t_1 \oplus ... \oplus t_{j-1} \oplus t_{j+1} \oplus ... \oplus t_n)) \\ \Longrightarrow\ & t_j \in 0 \oplus ((-t_1) \oplus ... \oplus (-t_{j-1}) \oplus (-t_{j+1}) \oplus ... \oplus (-t_n)) \end{aligned}$$

Moreover, for at least one v_j we have $v_j = (a_j^{-1}) \cdot t_j$. which means

$$\begin{aligned} v_j \in\ & (a_j^{-1}) \cdot (-t_1 \oplus ... \oplus (-t_{j-1}) \oplus (-t_{j+1}) \oplus ... \oplus (-t_n)) \in \\ & \in ((a_j^{-1}) \cdot (-t_1)) \oplus ((a_j^{-1}) \cdot (-t_{j-1})) \oplus ((a_j^{-1}) \cdot (-t_{j+1})) \oplus ... \oplus ((a_j^{-1}) \cdot (-t_n)) \\ & \in ((a_j^{-1}) \cdot (-a_1 \cdot v_1)) \oplus ... \oplus ((a_j^{-1}) \cdot (-a_1 \cdot v_{j-1})) \oplus ((a_j^{-1}) \cdot (-a_{j+1} \cdot v_{j+1})) \\ & \qquad \oplus ... \oplus ((a_j^{-1}) \cdot (-a_n \cdot t_n)) \\ & \in ((a_j^{-1} \star (-a_1)) \cdot v_1) \oplus ... \oplus ((a_j^{-1} \star (-a_1)) \cdot v_{j-1}) \oplus ((a_j^{-1} \star (-a_{j+1})) \cdot v_{j+1}) \\ & \qquad \oplus ... \oplus ((a_j^{-1} \star (-a_n)) \cdot t_n) \\ & \in (c_1 \cdot v_1) \oplus ... \oplus (c_j \cdot v_{j-i}) \oplus (c_j \cdot v_{j+1}) \oplus ... \oplus (c_j \cdot v_n) \end{aligned}$$

where $c_j = (a_j^{-1} \star (-a_n))$. Therefore v_j belongs to a linear combination of $v_1, ..., v_{j-1}, v_{j+1}, ..., v_n$ as desired. □

Definition 4.4. We call β a basis for V if it is a linearly independent subset of V and it spans V. We say that V has finite dimensional if it has a finite basis.

The following results are the generalization of the same results for vector spaces, also the methods here are adopted from those in the ordinary vector spaces.

Theorem 4.5. *Let V be a hypervector space. If W is a subhypervector space of V generated by $\beta = \{v_1, ..., v_n\}$, then W has a basis contained in β.*

Corolary 4.6. *If V is a hypervector space, then every generating subset of V, contains a basis of V, which means every independent subset of V is included in a finite basis.*

Theorem 4.7. *Let V be a hypervector space. If V has a finite basis with n elements, then the number of elements of every independent subset of V is smaller or equal to n.*

Corolary 4.8. *Let V be strongly left distributive and hypervector space. If V is finite dimensional then every two basis of V have the same elements.*

Lemma 4.9. *Let V be a hypervector space. If V is finite dimensional, then every linearly independent subset of V is contained in a finite basis.*

Now, we want to determine that what is a free object in the category of hypervector spaces. First, notice that if we denote the category of hypervector spaces by **KrH-vect**, we define the category as follows:
(i) the objects in this category are hypervector spaces over a hyperskew field K;
(ii) for the objects V, W of **KrH-vect**, the set of morphisms from V to $P^*(W)$ is the multivalued linear transformations which we show by $Home(V, W)$.
(iii) combination of morphism is defined as usual;
(iv) for all objects V in the category, the morphism $1_V : V \longrightarrow V$ is the identity.

According to the definition of a free object in the category of hypersets [2], and considering the category of hypervector spaces, if X is a basis for the hypervector space V, then we say that F is a free object in **KrH-vect** then for every function $f : X \longrightarrow V$, there exists a homomorphism $\bar{f} : F \longrightarrow V$, such that $\bar{f} \circ i = f$, where i is the inclusion function. Now, we have

$$(\bar{f} \circ i)(x) = \bar{f}(i(x)) = \bar{f}(x) \qquad (\star)$$

Since the homomorphism $\bar{f}$ is defined in **H-vect**, it is a multivalued transformation, then we define $\bar{f}(x) = \{f(x)\}$ we obtain $\bar{f} \circ i = f$.

Let $g : F \longrightarrow V$ be another homomorphism such that $g(x_i) = f(x_i)$, then for $t \in \sum_{i=1}^{n} a_i \cdot x_i$, let $\bar{f}$ be defined by $\bar{f}(t) = \sum_{i=1}^{n} a_i \cdot f(x_i)$, we have

$$g(t) \subseteq g(\sum_{i=1}^{n} a_i \cdot x_i) = \sum_{i=1}^{n} a_i \cdot g(x_i) = \bar{f}(t).$$

hence $\bar{f}$ defined above is the maximum homomorphism such that $(\star)$ is satisfied.

Suppose $t \in \sum_{i=1}^{n} a_i \cdot x_i$ and $t \in \sum_{i=1}^{n} b_i \cdot x_i$, for $a_i, b_i \in K$, we have $\bar{f}(t) = \sum_{i=1}^{n} a_i \cdot f(x_i)$, and also $\bar{f}(t) = \sum_{i=1}^{n} b_i \cdot f(x_i)$, then $\sum_{i=1}^{n} a_i \cdot f(x_i) = \sum_{i=1}^{n} b_i \cdot f(x_i)$, we obtain

$$0 \in \sum_{i=1}^{n} a_i \cdot f(x_i) - b_i \cdot f(x_i) = \sum_{i=1}^{n} (a_i - b_i) \cdot f(x_i)$$

So $a_i = b_i$. Therefore, $\bar{f}$ is a unique mv-transformation.
Hence we have the following corollary:

Corolary 4.10. *Every hypervector space with a basis is a free object in the category of hypervector spaces.*

Theorem 4.11. *Let $(V, \oplus, \cdot), (W, \oplus, \cdot)$ be two hypervector spaces on a hyperskewfiled K . If $T : V \longrightarrow P^*(W)$ and $U : V \longrightarrow P^*(W)$ be two mv-transformations. We define $L(V, W) = \{T | T : V \longrightarrow P^*(W)\}$ and the hyperoperation " $\boxplus$ " as follows:*

$$(T \boxplus U)(\alpha) = T(\alpha) \boxplus U(\alpha)$$

Also, we define the external composition as

$$(c \boxdot T)(\alpha) = c \boxdot T(\alpha)$$

Then $(L(V, W), \boxplus, \boxdot))$ as defined above is a hypervectorspace over a hyperskewfield K.

Proof. The external composition " $\boxdot$ " is defined as follows:

$$\boxdot : K \times L(V, W) \longrightarrow P^*(L(V, W))$$
$$(\alpha, T) \longmapsto \alpha \boxdot T$$

First we show that $(L(V, W), \boxplus)$ is a canonical hypergroup.
Communativity and associativity is obvious. We consider the transformation $0 : V \longrightarrow 0$ as a "0" for the group and $1 : V \longrightarrow P^*(V)$ as the identity. Then there exists a unique inverse $(-T)$ such that $0 \in (T \boxplus (-T))(\alpha)$.
Now, let T, U, Z be three linear transformations that belong to $L(V, W)$ then if $Z \in T \boxplus U$ then we have $Z(\alpha) \in (T \boxplus U)(\alpha)$, which means $Z(\alpha) \in T(\alpha) \boxplus U(\alpha)$. Now since W is hypervector space then we obtain $T(\alpha) \in Z(\alpha) \boxplus (-U)(\alpha)$, hence $T \in Z \boxplus (-U), \forall \alpha \in K$. Therefore, $(L(V, W), \boxplus)$ is a canonical hypergroup.
Now, we check that $L(V, W)$ is a hypervector space. Let $x, y \in K$ and $T, U \in L(V, W)$ then we have
(1) $(x \boxdot (T \boxplus U))(\alpha) = x \boxdot (T \boxplus U)(\alpha) = (x \boxdot T(\alpha)) \boxplus (x \boxdot U(\alpha))$
(2) $((x \boxplus y) \boxdot T)(\alpha) = \bigcup_{z \in x \boxplus y} z \boxdot T(\alpha) = (x \boxdot T(\alpha)) \boxplus (y \boxdot T(\alpha))$.

The other conditions will be obtained immediately. Therefore, $(L(V,W),\boxplus,\boxdot)$ is a hypervector space. $\square$

Theorem 4.12. *Let $(V,\oplus,\cdot)$, $(W,\oplus,\cdot)$ be two hypervector spaces on a hyperskewfield K, if $A=\{\alpha_1,...,\alpha_n\}$ be a basis for V and $\beta_1,...,\beta_n$ be any vectors in W, then there is a unique linear transformation $T:V\longrightarrow P^*(W)$ such that $T(\alpha_i)=\beta_i,\ 1\leq i\leq n$.*
In Other words, every linear transformation can be characterized by its operation on the basis of V.

Proof. Since for every $v\in V$, there exists scalars $c_1,...,c_n\in K$ such that

$$(*)\qquad v\in\sum_{i=1}^{n}c_i\cdot\alpha_i$$

then we define a map $T:V\longrightarrow P^*(W)$ as follows:

$$T(v)=\sum_{i=1}^{n}c_i\cdot T(\alpha_i)=\sum_{i=1}^{n}c_i\cdot\beta_i$$

Since $(*)$ is unique then T is well-defined. Now, we check that T is a linear transformation. Let $v,w\in V$ and scalars $d_1,...,d_n\in K$ then $v\in\sum_{i=1}^{n}c_i\cdot\alpha_i$ and $w\in\sum_{i=1}^{n}d_i\cdot\alpha_i$, then we have $T(v)=\sum_{i=1}^{n}c_i\cdot T(\alpha_i)$ and $T(w)=\sum_{i=1}^{n}d_i\cdot T(\alpha_i)$. Now since $v\oplus w\in\sum_{i=1}^{n}(c_i\oplus d_i)\cdot\alpha_i$, then we obtain

$$\begin{aligned}T(v\oplus w)&\subseteq T(\textstyle\sum_{i=1}^{n}(c_i\oplus d_i)\cdot\alpha_i)=\sum_{i=1}^{n}(c_i\oplus d_i)\cdot T(\alpha_i)\\&=\textstyle\sum_{i=1}^{n}c_i\cdot T(\alpha_i)\oplus\sum_{i=1}^{n}d_i\cdot T(\alpha_i)=T(v)\oplus T(w).\end{aligned}$$

Also, it is clear that $(c\circ T)(\alpha)=c\circ T(\alpha)$. Hence, T is a linear transformation.
Now, we shall check that T is unique. Let $S:V\longrightarrow P^*(W)$ be another linear transformation that satisfies $S(\alpha_i)=\beta_i$. We will show that $S=T$. We have

$$S(\alpha)=\sum_{i=1}^{n}c_i\cdot S(\alpha_i)=\sum_{i=1}^{n}c_i\cdot\beta_i=\sum_{i=1}^{n}c_i\cdot T(\alpha_i)=T(\alpha)$$

So, $S=T$ as desired. $\square$

Remark 4.13. Let $T:V\longrightarrow P^*(W)$ be a linear transformation. We denote $KerT=\{\alpha\in V|\ 0\in T(\alpha)\}$ by N_T and by ImT we mean $R_T=\{T(\alpha)|\alpha\in V\}$.
We call dimension of R_T, rank of T and it is denoted by $R(T)$. Notice that N_T is a subhypervector space of V and R_T is a subhypervector space of W.

Theorem 4.14. *Let V,W be two hypervector spaces over a field K. Let $T:V\longrightarrow P^*(W)$ be a linear transformation and $dimV=n<\infty$. Then*

$$dimR_T+dimKerT=dimV$$

Proof. Let $W = N_T$ and let $\beta_1 = \{\alpha_1, ..., \alpha_k\}$ be a basis for W. We extend β_1 to $\beta_2 = \{\alpha_1, ..., \alpha_k, \alpha_{k+1}, ..., \alpha_n\}$. We will show that $\beta = \{T(\alpha_{k+1}), ..., T(\alpha_n)\}$ is a basis for R_T. Let $c_1, ..., c_n$ be scalars in K such that

$$0 \in \sum_{i=k+1}^{n} c_i \cdot T(\alpha_i)$$

then there exists $\gamma \in \sum_{i=k+1}^{n}(c_i \cdot \alpha_i)$ such that $0 \in T(\gamma)$, this implies that $\gamma \in KerT = N_T$, hence $\gamma \in \sum_{i=1}^{k}(c_i \cdot \alpha_i)$. Therefore

$$0 = \gamma - \gamma \in \sum_{i=1}^{k}(c_i \cdot \alpha_i) \oplus \sum_{i=k+1}^{n}((-c_i) \cdot \alpha_i) \Longrightarrow c_i = 0$$

Now, we claim that β generates R_T because if for all $\alpha \in V$ we have $T(\alpha) = \beta$, and since $0 \in \sum_{i=1}^{k} c_i \cdot T(\alpha_i)$, hence
$\beta = T(\alpha) \subseteq T(\sum_{i=1}^{n} c_i \cdot \alpha_i) = \sum_{i=1}^{n} c_i \cdot T(\alpha_i) = \sum_{i=1}^{k} c_i \cdot T(\alpha_i) + \sum_{i=k+1}^{n} c_i \cdot T(\alpha_i) = \sum_{i=k+1}^{n} c_i \cdot T(\alpha_i)$
Therefore, $dimR_T + dimN_T = (n-k) + k = n = dimV$. □

For all $1 \leqslant j \leqslant n$ and $1 \leqslant p \leqslant m$, we define C_{pj} as the coordinator of $T(\alpha_j)$ on the ordered basis $B = \{\beta_1, ..., \beta_p\}$ which means

$$T(\alpha_j) = \sum_{p=1}^{m} C_{pj} \cdot \beta_p$$

where for $C_{pj} = (c_{pj}), \beta_p = (\beta_{p1})$. Now, if we notice the following matrix with a crisp product and hypersum, we will have a hypermatrix as the following:

$$\underbrace{\begin{pmatrix} c_{11} & ... & c_{1p} \\ ... & ... & ... \\ c_{j1} & ... & c_{jp} \end{pmatrix}}_{C_{pj}} \underbrace{\begin{pmatrix} \beta_{11} \\ ... \\ \beta_{p1} \end{pmatrix}}_{\beta_p} = \begin{pmatrix} c_{11} \cdot \beta_{11} \oplus ... \oplus c_{1p} \cdot \beta_{p1} \\ ... \\ c_{j1} \cdot \beta_{11} \oplus ... \oplus c_{jp} \cdot \beta_{p1} \end{pmatrix} = \underbrace{\begin{pmatrix} T(\alpha_1) \\ ... \\ T(\alpha_j) \end{pmatrix}}$$

Theorem 4.15. *Let V, W be two hypervector spaces. If $dimV = n$ and $dimW = m$, then $dimL(V, W) = mn$.*

Proof. Let $A = \{\alpha_1, ..., \alpha_n\}$ and $B = \{\beta_1, ..., \beta_m\}$ be the basis of V, W respectively. For all (p, q), where $p, q \in \mathbb{Z}$, and $1 \leqslant q \leqslant n$, $1 \leqslant p \leqslant m$ by Theorem 4.12 we have a unique linear transformation $T_{pq} : V \longrightarrow P^*(W)$ which we define by $T_{pq}(\alpha_i) = \beta_p$, when $i = q$ and otherwise it is defined 0.
Since we have mn linear transformation from V to $P^*(W)$, it is sufficient to

show that $\beta'' = \{T_{pq} | 1 \leqslant p \leqslant m,\ 1 \leqslant q \leqslant n\}$ is a basis for $L(V, W)$.
Let $T : V \longrightarrow P^*(W)$ be a linear transformation. For all $1 \leqslant j \leqslant n$, let $C_{1j}, ..., C_{mj}$ be the coordinate of $T(\alpha_j)$ in the ordered basis $\acute{\beta}$, i.e, $T(\alpha_j) = \sum_{p=1}^{m} C_{pj} \cdot \beta_p$. We will show that $T = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot T_{pq}$ generates $L(V, W)$. Because if we suppose $U = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot T_{pq}$, then if suppose $i = q$ we obtain

$U(\alpha_j) = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot T_{pq}(\alpha_j) = \sum_{p=1}^{m} \sum_{q=1}^{n} C_{pq} \cdot \beta_p = \sum_{p=1}^{m} A_{pj} \cdot \beta_p = T(\alpha_j)$.

Otherwise it will be 0. Also, it is obvious that β'' is independent. Hence the result. □

Remark 4.16. Let $T : V \longrightarrow P^*(W)$ and $S : W \longrightarrow P^*(Z)$ be two linear transformations and $\alpha \in V$, we define $(S \circ T)(\alpha) = S(T(\alpha)) = \bigcup_{\beta \in T(\alpha)} S(\beta)$ then $S \circ T$ is also a linear transformation.

Definition 4.17. Let $T : V \longrightarrow P^*(V)$ be a linear transformation, we call T a *linear operator*(or shortly an operator) on V, and If we have $T \circ T$, we denote it by T^2.

Lemma 4.18. *let V be a hypervector space on a field K. If U, T, S be three operators on V and $k \in K$, then the following results are immediate:*
(i) $I \circ U = U \circ I = U$;
(ii) $(S \oplus T) \circ U = S \circ U \oplus T \circ U$, $U \circ (S \oplus T) = U \circ S \oplus U \circ T$;
(iii) $k \oplus (U \circ T) = (kU) \circ T = U \circ (kT)$.□

Example 4.19. Let $\beta = \{\alpha_1, ..., \alpha_n\}$ be an ordered basis for the hypervector space V. Consider the operators $T_{(p,q)}$ regarding the proof of Theorem 4.15. These n^2 operators construct a basis for the space of operators of V. let S, U be two operators on V then we have

$$S = \sum_p \sum_q C_{pq} \cdot S_{pq}, \qquad U = \sum_r \sum_s B_{rs} \cdot S_{rs}.$$

Now by lemma 4.18, we have

$$(S \circ U)(\alpha_i) = S(U(\alpha_i)) = \bigcup_{\beta \in U(\alpha_i)} S(\beta) = \bigcup_{\beta \in \Sigma_r \Sigma_s B_{rs} \cdot T_{rs}(\alpha_i)} S(\beta)$$
$$= S(\sum_r \sum_s B_{rs} \cdot T_{(r,s)}(\alpha_i)) = S(\sum_r \sum_s B_{rs} \cdot \alpha_r)$$

when $i = s$ we have

$$\sum_r \sum_s B_{ri} \cdot S(\alpha_r) = \sum_r \sum_s B_{rs} \cdot (\sum_p \sum_q C_{pq} \cdot T_{pq}(\alpha_r))$$
$$= \sum_r \sum_s \sum_p \sum_q (B_{ri} A_{pq}) \circ \alpha_p$$

and when $r = q$ we have

$$= \sum_r \sum_s \sum_p \sum_q (B_{ri} C_{pr}) \cdot \alpha_p = \sum_r \sum_s \sum_p \sum_q (C_{pr} B_{ri}) \cdot \alpha_p$$

and since $1 \leqslant i \leqslant P$ then we have $\sum_r \sum_s \sum_p \sum_q (BC)_{n^2} \cdot \alpha_i$
Hence when we compose two operators S and U, the result is obtained by multiplying two matrices of them.□

Now, it is time to talk about the inverse of a transformation. As it is usual for defining an inverse we have:

Definition 4.20. let $T : V \longrightarrow P^*(W)$ be one to one and onto. T is said to have an inverse when there exists $U : W \longrightarrow P^*(V)$ such that $T \circ U = I_V$ and $U \circ T = I_W$. Also, the inverse of T is denoted by T^{-1} and obviously is not unique. We have $(U \circ T)^{-1} = T^{-1} \circ U^{-1}$.
We say that a linear transformation T is called *nonsingular* if $0 \in T(\alpha)$ implies that $\alpha = \{0\}$, which means that the null space of T is equal to $\{0\}$.

Lemma 4.21. *Let $T : V \longrightarrow P^*(W)$ be a linear transformation then T is one to one if and only if T is nonsingular if and only if $KerT = 0$*

Proof. Let T be one to one and suppose $0 \in T(\alpha)$, then since $T(0) = 0$, we have $T(0) \in T(\alpha)$ then

$$T(0) \in T(\alpha + 0) \subseteq T(\alpha) + T(0) \Longrightarrow T(\alpha) \in T(0) + (-T(0)) = 0$$

hence $\alpha = 0$. Conversely, let T is nonsingular and suppose for $x, y \in V$, we have $T(x) = T(y)$ then, $0 \in T(x) - T(y) = T(x - y)$ and since T is nonsingular we obtain $x - y = 0$, which means $x = y$.
Now let for all $\alpha \in KerT$ we have $0 \in T(\alpha)$, then since T is nonsingular we obtain $\alpha = 0$ which means $KerT = 0$. Conversely, if $KerT = 0$, then suppose $0 \in T(\alpha)$ implies that $\alpha \in KerT = 0$, hence $\alpha = 0$. □

Theorem 4.22. *Let V, W be two hypervector spaces on a hyperfiled K and let $T : V \longrightarrow P^*(W)$ be a linear transformation. If T is good reversible linear transformation, then the reverse of T is also a good linear transformation.*

Proof. Let $w_1, w_2 \in W$ and $k \in K$, then there exists $v_1, v_2 \in V$ such that $T^{-1}(w_1) = v_1, T^{-1}(w_2) = v_2$, where $T(v_1) = w_1$, and $T(w_2) = v_2$. We have

$$T^{-1}(w_1 \oplus w_2) = T^{-1}(T(v_1) \oplus T(v_2)) \supseteq T^{-1}(T(v_1 \oplus v_2)) = v_1 \oplus v_2 = T^{-1}(w_1) \oplus T^{-1}(w_2)$$

and when T is a good linear transformation, T^{-1} is also a good linear transformation. □

Theorem 4.23. *Let $T : V \longrightarrow P^*(W)$ be a linear transformation. T is nonsingular if and only if T corresponds every linearly independent subset of V onto a linearly independent subset of W.*

Proof. Let T be nonsingular and S be a linearly independent subset of V. We show that $T(S)$ is independent. Let $\acute{s_i} \in T(S)$ and for all i there exists $s_i \in S$ such that $T(s_i) = \acute{s_i}$. We assume

$$\sum_{i=1}^{n} c_i \cdot \acute{s_i} = 0 \Longrightarrow \sum_{i=1}^{n} c_i \cdot T(s_i) = 0 \Longrightarrow T(\sum_{i=1}^{n} c_i \cdot s_i) = 0$$

because T is nonsingular we have $\sum_{i=1}^{n} c_i \cdot s_i = 0$, and since s_i, for all i are

linearly independent then $c_i = 0$, hence $T(S)$ is linearly independent.
Conversely, let $0 \neq \alpha \in V$, then $\{0\}$ is an independent set. Hence by hypothesis T corresponds this independent set to a linearly in dependent set such as $T(\alpha) \in P^*(W)$, then we have $T(\alpha) \neq 0$. Therefore, T is nonsingular.
$\square$

Theorem 4.24. *Let V, W be two hypervector spaces with finite dimension on a hyperskewfield K and $dimV = dimW$. If $T : V \longrightarrow P^*(W)$ is a linear transformation, then the followings are equivalent:*
(i) T is reversible;
(ii) T is nonsingular;
(iii) T is onto.
(iv) If $\{\alpha_1, ..., \alpha_n\}$ is a basis for V, then $\{T(\alpha_1), ..., T(\alpha_n)\}$ is a basis for W.
(v) There exists a basis like $\{\alpha_1, ..., \alpha_n\}$ for V such that $\{T(\alpha_1), ..., T(\alpha_n)\}$ is a basis for W.

Lemma 4.25. *let V be a hypervector space with finite dimension on a hyperfield K, then $V \cong K^n$.*

5 Fundamental Relations

Let $(V, \oplus, \cdot)$ be a hypervector space, we define the relation ε^* as the smallest equivalence relation on V such that the set of all equivalence classes,V/ε^*, is an ordinary vector space. ε^* is called fundamental equivalence relation on V and V/ε^* is the fundamental ring.
Let $\varepsilon^*(v)$ is the equivalence class containing $v \in V$, then we define $\boxplus$ and $\boxdot$ on V/ε^* as follows:

$$\begin{aligned}
&\varepsilon^*(v) \boxplus \varepsilon^*(w) = \varepsilon^*(z), \text{ for all } z \in \varepsilon^*(v) \oplus \varepsilon^*(w)\\
&a \boxdot \varepsilon^*(v) = \varepsilon^*(z), \text{ for all } z \in a \cdot \varepsilon^*(v), \quad a \in K
\end{aligned}$$

Let $\mathbf{U}$ be the set of all finite linear combinations of elements of V with coefficients in K, which means

$$\mathbf{U} = \{\textstyle\sum_{i=1}^{n} a_i \cdot v_i;\ a_i \in K,\ v_i \in V,\ n \in \mathbf{N}\}$$

we define the relation ε as follows:

$$v\varepsilon w \Longleftrightarrow \exists u \in \mathbf{U}; \{v, w\} \subseteq u$$

Koskas [4] introduced the relation β^* on hypergroups as the smallest equivalence relation such that the quotient R/β^*_+ is a group. We will denote β_+ the relation in R as follows:

$$v\beta_+ w \Longleftrightarrow \exists (c_1, ..., c_n) \in V^n \text{ such that } \{v, w\} \subseteq c_1 \oplus ... \oplus c_n$$

Freni proved that for hyperrings we have $\beta_+^* = \beta_+$. Since in here $(V, \oplus)$ is a canonical hypergroup the we will have:

Theorem 5.1. *In the hypervector space $(V, \oplus, \cdot)$, we have $\varepsilon^* = \beta_+^*$.*

Vougiouklis [9] has proved that the sets $\{\varepsilon^*(z) : z \in \varepsilon^*(v) \oplus \varepsilon^*(w)\}$ and $\{\varepsilon^*(z) : z \in a \cdot \varepsilon^*(v)\}$ are singletons. With a similar method we can prove the following theorem:

Theorem 5.2. *Let $(V, \oplus, \cdot)$ be a hypervector space, then for all $a \in K$, $v, w \in V$, we have the followings:*
(i) $\varepsilon^*(v) \boxplus \varepsilon^*(w) = \varepsilon^*(z)$, $\forall z \in \varepsilon^*(v) \oplus \varepsilon^*(w)$
$a \boxdot \varepsilon^*(v) = \varepsilon^*(z)$, $\forall z \in a \cdot \varepsilon^*(w)$
(ii)$\varepsilon^*(0_V)$ *is the zero element of* $(V/\epsilon^*, \boxplus)$.
(iii) $(V/\varepsilon^*, \boxplus, \boxdot)$ *is a hypervector space and is called the fundamental hypervector space of V.*

Proof. (i) The proof is the same as [9], and we omit it.
(ii) since from (i) we obtain $\varepsilon^*(v) \boxplus \varepsilon^*(w) = \varepsilon^*(v \oplus w)$ and $a \boxdot \varepsilon^*(v) = \varepsilon^*(a \cdot v)$ we have

$$\varepsilon^*(v) \boxplus \epsilon^*(0) = \varepsilon(v \oplus 0) = \varepsilon^*(v)$$

(iii) The conditions for the vector space $(V/\epsilon^*, \boxplus, \boxdot)$ will be obtained from the hypervector space $(V, \oplus, \cdot)$. □

Theorem 5.3. *Let $(V, \oplus, \cdot, K)$ be a hypervector space and $(V/\epsilon^*, \boxplus, \boxdot)$ be the fundamental relation of it then $dimV = dimV/\epsilon^*$.*

Proof. Let $B = \{v_1, ..., v_n\}$ be a basis for V. We show that the set $B^* = \{\varepsilon^*(v_1), ..., \varepsilon^*(v_n)\}$ is a basis for V/ϵ^*. For this let $\varepsilon^*(v) \in V/\varepsilon^*$, then for every $v \in V$ there exists $a_1, ..., a_n \in K$ such that $x \in \sum_{i=1}^n a_i \cdot v_i$, then $v = t_1 \oplus ... \oplus t_n$, where $t_i = a_i \cdot v_i, i \in \{1, ..., n\}$. Now by Theorem 5.2 we have $\varepsilon^*(t_i) = a_i \cdot \varepsilon^*(v_i)$ then

$$\varepsilon^*(v) = \varepsilon^*(t_1 \oplus .... \oplus t_n) = \varepsilon^*(t_1) \boxplus .... \boxplus \varepsilon^*(t_n) = (a_1 \boxdot \varepsilon^*(v_1)) \boxplus (a_n \boxdot \varepsilon^*(v_n)).$$

hence, V/ε^* is spanned by B^*.
Now we show that B^* is linearly independent. For this let

$$\begin{aligned} &(a_1 \boxdot \varepsilon^*(v_1)) \boxplus ... \boxplus (a_n \boxdot \varepsilon^*(v_n)) = \varepsilon^*(0) \\ &\Longrightarrow \varepsilon^*(a_1 \cdot v_1) \boxplus ... \boxplus \varepsilon^*(a_n \cdot v_n) = \varepsilon^*(0) \\ &\Longrightarrow \varepsilon^*(a_1 \cdot v_1 \oplus ... \oplus a_n \cdot v_n) = \varepsilon^*(0) \\ &\Longrightarrow 0 \in a_1 \cdot v_1 \oplus ... \oplus a_n \cdot v_n \end{aligned}$$

since B in linearly independent in V, then $a_1 = ... = a_n = 0$. Therefore, B^* is also linearly independent. □

Lemma 5.4. *Let V, W be two hypervector spaces and $T : V \longrightarrow P^*(W)$ be a linear transformation, then*
(i) $T(\varepsilon^*(v)) \subseteq \varepsilon^*(T(v))$, for all $v \in V$;
(ii) The map $T^* : V/\varepsilon^* \longrightarrow W/\varepsilon^*$ defined as $T^*(\varepsilon^*(v)) = \varepsilon^*(T(v))$ is a linear transformation.

Proof. (i) straightforward.
(ii) It is obvious that T^* is well defined. Now we show that T^* is a linear transformation. Let $a \in K$, $x, y \in V$, then by Theorem 5.2 we have
$T^*(\varepsilon^*(x) \boxplus \varepsilon^*(y)) = T^*(\varepsilon^*(x \oplus y)) = \varepsilon^*(T(x \oplus y)) \subseteq \varepsilon^*(T(x) \oplus T(y)) = \varepsilon^*(T(x)) \boxplus \varepsilon^*(T(y)) = T^*(\varepsilon^*(x)) \boxplus T^*(\varepsilon^*(y))$
and $T^*(a \boxdot \varepsilon^*(x)) = T^*(\varepsilon^*(a \cdot x)) = \varepsilon^*(T(a \cdot x)) = \varepsilon^*(a \cdot T(x)) = a \boxdot \varepsilon^*(T(x)) = a \boxdot T^*(\varepsilon^*(x))$
hence, T^* is a linear transformation. $\square$

Theorem 5.5. *The map $F : \boldsymbol{HV} \longrightarrow \boldsymbol{V}$ defined by $F(V) = V/\varepsilon^*$ and $F(T) = T^*$ is a functor, where $\boldsymbol{HV}$ and $\boldsymbol{V}$ denote the category of hypervector spaces and vector spaces respectively. Moreover, F preserves the dimension.*

Proof. By Lemma 5.4 F is well-defined. Let$T : V \longrightarrow P^*(W)$ and $U : W \longrightarrow P^*(Z)$ be two linear transformations, then $F(U \circ T) = (U \circ T)^*$ such that for all $v \in V$ we have

$$\begin{aligned}(U \circ T)^*(\varepsilon^*(v)) &= \varepsilon^*((U \circ T)(v)) = \varepsilon^*(U(T(v)))\\ &= U^*\varepsilon^*(T^*(x)) = U^*T^*(\varepsilon^*(x)) = F(U)F(T)(\varepsilon^*(v))\\ &\Longrightarrow F(U \circ T) = F(U)F(T)\end{aligned}$$

Also, the identity is $F(1_V^*) : V/\varepsilon^* \longrightarrow V/\varepsilon^*$ such that $1_V^*(\varepsilon^*(v)) = \varepsilon^*(v)$. Hence, F is a functor And by Theorem 4.14 we have $dim(F(V)) = dim(V/\varepsilon^*) = dim(V)$. $\square$

Theorem 5.6. *Let $T : V \longrightarrow P^*(W)$ be a liner transformation in $\boldsymbol{HV}$. Then the following diagram is commutative:*

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \varphi_V \downarrow & & \downarrow \varphi_W \\ V/\varepsilon^* & \xrightarrow{T^*} & W/\varepsilon^* \end{array}$$

where β_V, β_W are the canonical projections of V and W.

Proof. Let $v \in V$ then $\varphi_W(T(v)) = \varepsilon^*(T(v)) = T^*(\varepsilon^*(v)) = T^*(\varphi_V(v)) = T^*\varphi_V(v)$.
Hence, the diagram is commutative. $\square$

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