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Visualization of Algebraic Properties of special H_V-structures

Achilles Dramalidis
Democritus University of Thrace, School of Education,
681 00 Alexandroupolis, Greece
adramali@psed.duth.gr

Abstract

The paper looks at visualization as it relates to special H_V-structures, focusing upon how it can be used to improve the perception and understanding of abstract algebraic concepts. Using position vectors into the plane IR^2, abstract algebraic properties of H_V-structures are gradually transformed into geometrical shapes.

Key words: Hyperstructures, H_V-structures, Visualization.

MSC2010: 20N20, 16Y99.

1 Introduction

In most branches of mathematical research, visualization has been an area of interest for mathematicians [1], [6], [9], specifying that visual thinking can be an alternative and powerful resource, as well as a serious tool, not only for specialists but also, for students doing mathematics. Mathematicians have always used their "mind's eye" to visualize the abstract objects and processes that arise in mathematical research. But it is only in recent years that remarkable improvements in computer technology have made it easy to externalize these vague and subjective pictures that we see in our heads, replacing them with precise and objective visualizations that can be shared with others [7]. The subject is of such recent research that searching the literature, in preparation for this paper, it was surprising to discover that no papers were specifically focused on visualization in hyperstructures.

According to [10], the term *visualization* has been used in various ways in the research literature, so it is necessary to clarify how it is used in this paper. *Thus visualization is taken to include processes of constructing and transforming both mental imagery and abstract algebraic concepts.*

This paper, looks at visualization as it relates to special H_V-structures, focusing upon how it can be used to improve the perception and understanding of abstract algebraic concepts, since, being able to "see" something in a geometrical shape, is a common metaphor for understanding it. According to Bruner [2], to understand a specific concept (algebraic), the first approach has to be intuitive. So, geometry or linear algebra into a two-dimensional real vector space, with constant references to the fundamental intuitively understood principles, are teaching and educative tools.

Using position vectors into the plane IR^2, abstract algebraic properties of H_V-structures are gradually transformed into geometrical shapes, which operate, not only as a translation of the algebraic concept but also, as a teaching process.

2 Basic definitions on hyperstructures

In 1934, F. Marty introduced the definitions of the hyperoperation and of the hypergroup as a generalization of the operation and the group respectively.

Definition 2.1 *In a set $H \neq \varnothing$, a hyperoperation is a map, such that:*

$$\circ : H \times H \to P(H) - \{\varnothing\} : (x, y) \mapsto x \circ y \subset H$$

Also, if $A, B \subset H$,then

$$A \circ B = \cup_{a \in A, b \in B}(a \circ b).$$

Properties of hyperoperations [3], [4], [12]:

i) A hyperoperation ($\circ$) in a set H is called associative, if

$$(x \circ y) \circ z = x \circ (y \circ z), \forall x, y, z \in H$$

ii) A hyperoperation ($\circ$) in a set H is called commutative, if

$$x \circ y = y \circ x, \forall x, y \in H$$

iii) A hyperoperation ($\circ$), in a set H, is having an identity or unit element if there exists $e \in H$, such that

$$x \in x \circ e \text{ and } x \in e \circ x, \forall x \in H$$

iv) A hyperoperation ($\circ$), in a set H, with a unit element e, is having an inverse element, if for every $x \in H$, there exists an element $x' \in H$, such that

$$e \in x \circ x' and e \in x' \circ x, \forall x \in H$$

v) In a set H, equipped with two hyperoperations ($\circ$) and ($*$), the ($*$) is called distributive with respect to ($\circ$), if

$$x * (y \circ z) = (x * y) \circ (x * z), \forall x, y, z \in H$$

An algebraic hyperstructure $(H, \circ)$, i.e. a set H equipped with a hyperoperation ($\circ$), is called hypergroupoid. If this hyperoperation is associative, then the hyperstructure is called semihypergroup. The semihypergroup $(H, \circ)$, is called hypergroup if it satisfies the reproduction axiom:

$$x \circ H = H \circ x, \forall x \in H.$$

One more complicated hyperstructure, is that $(H, \circ, *)$, which is called hyperring, where $(H, \circ)$ is a commutative hypergroup, the ($*$) is associative and distributive with respect to ($\circ$).

One of the topics of great interest, in the last years, is the H_v-stuctures, which was introduced by T. Vougiouklis in 1990 [11]. The class of H_v-stuctures is the largest class of algebraic hyperstructures. These structures satisfy weak axioms, where the non-empty intersection replaces the equality, as bellow [12]:
Let H be a set and $\circ : H \times H \rightarrow P(H) - \{\varnothing\}$ be a hyperoperation.

i) The ($\circ$)in H is called weak associative, we write WASS, if

$$(x \circ y) \circ z \cap x \circ (y \circ z) \neq \varnothing, \forall x, y, z \in H$$

ii) The ($\circ$) is called weak commutative, we write COW, if

$$(x \circ y) \cap (y \circ x) \neq \varnothing, \forall x, y \in H$$

iii) If H is equipped with two hyperoperations ($\circ$) and ($*$), then ($*$)is called weak distributive with respect to ($\circ$), if

$$[x * (y \circ z)] \cap [(x * y) \circ (x * z)] \neq \varnothing, \forall x, y, z \in H$$

The hyperstructure $(H, \circ)$ is called H_v-semigroup if it is WASS and it is called H_v-group if it is reproductive H_v-semigroup. It is called commutative H_v-group if $(\circ)$ is commutative and it is called H_v-commutative group if$(\circ)$ is weak commutative. The hyperstructure $(H, \circ, *)$ is called H_v-ring if both hyperstructures $(\circ)$ and $(*)$ are WASS, the reproduction axiom is valid for $(\circ)$ and $(*)$ is weak distributive with respect to $(\circ)$.
What it follows to the end of the paragraph comes from [5]:

Definition 2.2 *An H_v-ring $(R, +, \bullet)$ is called dual H_v-ring, if $(R, \bullet, +)$ is an H_v-ring, too.*

Definition 2.3 *Let V be a vector space over a field K. Then, define two hyperoperations in V as follows: For all $x, y \in V$ and $r \in K$,*

$$x \circ y = \{z/z = x + r(y - x), r \in [0, 1]\}$$

$$x \bullet y = \{z/z = x + ry, r \in [0, 1]\}$$

Remark 2.1 *Into the plane $IR^2 : x \circ y = [x, y]$, it is known as join operation [8] and $x \bullet y = [x, x+y]$. The $[\alpha, \beta]$ denotes the line segment which is bounded by the two end points α and β.*

Then, for the four hyperstructures occur, we get the following:

Proposition 2.1 *The hyperstructure $(V, *, \Box)$, where $*, \Box \in \{\circ, \bullet\}$, is a weak commutative dual H_v-ring.*

Let:

E_*	be the set of the unit elements with respect to $(*)$.
E_*^r	be the set of the right unit elements with respect to $(*)$.
E_*^l	be the set of the left unit elements with respect to $(*)$.
$I_*(x, e)$	be the set of the inverse elements of x associated with the unit e (left or right), with respect to $(*)$.
$I_*^r(x, e)$	be the set of the right inverse elements of x associated with the right unit e, with respect to $(*)$.
$I_*^l(x, e)$	be the set of the left inverse elements of x associated with the left unit e, with respect to $(*)$.

Proposition 2.2 *i) $E_\circ = V$, ii)$I_\circ(x, e) = \{z/z = (1 - r)x + re, r \geq 1\}$*

Proposition 2.3 *i)$E_\bullet^r = V$, ii)$I_\bullet^r(x, e) = \{z/z = r(e - x), r \geq 1, e \in E_\bullet^r\}$, iii)$E_\bullet^l = \{O\} \subset E_\bullet$, iv)$I_\bullet^l(x, e) = [e, e - x], e \in E_\bullet^r$.*

3 Visualization in H_v-groups

Now, let us introduce a coordinate system into the IR^2. We place a given vector p so that its initial point P determines an ordered pair (a_1, a_2). Conversely, a point P with coordinates (a_1, a_2) determines the vector $p = OP$, where O the origin of the coordinate system. We shall refer to the elements x, y, z,... of the set IR^2 , as vectors whose initial point is the origin. These vectors are very well known as position vectors.

i) **The hyperoperation**: $x \bullet y = \{z/z = x + ry, r \in [0, 1]\} = [x, x + y]$

In Figure 3.1, to every point x and y of the plane, i.e. to every ordered pair (x, y) we map an infinite number of points (hyperstructure) instead of one point (operation). The infinite number of points is the line segment $[x, x+y]$ which is bounded by the two end points x and $x+y$. Graphically, having the points O, x, y, draw the parallelogram with vertices $O, x, y, x + y$. Then, the side $[x, x + y]$ is the hyperoperation $x \bullet y$.

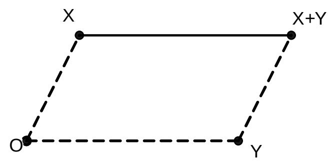

Fig.3.1

ii) **Reproduction** : $x \bullet IR^2 = \cup_{r \in IR^2}(x \bullet r) = IR^2$

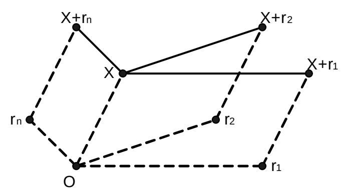

Fig.3.2

In Figure 3.2, take any point x of the plane. For any of the infinite points r_i of the plane, draw the parallelogram with vertices $O, x, r_i, x + r_i$. Unite all these infinite line segments $[x, x + r_i]$, then all these segments cover the plane.

iii) **Weak Associativity**: $x \bullet (y \bullet z) \cap (x \bullet y) \bullet z \neq \varnothing$

In Figure 3.3a, take three points x, y, z of the plane, then the side $[y, y+z]$ of the parallelogram with vertices $O, y, z, y+z$ is the hyperoperation $y \bullet z$. With the points O, x and every point r_i of the line segment $[y, y+z]$ draw, each time, the parallelogram with vertices $O, x, r_i, x + r_i$. All these infinite line segments $[x, x + r_i]$, create the triangle with vertices $x, x + y, x + y + z$. Then the area of this triangle is the first part of the above intercection, i.e. $x \bullet (y \bullet z)$. Similarly, in Figure 3.3b, the side $[x, x + y]$ of the parallelogram with vertices $O, x, y, x + y$ is the hyperoperation $x \bullet y$. With the points O, z and every point r_i of the line segment $[x, x+y]$ draw, each time, the parallelogram with vertices $O, z, r_i, r_i + z$. All these infinite line segments $[r_i, r_i + z]$ create the parallelogram with vertices $x, x + y, x + y + z, x + z$. Then the area of this parallelogram is the second part of the above intercection, i.e. $(x \bullet y) \bullet z$. Notice that the triangle with vertices $x, x + y, x + y + z$ is part of the parallelogram with vertices $x, x+y, x+y+z, x+z$, i.e. the intersection of these two figures is not equal to the empty set.

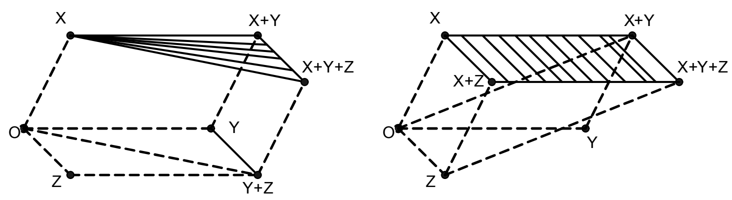

Fig.3.3a Fig.3.3b

iv) **Weak Commutativity**: $(x \bullet y) \cap (y \bullet x) \neq \varnothing$ In Figure 3.4, take two

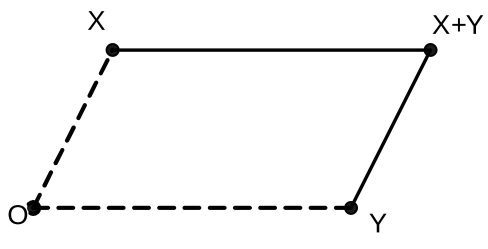

Fig.3.4

points x and y of the plane. Then draw the parallelogram with vertices

$O, x, x+y, y$. The side $[x, x+y]$ is the hyperoperation $x \bullet y$ and the side $[y, x+y]$ is the hyperoperation $y \bullet x$. Notice that the only common point of these two sides is the point $x+y$, i.e. the intersection of $x \bullet y$ and $y \bullet x$ is not equal to the empty set.

v) **The set of the right unit elements**: $(x \in x \bullet e, \forall x \in IR^2)$
$E^r_\bullet = IR^2$

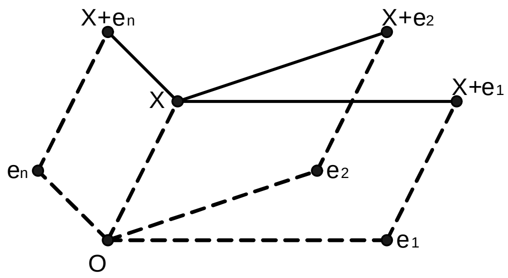

Fig.3.5

In Figure 3.5, take any point x of the plane. Then draw the parallelograms with vertices $O, x, x+e_i, e_i$, where e_i any point of the plane. The side $[x, x+e_i]$ is the hyperoperation $x \bullet e_i$. Notice that x belongs to every line segment $[x, x+e_i]$, i.e. x belongs to every $x \bullet e_i$. Since all these e_i's, having the above property, are infinite, we get that the set $E^r_\bullet$ of the right unit elements with respect to $(\bullet)$ is equal to IR^2.

vi) **The set of the right inverse elements**:

$$I^r_\bullet(x, e) = \{z/z = r(e-x), r \geq 1\}, e \in E^r_\bullet$$

Having any point x of the plane, we want to find at least one point x' of the plane, such that, for a right unit point e of the plane (i.e. any point of the plane) the following to be valid: $e \in x \bullet x'$, i.e. we want e to be point of the line segment $[x, x+x']$. In Figure 3.6a, notice that all the infinite points x' of the half-line $[e-x, +\infty)$ have the above property.

Indeed, in Figure 3.6b, for a given x and e, take any x' belonging to the half-line $[e-x, +\infty)$. Draw the parallelogram with vertices $O, x, x+x', x'$. Then e belongs to the line segment $[x, x+x']$, i.e., e belongs to the hyperoperation $x \bullet x'$.

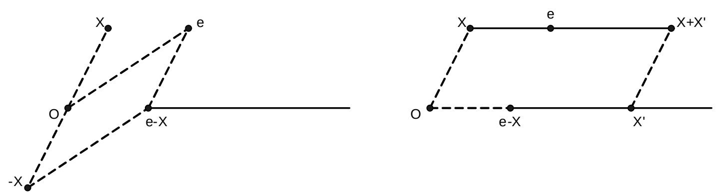

Fig.3.6

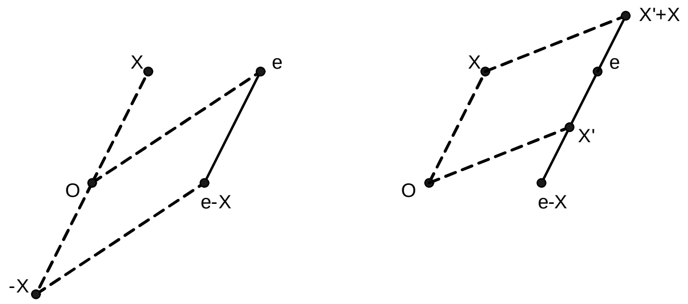

Fig.3.7a Fig.3.7b

vii) **The set of the left inverse elements**: $I^l_\bullet(x,e) = [e, e-x], e \in E^r_\bullet$

Take any point x of the plane, we want to find at least one point x' of the plane, such that, for a right unit point e of the plane (i.e. any point of the plane) the following to be valid: $e \in x' \bullet x$, i.e. we want e to be point of the line segment $[x', x'+x]$. In Figure 3.7b, notice that the points x' of the line segment $[e, e-x]$ have the above property. Indeed, in Figure 3.7b, for a given x and e, take any x' belonging to the line segment $[e, e-x]$. Draw the parallelogram with vertices $O, x, x'+x, x'$. Then e belongs to the line segment $[x', x'+x]$, i.e., e belongs to the hyperoperation $x' \bullet x$. Since, $E^l_\bullet = \{O\} \subset E_\bullet$ (that means that the origin O of the coordinate system is simultaneously left and right unit element), set $O \equiv e$, then $I^e_\bullet(x, O) = [O, O-x]$.

Remark 3.1 *Into the plane IR^2, the hyperoperation (i), together with the axioms (ii) and (iii) are giving the concept of H_v-group. Furthermore, by putting together the axiom (iv) we get the concept of H_v-commutative group.*

4 Visualization in H_v-rings

i) **Distributivity of** $(\circ)$ **with respect to** $(\circ)$:

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$$

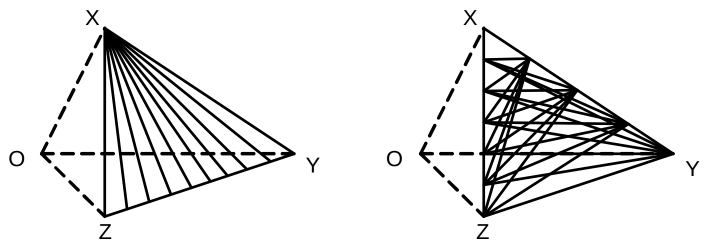

Fig.4.1a Fig.4.1b

In Figure 4.1a, take three points x, y, z of the plane, then the line segment $[y, z]$ is the hyperoperation $y \circ z$. Join the point x to each point of the segment $[y, z]$. Then the area of the triangle with vertices x, y, z is the first part of the above equality, i.e. $x \circ (y \circ z)$. Similarly, in Figure 4.1b, the line segment $[x, y]$ is the hyperoperation $x \circ y$ and the line segment $[x, z]$ is the hyperoperation $x \circ z$. Join every point of the segment $[x, y]$ to every point of the segment $[x, z]$. Then the area of the triangle with vertices x, y, z is the second part of the above equality, i.e. $(x \circ y) \circ (x \circ z)$.

ii) **Weak Distributivity of** $(\bullet)$ **with respect to** $(\bullet)$:

$$x \bullet (y \bullet z) \cap (x \bullet y) \bullet (x \bullet z) \neq \varnothing$$

In Figure 4.2a, take three points x, y, z of the plane, then the side $[y, y+z]$ of the parallelogram with vertices $O, y, z, y+z$ is the hyperoperation $y \bullet z$. With the points O, x and every point r_i of the line segment $[y, y+z]$ draw, each time, the parallelogram with vertices $O, x, r_i, x+r_i$. All these infinite line segments $[x, x+r_i]$ create the triangle with vertices $x, x+y, x+y+z$. Then the area of this triangle is the first part of the above inersection, i.e. $x \bullet (y \bullet z)$.
In Figure 4.2b, the side $[x, x+y]$ of the parallelogram with vertices $O, x, y, x+y$ is the hyperoperation $x \bullet y$ and the side $[x, x+z]$ of the parallelogram with vertices $O, x, z, x+z$ is the hyperoperation $x \bullet z$.

With the points: O, every point r_i of the side $[x, x+y]$ and every point t_i of the side $[x, x+z]$ draw, each time, the parallelogram with vertices O, r_i, t_i, r_i+t_i. All these infinite line segments $[r_i, r_i+t_i]$ create the pentagon with vertices $x, 2x, 2x+y, 2x+y+z, x+y$. Then the area of this pentagon is the second part of the above intersection, i.e. $(x \bullet y) \bullet (x \bullet z)$. Notice that the line segment $[x, x+y]$ is the common part of the triangle area with vertices $x, x+y, x+y+z$ and the pentagon area with vertices $x, 2x, 2x+y, 2x+y+z, x+y$, i.e. the intersection of these two figures is not equal to the empty set.

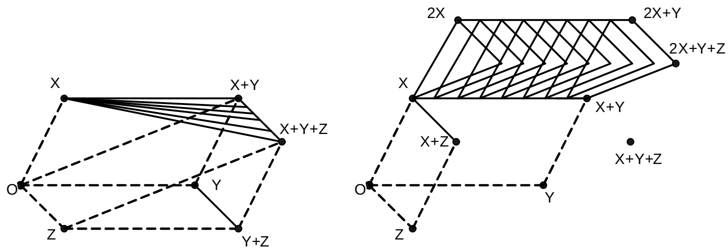

Fig.4.2a Fig.4.2b

iii) **Weak Distributivity of** $(\circ)$ **with respect to** $(\bullet)$:

$$x \circ (y \bullet z) \cap (x \circ y) \bullet (x \circ z) \neq \varnothing$$

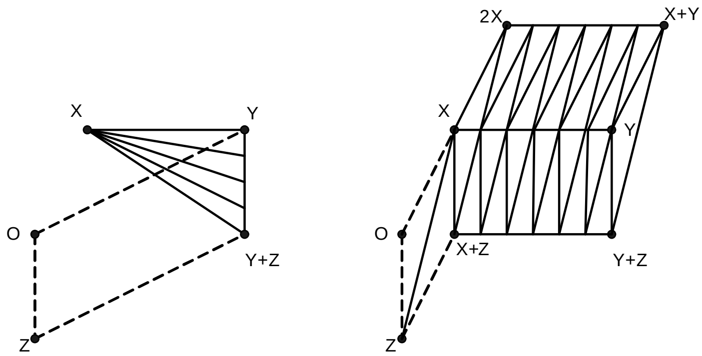

Fig.4.3a Fig.4.3b

In Figure 4.3a, take three points x, y, z of the plane, then the side $[y, y+z]$ of the parallelogram with vertices $O, y, z, y+z$ is the hyperoperation $y \bullet z$. Join the point x to each point of the segment $[y, y+z]$. Then the area of the triangle with vertices $x, y, y + z$ is the first part of the above inersection, i.e. $x \circ (y \bullet z)$. In Figure 4.3b, take three points x, y, z of the plane, then the line segment $[x, y]$ is the hyperoperation $x \circ y$ and the line segment $[x, z]$ is the hyperoperation $x \circ z$. With the points: O, every point r_i of the line segment $[x, y]$ and every point t_i of the line segment $[x, z]$ draw, each time, the parallelogram with vertices $O, r_i, t_i, r_i + t_i$. All these infinite line segments $[r_i, r_i + t_i]$ create the pentagon with vertices $x, 2x, x + y, y + z, x + z$. Then the area of this pentagon is the second part of the above intersection, i.e. $(x \circ y) \bullet (x \circ z)$. Notice that the triangle with vertices $x, y, y + z$ is part of the pentagon with vertices $x, 2x, x + y, y + z, x + z$, i.e. the intersection of these two figures is not equal to the empty set.

iv) **Distributivity of** $(\bullet)$ **with respect to** $(\circ)$:

$$x \bullet (y \circ z) = (x \bullet y) \circ (x \bullet z)$$

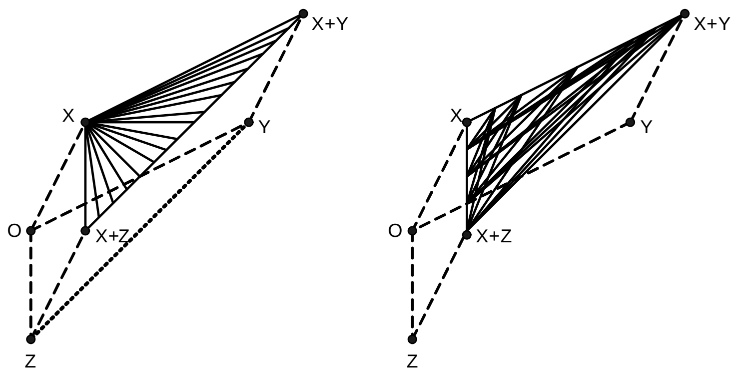

Fig.4.4a Fig.4.4b

In Figure 4.4a, take three points x, y, z of the plane, then the line segment $[y, z]$ is the hyperoperation $y \circ z$. With the points O, x and every point r_i of the line segment $[y, z]$ draw, each time, the parallelogram with vertices $O, x, r_i, x + r_i$. All these infinite line segments $[x, x + r_i]$ create the triangle with vertices $x, x + y, x + z$. Then the area of this triangle is the first part of the above equality, i.e. $x \bullet (y \circ z)$.
In Figure 4.4b, the side $[x, x + y]$ of the parallelogram with vertices $O, x, y, x + y$ is the hyperoperation $x \bullet y$ and the side $[x, x + z]$ of the parallelogram with vertices $O, x, z, x + z$ is the hyperoperation $x \bullet z$. Join every point of the side $[x, x+y]$ to every point of the side $[x, x+z]$.

Then the area of the triangle with vertices $x, x+y, x+z$ is the second part of the above equality, i.e. $(x \bullet y) \circ (x \bullet z)$.

Remark 4.1 *It is known that* $(IR^2, \circ)$ *is a commutative hypergroup. Into the plane* IR^2*, the hyperoperations* ($\circ$) *and* ($\bullet$) *together with the axioms 3ii), 3iii), 4i), 4ii), 4iii) and 4iv) are giving the concepts of hyperring,* H_v*-ring and dual* H_v*-ring.*

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