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CAS wavelet approximation of functions of Hölder's class $H^{\alpha}[0,1)$ and Solution of Fredholm Integral Equations

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Abstract

In this paper, cosine and sine wavelet is considered. Two new CAS wavelet estimators $E^{(1)}_{2^k,2M+1}(f)$ and $E^{(2)}_{2^k,2M+1}(f)$ for the approximation of a function f whose first derivative f' and second derivative f'' belong to Hölder's class $H^{\alpha}[0,1)$ of order $0<\alpha\leqslant 1$, have been obtained. These estimators are sharper and best in wavelet analysis. Using CAS wavelet, a computational method has been developed to solve Fredholm integral equation of second kind. In this process, Fredholm integral equations are reduced into a system of linear equations. Approximation of functions by CAS wavelet method is applied in obtaining the solution of Fredholm integral equation of second kind. CAS wavelet coefficient matrices are prepared using the properties of CAS wavelets. Two examples are illustrated to show the validity and efficiency of the technique discussed in this paper.

Keywords: CAS wavelet, CAS Wavelet Approximation, Function of Hölder's class, Orthonormal basis, Fredholm integral equation.
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[1]

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1 Introduction

Wavelet is a very recent and powerful tool in pure as well as applied mathematical research area. It has wide range applications in engineering, science and technology, signal analysis, time-frequency analysis, fast numerical algorithm. Several problems of Physics, Engineering, science and Technology are found in the form of integral equations. In some cases, integral equations are reformulated into ordinary differential equations and partial differential equations. In many cases, it is very difficult to solve integral equations analytically and hence there is a need of approximate solution of integral equations. In recents years, the approximate solutions of integral equations have been obtained by orthogonal basis functions as well as orthogonal wavelets. The main advantage of using orthonormal basis is that it converts the mathematical problems to a system of algebraic equations. Working in same direction, several researchers like [2], Sahu [3] etc. have been solved integral equations. It is known that wavelets are considerably useful in the solution of integral equations. In science and Technology, some problems are available in the form of Fredholm integral equations of second kind:

$$u(x) = f(x) + \int_0^1 K(x,y)u(y)dy \tag{1}$$

where $f \in L^2[0,1)$ and $K \in L^2[0,1) \times L^2[0,1)$ are known functions and u is unknown function to be determined (Ray and Sahu [3]).

In best of our knowledge, there is no work associated with the solution of Fredholm integral eq^n (1) by CAS wavelet method. The main objectives of the research paper are as follows:

1. To estimate the approximation of functions belonging to Hölder's class $H^\alpha[0,1)$ of order $0 < \alpha \leqslant 1$ by CAS wavelet method.

2. To develop a procedure to solve Fredholm integral equation of second kind by using CAS wavelet approximation.

3. To compare the solutions of Fredholm integral eq^n (1) obtained by CAS wavelet, Legendre wavelet and Haar wavelet method with their exact solutions.

It is remarkable to note that the solution of Fredholm integral eq^n (1) obtained by CAS wavelet method and its exact solution are almost same. The solution of Fredholm integral eq^n (1) obtained by CAS wavelet method is better and more closed to its exact solution than the solutions obtained by Legendre wavelet and Haar wavelet method. It is observed in numerical comparison of these solutions. It is a significant achievement of the proposed method.

2 Definitions and Preliminaries

2.1 Basic Wavelets And CAS Wavelets

Let $\psi \in L^2(\mathbb{R})$. ψ is called a basic wavelet if it satisfies the admissibility condition:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}|^2}{|w|} dw < \infty \text{ (Chui [1])} \tag{2}$$

The integral wavelet transform, relative to a basic wavelet ψ, is defined by

$$(W_\psi f)(b,a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t)\overline{\psi(\frac{b-a}{a})}dt\ , f \in L^2(\mathbb{R}) \tag{3}$$

where $a, b \in \mathbb{R}, a \neq 0$. Set

$$\psi_{b,a}(t) = |a|^{-1/2}\psi(\frac{b-a}{a}). \tag{4}$$

This is a family of wavelets. If we restrict the parameters a and b to discrete values

$$a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$$

where n and k are positive integers, then

$$\psi_{b,a}(t) = \psi_{n,k}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0). \tag{5}$$

Taking $a_0 = 2, b_0 = 1$ in eq^n (5),

$$\psi_{n,k}(t) = 2^{k/2}\psi(2^k t - n). \tag{6}$$

If

$$\begin{aligned} \psi(2^k t - n) &= \cos(2m\pi(2^k t - n + 1)) + \sin(2m\pi(2^k t - n + 1)) && (7) \\ &= CAS_m(2^k t - n + 1). && (8) \end{aligned}$$

Using eq^n(7), eq^n (6) becomes

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\{\cos(2m\pi(2^k t - n + 1)) + \sin(2m\pi(2^k t - n + 1))\}, & if\ \frac{n-1}{2^k} \leqslant t < \frac{n}{2^k}, \\ 0, & otherwise. \end{cases}$$

$\{\psi_{n,m}\}_{n,m\in\mathbb{Z}}$ are orthonormal CAS wavelets defined on [0,1) .

3 Function belonging to Hölder's class $H^{\alpha}[0,1)$

A function f is said to belong to Hölder's class $H^{\alpha}[0,1)$ of order $0<\alpha\leqslant 1$ if f satifies the following condition :

$$|f(x)-f(y)|\leqslant A|x-y|^{\alpha},\ \ \forall x,y\in\mathbb{R} \tag{9}$$

for some positive constant A (Zheng, Wei [4]).

3.1 Proposition

Let f be a function such that its second derivative $f^{''}$ is in $H^{\alpha}[0,1)$, then its first derivative $f^{'}$ is in $H^{\alpha}[0,1)$.
Proof : Let $\phi^{''}\in H^{\alpha}[0,1)$.

$$\begin{aligned}
f(x) &= \int_0^{x^{\alpha}}\phi^{'}(t)\,dt\\
f^{'}(x) &= \int_0^{x^{\alpha}}\phi^{''}(t)\,dt \text{ and } f^{'}(y)=\int_0^{y^{\alpha}}\phi^{''}(t)\,dt\\
|f^{'}(x)-f^{'}(y)| &= |\int_0^{x^{\alpha}}\phi^{''}(t)\,dt-\int_0^{y^{\alpha}}\phi^{''}(t)\,dt|=|\int_{y^{\alpha}}^{x^{\alpha}}\phi^{''}(t)\,dt|\\
&\leq M|x^{\alpha}-y^{\alpha}|\leq M|x-y|^{\alpha}, \qquad M=\sup_{t\in[0,1)}\{\phi^{''}(t)\}
\end{aligned}$$

Converse is not true. Consider the example $f(x)=\frac{x^{\alpha+1}}{\alpha+1}$ $0<\alpha<1$.Then, $f^{'}(x)=x^{\alpha}$ and $f^{''}(x)=\alpha x^{\alpha-1}$. For $x=\frac{1}{N^{\frac{1}{1-\alpha}}}$, $y=\frac{1}{(1+N)^{\frac{1}{1-\alpha}}}$, we have $|x-y|\leq\frac{1}{N^{\frac{1}{1-\alpha}}}-\frac{1}{(1+N)^{\frac{1}{1-\alpha}}}\leq\frac{1}{N^{\frac{1}{1-\alpha}}}=\delta$.
And $|f^{''}(x)-f''(y)|=\alpha(1+N-N)=\alpha$
If $0<\epsilon<\alpha$, then $|f^{''}(x)-f^{''}(y)|\nleq\epsilon$ whenever $|x-y|\leq\delta=\frac{1}{N^{\frac{1}{1-\alpha}}}$. Hence, $f^{'}\in H^{\alpha}[0,1)$ but $f^{''}\notin H^{\alpha}[0,1)$.

3.2 Difference between Hölder's class and Lipschitz class

1. Consider the function $f(x)=\sqrt{x^2+5}$ $\quad\forall x\in[0,1]$. Then

$$|f(x)-f(y)|\leq|\sqrt{x^2+5}-\sqrt{y^2+5}|\leq|\sqrt{x^2-y^2}|\leq\sqrt{2}|x-y|^{\frac{1}{2}} \tag{10}$$

Eq^n(10) shows that $f\in H^{\frac{1}{2}}[0,1)$. And also, we have

$$|f^{'}(x)|\leq|\frac{x}{x^2+5}|\leq 1,\qquad\forall\, x\in[0,1] \tag{11}$$

Eq^n(10) and Eq^n(11) shows that $f \in Lip_{\frac{1}{2}}[0,1)$.

2. Define the function $f(x) = \sqrt{x} \qquad \forall x \in [0,1]$, then we have

$$|f(x) - f(y)| \leq |\sqrt{x} - \sqrt{y}| \leq |x-y|^{\frac{1}{2}} \implies f \in H^{\frac{1}{2}}[0,1).$$

And since, $f^{'}(x) = \frac{1}{2\sqrt{x}} \to \infty$ as $x \to 0^{+}$. Hence, f is not bounded.
$\therefore f \notin Lip_{\frac{1}{2}}[0,1)$. Hence, we conclude that $Lip_{\alpha}[0,1] \subset H^{\alpha}[0,1]$.

4 Approximation of function

Since $\{\psi_{n,m}\}_{n,m\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[0,1]$, therefore a function $f \in L^2[0,1)$ can be expressed into CAS wavelet series as:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} c_{n,m}\psi_{n,m}(t) \tag{12}$$

where the coefficients $c_{n,m}$ are given by

$$c_{n,m} =< f, \psi_{n,m} > \tag{13}$$

$(2^k, 2M+1)^{\text{th}}$ partial sum $S_{2^k,2M+1}(f)(t)$ of (12) is given by

$$S_{2^k,2M+1}(f)(t) = \sum_{n=1}^{2^k} \sum_{m=-M}^{M} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t) \tag{14}$$

where C and $\Psi(t)$ are given by
$C = [c_{1,(-M)}, c_{1,(-M+1)}, ..., c_{1,M}, c_{2,(-M)}, ..., c_{2,M}, ..., c_{2^k,(-M)}, ..., c_{2^k,M}]^T$
and

$$\begin{aligned}\Psi(t) \;\; = \;\; & [\psi_{1,(-M)}(t), \psi_{1,(-M+1)}(t), ..., \psi_{1,M}(t), \psi_{2,(-M)}(t), ..., \psi_{2,M}(t), ..., \\ & \psi_{2^k,(-M)}(t), ..., \psi_{2^k,M}(t)]^T.\end{aligned}$$

Extended Legendre Wavelet expansion of function $f \in L^2[0,1)$ is

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(\mu)}(x),$$

and its $(\mu^k, M)^{th}$ partial sum is

$$S_{\mu^k,M}(f)(x) = \sum_{n=1}^{\mu^k}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}^{(\mu)}(x).$$

The extended Legendre wavelet approximation $E_{\mu^k,M}(f)$ of f by $(\mu^k, M)^{th}$partial sum $S_{\mu^k,M}(f)$ is defined by

$$E_{\mu^k,M}(f) = \min_{S_{\mu^k,M}(f)} ||f - S_{\mu^k,M}(f)||_2 \ .$$

In our case, the CAS wavelet approximation $E_{2^k,2M+1}(f)$ of f by $(2^k, 2M+1)^{\text{th}}$ partial sum $S_{2^k,2M+1}(f)$ of series (12) is defined by

$$E_{2^k,2M+1}(f) = \min_{S_{2^k,2M+1}(f)} ||f - S_{2^k,2M+1}(f)||_2 \ . \tag{15}$$

5 Theorems

In this paper, we prove the following theorems:

Theorem 5.1. *If $f \in L^2[0,1)$ is a function such that $f' \in H^\alpha[0,1)$ and its CAS wavelet expansion is*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} c_{n,m}\psi_{n,m}(t) \tag{16}$$

then the approximation error $E^{(1)}_{2^k,2M+1}(f)$ of f by $(2^k, 2M+1)^{th}$ partial sum

$$S_{2^k,2M+1}(f)(t) = \sum_{n=1}^{2^k} \sum_{m=-M}^{M} c_{n,m}\psi_{n,m}(t) \tag{17}$$

of expansion 16 is given by

$$E^{(1)}_{2^k,2M+1}(f) = \min_{S_{2^k,2M+1}(f)} ||f - (S_{2^k,2M+1}f)||_2 = O(\frac{1}{\sqrt{M+1}\ 2^{k(\alpha+1)}}) \tag{18}$$

Theorem 5.2. *If $f \in L^2[0,1)$ is a function such that $f'' \in H^\alpha[0,1)$ and its CAS wavelet expansion is given by the series (16) , then the approximation error $E^{(2)}_{2^k,2M+1}(f)$ of f by $(2^k, 2M+1)^{th}$ partial sum $S_{2^k,2M+1}(f)(t)$ of series (16) is given by*

$$E^{(2)}_{2^k,2M+1}(f) = \min_{S_{2^k,2M+1}(f)} ||f - (S_{2^k,2M+1}f)||_2 = O(\frac{1}{(M+1)^{\frac{3}{2}}\ 2^{k(\alpha+2)}}) \tag{19}$$

Proof of theorem (5.1) Since

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}c_{n,m}\psi_{n,m}(t)$$

and

$$S_{2^k,2M+1}(f)(t)=\sum_{n=1}^{2^k}\sum_{m=-M}^{M}c_{n,m}\psi_{n,m}(t)$$

$$\begin{aligned}
\therefore f(t)-S_{2^k,2M+1}(f)(t) &= \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}c_{n,m}\psi_{n,m}(t)-\sum_{n=1}^{2^k}\sum_{m=-M}^{M}c_{n,m}\psi_{n,m}(t)\\
&= (\sum_{n=1}^{2^k}+\sum_{n=2^k+1}^{\infty})(\sum_{m=-\infty}^{-M-1}+\sum_{m=-M}^{M}+\sum_{m=M+1}^{\infty})c_{n,m}\psi_{n,m}(t)\\
&\quad -\sum_{n=1}^{2^k}\sum_{m=-M}^{M}c_{n,m}\psi_{n,m}(t)\\
&= \sum_{n=1}^{2^k}\sum_{m=-\infty}^{-M-1}c_{n,m}\psi_{n,m}(t)+\sum_{n=1}^{2^k}\sum_{m=M+1}^{\infty}c_{n,m}\psi_{n,m}(t)\\
(f(t)-S_{2^k,2M+1}(f)(t))^2 &= \sum_{n=1}^{2^k}\sum_{m=-\infty}^{-M-1}c_{n,m}^2\psi_{n,m}^2(t)+\sum_{n=1}^{2^k}\sum_{m=M+1}^{\infty}c_{n,m}^2\psi_{n,m}^2(t)\\
&\quad +2\sum\sum_{1\leqslant\ n\neq n'\leq\ 2^k}\ \sum\sum_{-\infty\leq m\neq m'\leq -M-1}c_{n,m}c_{n',m'}\psi_{n,m}^T(t)\psi_{n',m'}(t)\\
&\quad +2\sum\sum_{1\leqslant\ n\neq n'\leq\ 2^k}\ \sum\sum_{M+1\leq m\neq m'\leq\infty}c_{n,m}c_{n',m'}\psi_{n,m}^T(t)\psi_{n',m'}(t)\\
||f-S_{2^k,2M+1}(f)||_2^2 &= \int_0^1|f(t)-S_{2^k,2M+1}(f)(t)|^2dt\\
&\leqslant \sum_{n=1}^{2^k}\sum_{m=-\infty}^{-M-1}|c_{n,m}|^2\int_0^1|\psi_{n,m}(t)|^2dt\\
&+ \sum_{n=1}^{2^k}\sum_{m=M+1}^{\infty}|c_{n,m}|^2\int_0^1|\psi_{n,m}(t)|^2dt\\
&+ 2\sum\sum_{1\leqslant\ n\neq n'\leq\ 2^k}\ \sum\sum_{-\infty\leq m\neq m'\leq -M-1}|c_{n,m}||c_{n',m'}|\\
&\quad \int_0^1|\psi_{n,m}^T(t)\psi_{n',m'}(t)|dt
\end{aligned}$$

$$+2\sum\sum_{1\leqslant\ n\neq n'\leq\ 2^k}\ \sum\sum_{M+1\leq m\neq m'\leq\infty}|c_{n,m}||c_{n',m'}|\int_0^1|\psi^T_{n,m}(t)\psi_{n',m'}(t)|dt$$

$$=\sum_{n=1}^{2^k}\sum_{m=-\infty}^{-M-1}|c_{n,m}|^2+\sum_{n=1}^{2^k}\sum_{m=M+1}^{\infty}|c_{n,m}|^2\ ,\ \text{by orthonormality of }\{\psi_{n,m}\}_{n,m\in\mathbb{Z}}$$

$$||f-S_{2^k,2M+1}(f)||_2^2\ \leq\ \sum_{n=1}^{2^k}(\sum_{m=-\infty}^{-M-1}+\sum_{m=M+1}^{\infty})|c_{n,m}|^2 \tag{20}$$

$$\begin{aligned}
c_{n,m} &= \ <f,\psi_{n,m}> \\
&= \int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} f(t)\ 2^{\frac{k}{2}}\{\cos(2m\pi(2^kt-n+1))+\sin(2m\pi(2^kt-n+1))\}\ dt \\
&= \frac{1}{2^{\frac{k}{2}}}\int_0^1 f(\frac{x+n-1}{2^k})\ (\cos(2m\pi x)+\sin(2m\pi x))\ dx,\ \ 2^kt-n+1=x \\
&= \frac{1}{(2m\pi)\ 2^{\frac{3k}{2}}}\int_0^1 f^{'}(\frac{x+n-1}{2^k})(\cos(2m\pi x)-\sin(2m\pi x))dx,\ \ \text{integrating by part} \\
&= \frac{1}{(2m\pi)\ 2^{\frac{3k}{2}}}[\int_0^1\{f^{'}(\frac{x+n-1}{2^k})-f^{'}(\frac{n-1}{2^k})\}(\cos(2m\pi x)-\sin(2m\pi x))dx \\
&\quad -f^{'}(\frac{n-1}{2^k})\ \int_0^1(\cos(2m\pi x)-\sin(2m\pi x))dx] \\
&= \frac{1}{(2m\pi)\ 2^{\frac{3k}{2}}}\int_0^1\{f^{'}(\frac{x+n-1}{2^k})-f^{'}(\frac{n-1}{2^k})\}(\cos(2m\pi x)-\sin(2m\pi x))dx \\
|c_{n,m}| &\leqslant \frac{1}{(2m\pi)\ 2^{\frac{3k}{2}}}\int_0^1|f^{'}(\frac{x+n-1}{2^k})-f^{'}(\frac{n-1}{2^k})|\ |\cos(2m\pi x)-\sin(2m\pi x)|dx \\
&\leqslant \frac{A}{(2m\pi)\ 2^{\frac{3k}{2}}}\int_0^1\ |\frac{x}{2^k}|^{\alpha}\ |\cos(2m\pi x)-\sin(2m\pi x)|\ dx,\ \ \text{since}\ \ f^{'}\in H^{\alpha}[0,1)
\end{aligned}$$

Now by Cauchy Schwarz inequality, we have

$$\begin{aligned}
|c_{n,m}| &\leqslant \frac{A}{(2m\pi)\ 2^{\frac{3k}{2}}}\{\int_0^1\ |\frac{x}{2^k}|^{2\alpha}\ dx\}^{\frac{1}{2}}\ \{\int_0^1\ |\cos(2m\pi x)-\sin(2m\pi x)|^2dx\}^{\frac{1}{2}} \\
&= \frac{A}{(2m\pi)\ 2^{(\frac{3k}{2}+k\alpha)}}\{\int_0^1\ |x|^{2\alpha}\ dx\}^{\frac{1}{2}} \\
&= \frac{A}{(2m\pi)\ 2^{(\frac{3}{2}+\alpha)k}}\ \frac{1}{\sqrt{2\alpha+1}} \\
|c_{n,m}| &\leqslant \frac{A}{2m\pi\ \sqrt{2\alpha+1}\ 2^{(\frac{3}{2}+\alpha)k}}
\end{aligned}$$

By eq^n (20) and (21) , we have

$$\begin{aligned}
||f-S_{2^k,2M+1}(f)||_2^2 &\leq \sum_{n=1}^{2^k}(\sum_{m=-\infty}^{-M-1}+\sum_{m=M+1}^{\infty})\frac{A^2}{4m^2\pi^2(2\alpha+1)\,2^{(3+2\alpha)k}}\,,\\
&= \frac{A^2}{4\pi^2(2\alpha+1)}(\sum_{m=-\infty}^{-M-1}+\sum_{m=M+1}^{\infty})\frac{2^k}{2^{(3+2\alpha)k}\,m^2}\\
&= \frac{A^2}{4\pi^2(2\alpha+1)}\,\frac{1}{2^{(2+2\alpha)k}}(\sum_{m=-\infty}^{-M-1}\frac{1}{m^2}+\sum_{m=M+1}^{\infty}\frac{1}{m^2})\\
&= \frac{A^2}{4\pi^2(2\alpha+1)}\,\frac{1}{2^{(1+\alpha)2k}}\,(\frac{1}{M+1}+\frac{1}{M+1})\\
&= \frac{A^2}{2\pi^2(2\alpha+1)}\,\frac{1}{2^{(1+\alpha)2k}}\frac{1}{M+1}\\
\therefore \min_{S_{2^k,2M+1}(f)}||f-S_{2^k,2M+1}(f)||_2 &\leqslant \frac{A}{\pi\sqrt{2(2\alpha+1)}}\,\frac{1}{2^{k(\alpha+1)}}\frac{1}{\sqrt{M+1}}
\end{aligned}$$

$$\therefore E^{(1)}_{2^k,2M+1}(f)=\min_{S_{2^k,2M+1}(f)}||f-S_{2^k,2M+1}(f)||_2=O(\frac{1}{\sqrt{M+1}\,2^{k(\alpha+1)}})$$

Thus, theorem (5.1) is completely established.

Proof of theorem (5.2) Following the steps of the proof of theorem (5.1)

$$\begin{aligned}
c_{n,m} &= \frac{1}{(2m\pi)\,2^{\frac{3k}{2}}}\int_0^1 f^{'}(\frac{x+n-1}{2^k})(\cos(2m\pi x)-\sin(2m\pi x))dx\\
&= \frac{-1}{(4m^2\pi^2)\,2^{\frac{5k}{2}}}\int_0^1 f^{''}(\frac{x+n-1}{2^k})(\cos(2m\pi x)+\sin(2m\pi x))dx,\\
&= \frac{-1}{(4m^2\pi^2)\,2^{\frac{5k}{2}}}[\int_0^1\{f^{''}(\frac{x+n-1}{2^k})-f^{''}(\frac{n-1}{2^k})\}(\cos(2m\pi x)+\sin(2m\pi x))dx\\
&\quad -f^{''}(\frac{n-1}{2^k})\int_0^1(\cos(2m\pi x)-\sin(2m\pi x))dx]\\
|c_{n,m}| &\leqslant \frac{1}{(4m^2\pi^2)\,2^{\frac{5k}{2}}}\int_0^1|f^{''}(\frac{x+n-1}{2^k})-f^{''}(\frac{n-1}{2^k})|\,|\cos(2m\pi x)+\sin(2m\pi x)|dx\\
&\leqslant \frac{B}{(4m^2\pi^2)\,2^{\frac{5k}{2}}}\int_0^1|\frac{x}{2^k}|^{\alpha}\,|\cos(2m\pi x)+\sin(2m\pi x)|dx,\ \ \text{since} f^{''}\in H^{\alpha}[0,1)
\end{aligned}$$

Now by Cauchy Schwarz inequality, we have

$$|c_{n,m}| \leqslant \frac{B}{(4m^2\pi^2)\,2^{(\frac{5k}{2}+k\alpha)}}\{\int_0^1|\frac{x}{2^k}|^{2\alpha}dx\}^{\frac{1}{2}}\{\int_0^1|\cos(2m\pi x)+\sin(2m\pi x)|^2dx\}^{\frac{1}{2}}$$

$$
\begin{aligned}
|c_{n,m}| &\leqslant \frac{B}{(4m^2\pi^2)\, 2^{(\frac{5}{2}+\alpha)k}}\,(\frac{1}{\sqrt{2\alpha+1}}) \\
|c_{n,m}| &\leqslant \frac{B}{4m^2\pi^2\,\sqrt{2\alpha+1}\,2^{(\frac{5}{2}+\alpha)k}} \qquad (21)
\end{aligned}
$$

From eq^n (20) and (21), we have

$$
\begin{aligned}
||f - S_{2^k,2M+1}(f)||_2^2 &= \sum_{n=1}^{2^k}(\sum_{m=-\infty}^{-M-1} + \sum_{m=M+1}^{\infty})|c_{n,m}|^2 \\
&\leqslant \sum_{n=1}^{2^k}(\sum_{m=-\infty}^{-M-1} + \sum_{m=M+1}^{\infty})\frac{B^2}{16m^4\pi^4\,(2\alpha+1)\,2^{(5+2\alpha)k}}\,, \\
&= \frac{B^2}{16\pi^4\,(2\alpha+1)\,2^{(5+2\alpha)k}}(\sum_{m=-\infty}^{-M-1} + \sum_{m=M+1}^{\infty})\frac{2^k}{m^4} \\
&= \frac{B^2}{16\pi^4\,(2\alpha+1)\,2^{(4+2\alpha)k}}(\sum_{m=-\infty}^{-M-1}\frac{1}{m^4} + \sum_{m=M+1}^{\infty}\frac{1}{m^4}) \\
&= \frac{B^2}{16\pi^4\,(2\alpha+1)\,2^{(4+2\alpha)k}}(\frac{1}{3(M+1)^3} + \frac{1}{3(M+1)^3}) \\
&= \frac{B^2}{24\pi^4\,(2\alpha+1)\,2^{2k(\alpha+2)}}\frac{1}{(M+1)^3} \\
\therefore \min_{S_{2^k,M}(f)}||f - S_{2^k,2M+1}(f)||_2 &\leqslant \frac{B}{2\sqrt{6}\pi^2\,\sqrt{(2\alpha+1)}\,2^{k(\alpha+2)}}\frac{1}{(M+1)^{\frac{3}{2}}}
\end{aligned}
$$

$$
\therefore E^{(2)}_{2^k,2M+1}(f) = \min_{S_{2^k,2M+1}(f)}||f - S_{2^k,2M+1}(f)||_2 = O(\frac{1}{(M+1)^{\frac{3}{2}}\,2^{k(\alpha+2)}})
$$

Hence, theorem (5.2) has been proved.

6 Solution of the Fredholm integral equation of second kind

Consider the Fredholm integral equation of second kind given by eq^n (1). Using CAS wavelet approximations,

$$
\begin{aligned}
u(x) &= U^T\Psi(x) = \Psi^T(x)U\,, \qquad (22) \\
f(x) &= F^T\Psi(x) = \Psi^T(x)F\,, \\
\text{and}\ \ K(x,y) &= \Psi^T(x)\mathbf{K}\Psi(y)\,,
\end{aligned}
$$

where **K** is a square matrix of order $2^k(2M+1)$, which is calculated as follows

$$\int_0^1\int_0^1 \psi_{n,m}(x)\psi_{n',m'}(y)K(x,y)dxdy\ , \tag{23}$$

where $1 \leqslant n, n^{'} \leqslant 2^k$ and $-\ M \leqslant m, m^{'} \leqslant M$, equation (1) becomes

$$\Psi^t(x)U = \Psi^t(x)F + \Psi^t(x)\mathbf{K}\int_0^1 \Psi(y)\Psi^t(y)Udy \tag{24}$$

By orthonormality of CAS wavelets, equation (24) reduces to

$$U = (I - \mathbf{K})^{-1}F \tag{25}$$

where I is identity matrix of order $2^k(2M+1)$. Subtituting the value of U from eq^n (25) in eq^n (22) , the solution u(x) of Fredholm integral equation of second kind (1) can be obtained.

6.1 Solution of integral eq^n (1) by Haar wavelet method

Let Haar wavelet solution of intgral eq^n (1) be of the form

$$u(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{26}$$

Subtituting the eq^n (26) in eq^n (1) , we have

$$\sum_{i=1}^{2M} a_i(h_i(x) - g_i(x)) = f(x) \tag{27}$$

where

$$g_i(x) = \int_0^1 k(x,y)h_i(y)dy \tag{28}$$

Taking the collocation points $x_k = \frac{k-\frac{1}{2}}{2M},\ k = 1, 2, ..., 2M$, in eq^{ns} (27) and (26), we obtain

$$\sum_{i=1}^{2M} a_i(h_i(x_k) - g_i(x_k)) = f(x_k) \tag{29}$$

$$\text{and } u(x_k) = \sum_{i=1}^{2M} a_i h_i(x_k) \tag{30}$$

The wavelet coefficients $a_i, i = 1, 2, ..., 2M$ are obtained by solving $2M$ system of equations in (29). Subtituting these coefficients in the eq^n(30) we can obtain the Haar wavelet solution of the integral eq^n (1).

7 Illustrated Numerical Examples

Two Fredholm integral equations have been solved by proposed method ie. CAS wavelet method discussed in this paper. Exact solutions of considered integral eq^n are compared with their approximate solutions obtained by CAS wavelet, Legendre wavelet and Haar wavelet method. The graphs of these solutions are plotted. It is observed that exact solution and approximate solutions of Fredholm integral equations obtained by CAS wavelet method are almost equal. The solutions of Fredholm integral equation derived by the help of CAS wavelet method are more closed than the solutions of this integral equation obtained by Legendre wavelet and Haar wavelet method. This comparison shows the advantages of proposed method of this paper. This is illustrated in following two examples.

Example 1

Subtituting $f(x) = \sin(8\pi x)$ and $K(x, y) = y^2$, in the Fredholm integral equation (1), it reduces to

$$u(x) = \sin(8\pi x) + \int_0^1 y^2 u(y) dy \tag{31}$$

The exact solution of integral eq^n (31) is given by

$$u(x) = \sin(8\pi x) - \frac{3}{16\pi} \tag{32}$$

CAS wavelet solution

For CAS wavelet solution, take $k = 2, M = 1$ in the eq^n (14) . In this case,

$$\begin{aligned}\Psi(x) \;=\; & [\psi_{1,-1}(x), \psi_{1,0}(x), \psi_{1,1}(x), \psi_{2,-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), \\ & \psi_{3,-1}(x), \psi_{3,0}(x), \psi_{3,1}(x), \psi_{4,-1}(x), \psi_{4,0}(x), \psi_{4,1}(x)]^T \end{aligned} \tag{33}$$

where

$$\left.\begin{aligned} \psi_{1,-1}(x) &= 2(\cos(8\pi x) - \sin(8\pi x)) \\ \psi_{1,0}(x) &= 2 \\ \psi_{1,1}(x) &= 2(\cos(8\pi x) + \sin(8\pi x)) \end{aligned}\right\} \quad 0 \leqslant x < \frac{1}{4}\,,$$

$$\left.\begin{aligned} \psi_{2,-1}(x) &= 2(\cos(8\pi x) - \sin(8\pi x)) \\ \psi_{2,0}(x) &= 2 \\ \psi_{2,1}(x) &= 2(\cos(8\pi x) + \sin(8\pi x)) \end{aligned}\right\} \quad \frac{1}{4} \leqslant x < \frac{1}{2}\,,$$

$$\left.\begin{aligned} \psi_{3,-1}(x) &= 2(\cos(8\pi x) - \sin(8\pi x)) \\ \psi_{3,0}(x) &= 2 \\ \psi_{3,1}(x) &= 2(\cos(8\pi x) + \sin(8\pi x)) \end{aligned}\right\} \quad \frac{1}{2} \leqslant x < \frac{3}{4}\,,$$

and

$$\left.\begin{aligned} \psi_{4,-1}(x) &= 2(\cos(8\pi x) - \sin(8\pi x)) \\ \psi_{4,0}(x) &= 2 \\ \psi_{4,1}(x) &= 2(\cos(8\pi x) + \sin(8\pi x)) \end{aligned}\right\} \quad \frac{3}{4} \leqslant x < 1\,.$$

$$F \;=\; [\frac{-1}{4}, 0, \frac{1}{4}, \frac{-1}{4}, 0, \frac{1}{4}, \frac{-1}{4}, 0, \frac{1}{4}, \frac{-1}{4}, 0, \frac{1}{4}]^T,$$

The matrix **K** is calculated as follows:

$$\begin{aligned}
K_{i,j} &= \int_0^1 \int_0^1 \psi_i(x) K(x,y) \psi_j(y) dy dx \\
&= \int_0^1 \psi_i(x) \; (\int_0^1 y^2 \psi_j(y) dy) \; dx \\
&= (\int_0^1 \psi_i(x) dx) \; (\int_0^1 y^2 \psi_j(y) dy)
\end{aligned}$$

$$\mathbf{K} = \begin{bmatrix} \frac{\pi+1}{64\pi^2} \\ \frac{1}{96} \\ -\frac{\pi+1}{64\pi^2} \\ \frac{3\pi+1}{64\pi^2} \\ \frac{7}{96} \\ -\frac{3\pi+1}{64\pi^2} \\ \frac{5\pi+1}{64\pi^2} \\ \frac{19}{96} \\ -\frac{5\pi+1}{64\pi^2} \\ \frac{7\pi+1}{64\pi^2} \\ \frac{37}{96} \\ -\frac{7\pi+1}{96\pi^2} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0 & \frac{\pi+1}{128\pi^2} & 0 & 0 & \frac{\pi+1}{128\pi^2} & 0 & 0 & \frac{\pi+1}{128\pi^2} & 0 & 0 & \frac{\pi+1}{128\pi^2} & 0 \\ 0 & \frac{1}{192} & 0 & 0 & \frac{1}{192} & 0 & 0 & \frac{1}{192} & 0 & 0 & \frac{1}{192} & 0 \\ 0 & \frac{-\pi+1}{128\pi^2} & 0 & 0 & \frac{-\pi+1}{128\pi^2} & 0 & 0 & \frac{-\pi+1}{128\pi^2} & 0 & 0 & \frac{-\pi+1}{128\pi^2} & 0 \\ 0 & \frac{3\pi+1}{128\pi^2} & 0 & 0 & \frac{3\pi+1}{128\pi^2} & 0 & 0 & \frac{3\pi+1}{128\pi^2} & 0 & 0 & \frac{3\pi+1}{128\pi^2} & 0 \\ 0 & \frac{7}{192} & 0 & 0 & \frac{7}{192} & 0 & 0 & \frac{7}{192} & 0 & 0 & \frac{7}{192} & 0 \\ 0 & \frac{-3\pi+1}{128\pi^2} & 0 & 0 & \frac{-3\pi+1}{128\pi^2} & 0 & 0 & \frac{-3\pi+1}{128\pi^2} & 0 & 0 & \frac{-3\pi+1}{128\pi^2} & 0 \\ 0 & \frac{5\pi+1}{128\pi^2} & 0 & 0 & \frac{5\pi+1}{128\pi^2} & 0 & 0 & \frac{5\pi+1}{128\pi^2} & 0 & 0 & \frac{5\pi+1}{128\pi^2} & 0 \\ 0 & \frac{19}{192} & 0 & 0 & \frac{19}{192} & 0 & 0 & \frac{19}{192} & 0 & 0 & \frac{19}{192} & 0 \\ 0 & \frac{-5\pi+1}{128\pi^2} & 0 & 0 & \frac{-5\pi+1}{128\pi^2} & 0 & 0 & \frac{-5\pi+1}{128\pi^2} & 0 & 0 & \frac{-5\pi+1}{128\pi^2} & 0 \\ 0 & \frac{7\pi+1}{128\pi^2} & 0 & 0 & \frac{7\pi+1}{128\pi^2} & 0 & 0 & \frac{7\pi+1}{128\pi^2} & 0 & 0 & \frac{7\pi+1}{128\pi^2} & 0 \\ 0 & \frac{37}{192} & 0 & 0 & \frac{37}{192} & 0 & 0 & \frac{37}{192} & 0 & 0 & \frac{37}{192} & 0 \\ 0 & \frac{-7\pi+1}{128\pi^2} & 0 & 0 & \frac{-7\pi+1}{128\pi^2} & 0 & 0 & \frac{-7\pi+1}{128\pi^2} & 0 & 0 & \frac{-7\pi+1}{128\pi^2} & 0 \end{bmatrix}$$

$$\begin{aligned} \therefore U &= (I-\mathbf{K})^{-1}F \\ &= [\frac{-1}{4},0,\frac{1}{4},\frac{-1}{4},0,\frac{1}{4},\frac{-1}{4},0,\frac{1}{4},\frac{-1}{4},0,\frac{1}{4}]^T \end{aligned} \tag{34}$$

Putting the values of $\Psi(x)$ and U from eq[ns] (33) and (34) in eq[n] (22), we have

$$u(x) = \sin(8\pi x) \tag{35}$$

which is the CAS wavelet solution of the integral equation (31) .

Legendre wavelet solution

Legendre wavelets $\psi^{(L)}_{n,m}(t) = \psi^{(L)}(k,n,m,t)$ having four arguments; $k = 2,3,...,$
$2n-1$, $n = 1,2,3,...,2^{k-1}$, m is the order of the Legendre polynomial and t is the normalised time, are defined by :

$$\psi^{(L)}_{n,m}(t) = \begin{cases} (m+\frac{1}{2})^{\frac{1}{2}} 2^{\frac{k}{2}} P_m(2^k t - 2n + 1), & if\ \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ 0, & otherwise. \end{cases} \tag{36}$$

where $P_m(t)$ are Legendre ploynomials of order m (Rehman and Khan [7]). The set $\{\psi^{(L)}_{n,m}\}_{n,m\in\mathbb{Z}}$ of Legendre wavelets forms an orthonormal set. A function $f \in L^2[0,1)$ may be expanded into Legendre wavelet series as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(L)}_{n,m}(t), \tag{37}$$

where $c_{n,m} =< f, \psi^{(L)}_{n,m} >$ .The series (37) may be truncated as:

$$(f)(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi^{(L)}_{n,m}(t) = C^T\Psi^{(L)}(t) \tag{38}$$

where C and $\Psi^{(L)}(t)$ are $2^{k-1}M \times 1$ matrices given by:

$$\begin{aligned} C &= [c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, ..., c_{2,M-1}, ..., \\ &\quad c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}]^T \end{aligned}$$

and

$$\begin{aligned} \Psi^{(L)}(t) &= [\psi^{(L)}_{1,0}(t), \psi^{(L)}_{1,1}(t), ..., \psi^{(L)}_{1,M-1}(t), \psi^{(L)}_{2,0}(t), ..., \psi^{(L)}_{2,M-1}(t), ..., \\ &\quad \psi^{(L)}_{2^{k-1},0}(t), ..., \psi^{(L)}_{2^{k-1},M-1}(t)]^T \end{aligned}$$

Similarly, a function $K \in L^2[0,1) \times L^2[0,1)$ may be approximated as:

$$K(x,y) \approx (\Psi^{(L)})^T(x)\mathbf{K}^{(L)}\Psi^{(L)}(y),$$

where $\mathbf{K}^{(L)}$ is $2^{k-1}M \times 2^{k-1}M$ matrix, whose entries are given by

$$\mathbf{K}^{(L)}_{i,j} =< \psi^{(L)}_i(x), < K(x,y), \psi^{(L)}_j(y) >> . \tag{39}$$

For Legendre wavelet solution, take $M = 3, k = 3$ in eq^n (38), then twelve basis functions are given by

$$\begin{aligned}\Psi^{(L)}(x) &= [\psi_{1,0}^{(L)}(x), \psi_{1,1}^{(L)}(x), \psi_{1,2}^{(L)}(x), \psi_{2,0}^{(L)}(x), \psi_{2,1}^{(L)}(x), \psi_{2,2}^{(L)}(x), \\ &\quad \psi_{3,0}^{(L)}(x), \psi_{3,1}^{(L)}(x), \psi_{3,2}^{(L)}(x), \psi_{4,0}^{(L)}(x), \psi_{4,1}^{(L)}(x), \psi_{4,2}^{(L)}(x)]^T \end{aligned} \tag{40}$$

where

$$\left.\begin{aligned}&\psi_{1,0}^{(L)}(x) = 2\\ &\psi_{1,1}^{(L)}(x) = 2\sqrt{3}(8x-1)\\ &\psi_{1,2}^{(L)}(x) = \sqrt{5}(3(8x-1)^2-1)\end{aligned}\right\} \quad 0 \leqslant x < \frac{1}{4}\,,$$

$$\left.\begin{aligned}&\psi_{2,0}^{(L)}(x) = 2\\ &\psi_{2,1}^{(L)}(x) = 2\sqrt{3}(8x-3)\\ &\psi_{2,2}^{(L)}(x) = \sqrt{5}(3(8x-3)^2-1)\end{aligned}\right\} \quad \frac{1}{4} \leqslant x < \frac{1}{2}\,,$$

$$\left.\begin{aligned}&\psi_{3,0}^{(L)}(x) = 2\\ &\psi_{3,1}^{(L)}(x) = 2\sqrt{3}(8x-5)\\ &\psi_{3,2}^{(L)}(x) = \sqrt{5}(3(8x-5)^2-1)\end{aligned}\right\} \quad \frac{1}{2} \leqslant x < \frac{3}{4}\,,$$

and

$$\left.\begin{aligned}&\psi_{4,0}^{(L)}(x) = 2\\ &\psi_{4,1}^{(L)}(x) = 2\sqrt{3}(8x-7)\\ &\psi_{4,2}^{(L)}(x) = \sqrt{5}(3(8x-7)^2-1)\end{aligned}\right\} \quad \frac{3}{4} \leqslant x < 1\,.$$

$$\mathbf{K}^{(L)} = \begin{bmatrix} \frac{1}{192} & 0 & 0 & \frac{1}{192} & 0 & 0 & \frac{1}{192} & 0 & 0 & \frac{1}{192} & 0 & 0 \\ \frac{\sqrt{3}}{384} & 0 & 0 & \frac{\sqrt{3}}{384} & 0 & 0 & \frac{\sqrt{3}}{384} & 0 & 0 & \frac{\sqrt{3}}{384} & 0 & 0 \\ \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 \\ \frac{7}{192} & 0 & 0 & \frac{7}{192} & 0 & 0 & \frac{7}{192} & 0 & 0 & \frac{7}{192} & 0 & 0 \\ \frac{\sqrt{3}}{128} & 0 & 0 & \frac{\sqrt{3}}{128} & 0 & 0 & \frac{\sqrt{3}}{128} & 0 & 0 & \frac{\sqrt{3}}{128} & 0 & 0 \\ \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 \\ \frac{19}{192} & 0 & 0 & \frac{19}{192} & 0 & 0 & \frac{19}{192} & 0 & 0 & \frac{19}{192} & 0 & 0 \\ \frac{5\sqrt{3}}{384} & 0 & 0 & \frac{5\sqrt{3}}{384} & 0 & 0 & \frac{5\sqrt{3}}{384} & 0 & 0 & \frac{5\sqrt{3}}{384} & 0 & 0 \\ \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 \\ \frac{37}{92} & 0 & 0 & \frac{37}{92} & 0 & 0 & \frac{37}{92} & 0 & 0 & \frac{37}{92} & 0 & 0 \\ \frac{7\sqrt{3}}{384} & 0 & 0 & \frac{7\sqrt{3}}{384} & 0 & 0 & \frac{7\sqrt{3}}{384} & 0 & 0 & \frac{7\sqrt{3}}{384} & 0 & 0 \\ \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 & \frac{\sqrt{5}}{1920} & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} F^{(L)} &= [0, \frac{-\sqrt{3}}{2\pi}, 0, 0, \frac{-\sqrt{3}}{2\pi}, 0, 0, \frac{-\sqrt{3}}{2\pi}, 0, 0, \frac{-\sqrt{3}}{2\pi}, 0]^T, \\ U^{(L)} &= (I - \mathbf{K}^{(L)})^{-1} F^{(L)} \\ &= [0, \frac{-\sqrt{3}}{2\pi}, 0, 0, \frac{-\sqrt{3}}{2\pi}, 0, 0, \frac{-\sqrt{3}}{2\pi}, 0, 0, \frac{-\sqrt{3}}{2\pi}, 0]^T. \end{aligned} \tag{41}$$

Putting the values of $\Psi^{(L)}(x)$ and $U^{(L)}$ from eq^{ns} (40) and (41) in eq^{n} (22), we get the Legendre wavelet solution of the integral equation (31) as:

$$u(x) = -\frac{\sqrt{3}}{2\pi}\psi_{1,1}^{(L)}(x) - \frac{\sqrt{3}}{2\pi}\psi_{2,1}^{(L)}(x) - \frac{\sqrt{3}}{2\pi}\psi_{3,1}^{(L)}(x) - \frac{\sqrt{3}}{2\pi}\psi_{4,1}^{(L)}(x) \tag{42}$$

Haar wavelet solution

The Haar wavelet family for $x \in [0,1]$ is defined as follows:

$$h_i(x) = \begin{cases} 1 & if \ \ x \in [\frac{k}{m}, \frac{k+\frac{1}{2}}{m}), \\ -1 & if x \in [\frac{k+\frac{1}{2}}{m}, \frac{k+1}{m}), \\ 0, & otherwise \end{cases} \tag{43}$$

where $m = 2^b, \ b = 0,1,...,J$ is the level of wavelet; $k = 0,1,...,m-1$ is the translation parameter. J is the maximum level of resulution. i is calculated by $i = m+k+1$. The minimum value of i for $m = 1, k = 0$ is 2. The maximum value of i is $i = 2M = 2^{J+1}$ (Arbabi and Darvishi [6]).

For $i = 1, \ h_1(x)$ is taken to be scaling function which is defined as follows:

$$h_1(x) = \begin{cases} 1 & if \ \ x \in [0,1), \\ 0, & otherwise \end{cases}$$

Any function $f(x)$ can be expressed in terms of Haar wavelets as follows:

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x), \tag{44}$$

where the wavelet coefficients $a_i, \ i = 1,2,...,2M$ are to be determined. For Haar wavelet solution take $J = 3$ in eq^{n} (43), $b = 0,1,2,3$, then $m = 2^b = 1,2,4,8$. By eq^{ns} (28) the Haar wavelet coefficients $a_i, \ i = 1,2,...,16$ are given by

$$[-0.008071, 0.001459, 0.002497, 0.001447, 0.000485, 0.006380, \\ 0.000488, -0.000476, 1.000010, 1, 1, 0.988178, 1, 1, 0.999039, 1] \tag{45}$$

Putting these values of a_i in the eq^{n} (26), we get the solution of integral equation (31) by Haar wavelet method. The Haar wavelet solutions of integral eq^{n} 31 are shown in the Table (1).

The exact solution and approximate solutions of Fredholm integral equation (31) obtained by CAS wavelet, Legendre wavelet and Haar wavelet method for different values of x are given in the Table (1).

Table (1)

x	Exact sol^n by eq^n 32	CAS wavelet sol^n by eq^n (35)	Legendre wavelet sol^n by eq^n (42)	Haar wavelet sol^n by eq^n (26)
0	-0.059680	0	0.954930	0.996370
0.1	0.528105	0.587785	0.190986	-1.003630
0.2	-1.010736	-0.951056	-0.572958	0.995399
0.3	0.891376	0.951056	0.572958	0.995399
0.4	-0.647465	-0.587785	-0.190986	0.972689
0.5	-0.059680	0	-0.954930	0.992404
0.6	0.528105	0.587785	0.190986	-1.008083
0.7	-1.010736	-0.951056	-0.572958	-1.007595
0.8	0.891376	0.951056	0.572958	0.987585
0.9	-0.647465	-0.587785	-0.190986	0.989498

The graphs of the exact solution and approximate solutions of integral equation (31) obtained by CAS wavelet, Legendre wavelet and Haar wavelet method are shown in the Fig.(1).

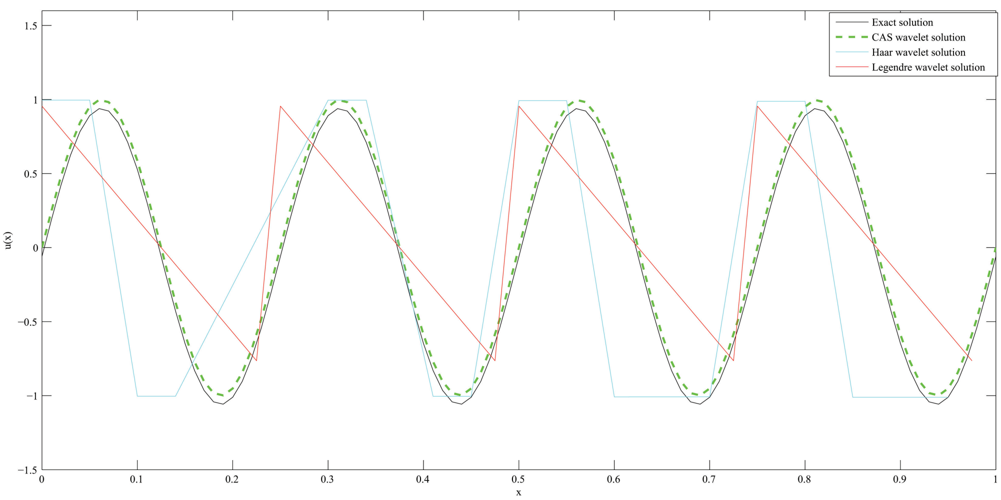

Fig.(1)

By numerical comparison in Table(1) and graphs shown in Fig.(1), it is clear that the solution of Fredholm integral equation (31) by CAS wavelet method is better than solutions obtained by Legendre wavelet and Haar wavelet methods.

Example 2

Consider the Fredholm integral equation:

$$u(x) = \sin(4\pi x) + \int_0^1 xyu(y)dy \ . \tag{46}$$

It is obtained by subtituting $f(x) = \sin(4\pi x)$ and $K(x, y) = xy$, in the Fredholm integral equation (1). The exact solution of Fredholm integral equation (46) is given by

$$u(x) = \sin(4\pi x) - \frac{3x}{8\pi} \tag{47}$$

CAS wavelet solution

For CAS wavelet solution, take $k = 1, M = 1$ in eq^n (14), then following the procedure of example (31), we have

$$F^* = [\frac{-1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}]^T,$$

The matrix $\mathbf{K}^*$ is calculated as follows:

$$\begin{aligned} K^*_{i,j} &= \int_0^1 \int_0^1 \psi_i(x)K(x,y)\psi_j(y)dydx \\ &= \int_0^1 \psi_i(x) \ (\int_0^1 xy\psi_j(y)dy) \ dx \\ &= (\int_0^1 x\psi_i(x)dx) \ (\int_0^1 y\psi_j(y)dy) \end{aligned}$$

$$\mathbf{K} = \begin{bmatrix} \frac{\sqrt{2}}{8\pi} \\ \frac{\sqrt{2}}{8} \\ -\frac{\sqrt{2}}{8\pi} \\ \frac{\sqrt{2}}{8\pi} \\ \frac{\sqrt{2}}{8} \\ -\frac{\sqrt{2}}{8\pi} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{8\pi} & \frac{\sqrt{2}}{8} & -\frac{\sqrt{2}}{8\pi} & \frac{\sqrt{2}}{8\pi} & \frac{\sqrt{2}}{8} & -\frac{\sqrt{2}}{8\pi} \end{bmatrix}$$

$$\mathbf{K}^* = \begin{bmatrix} \frac{1}{32\pi^2} & \frac{1}{32\pi} & \frac{-1}{32\pi^2} & \frac{1}{32\pi^2} & \frac{1}{32\pi} & \frac{-1}{32\pi^2} \\ \frac{1}{32\pi} & \frac{1}{32} & \frac{-1}{32\pi} & \frac{1}{32\pi} & \frac{1}{32} & \frac{-1}{32\pi} \\ \frac{-1}{32\pi^2} & \frac{-1}{32\pi} & \frac{1}{32\pi^2} & \frac{-1}{32\pi^2} & \frac{-1}{32\pi} & \frac{1}{32\pi^2} \\ \frac{1}{32\pi^2} & \frac{1}{32\pi} & \frac{-1}{32\pi^2} & \frac{1}{32\pi^2} & \frac{1}{32\pi} & \frac{-1}{32\pi^2} \\ \frac{3}{32\pi} & \frac{3}{32} & \frac{-3}{32\pi} & \frac{3}{32\pi} & \frac{3}{32} & \frac{-3}{32\pi} \\ \frac{-1}{32\pi^2} & \frac{-1}{32\pi} & \frac{1}{32\pi^2} & \frac{-1}{32\pi^2} & \frac{-1}{32\pi} & \frac{1}{32\pi^2} \end{bmatrix}$$

and

$$U^* = [\frac{-1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}]^T.$$

$$u(x) = 1.0188\,\sin(4\pi x) - 0.0294 \tag{48}$$

This is the approximate solution of the integral equation (46) by CAS wavelet method.

Legendre wavelet solution

For Legendre wavelet solution, take $M = 3, k = 2$ in eq^n (38), then we have

$$\Psi^{(L)}(x) = [\psi_{1,0}^{(L)}(x), \psi_{1,1}^{(L)}(x), \psi_{1,2}^{(L)}(x), \psi_{2,0}^{(L)}(x), \psi_{2,1}^{(L)}(x), \psi_{2,2}^{(L)}(x)]. \tag{49}$$

Following the procedure of the example (1), we have

$$\begin{aligned} (F^*)^{(L)} &= [0, \frac{-\sqrt{6}}{2\pi}, 0, 0, \frac{-\sqrt{6}}{2\pi}, 0]^T, \\ (U^*)^{(L)} &= [-0.0211, -0.4020, 0, -0.0633, -0.4020, 0]^T \end{aligned} \tag{50}$$

Putting the values of $\Psi^{(L)}(x)$ and $(U^*)^{(L)}$ from eq^{ns} (49) and (50) in eq^n (22), we get the solution of the integral equation (46) by Legendre wavelet method as

$$u(x) = -0.0211\psi_{1,0}^{(L)}(x) - 0.4020\psi_{1,1}^{(L)}(x) - 0.0633\psi_{2,0}^{(L)}(x) - 0.4020\psi_{2,1}^{(L)}(x) \tag{51}$$

$$(\mathbf{K}^*)^{(L)} = \begin{bmatrix} \frac{1}{32} & \frac{\sqrt{3}}{96} & 0 & \frac{3}{32} & \frac{\sqrt{3}}{96} & 0 \\ \frac{\sqrt{3}}{96} & \frac{1}{96} & 0 & \frac{\sqrt{3}}{32} & \frac{1}{96} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{32} & \frac{\sqrt{3}}{32} & 0 & \frac{9}{32} & \frac{\sqrt{3}}{32} & 0 \\ \frac{\sqrt{3}}{96} & \frac{1}{96} & 0 & \frac{\sqrt{3}}{32} & \frac{1}{96} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Haar wavelet solution

For Haar wavelet solution, take $J = 2$ in eq^n (43),$b = 0, 1, 2$ then $m = 2^b = 1, 2, 4$. The Haar wavelet coefficients $a_i, i = 1, 2, ..., 8$ are given by

$$[0.061361, 0.027885, 0.616015, 0.670955, 0.000922, 1.465270, 0.000264, 0.004906]$$

Putting these values of a_i in the eq^n (26), we get the solution of integral equation (46) by Haar wavelet method. The Haar wavelet solutions of integral eq^n 46 are given in the Table (2).

The exact solution and approximate solutions of Fredholm integral equation (46) obtained by CAS wavelet, Legendre wavelet and Haar wavelet method for different values of x are given in the Table (2).

Table (2)

x	Exact sol^n by eq^n (47)	CAS wavelet sol^n by eq^n (48)	Legendre wavelet sol^n by eq^n (51)	Haar wavelet sol^n by eq^n (26)
0	0	-0.0294	0.9549	0.6955
0.1	0.9391	0.9395	0.5610	0.6955
0.2	0.5639	0.5694	0.1671	0.6722
0.3	-0.6236	-0.6282	-0.2268	-0.7701
0.4	-0.9988	-0.9983	-0.6207	-1.7239
0.5	-0.0597	-0.0294	0.8952	0.6194
0.6	0.8794	0.9395	0.5013	0.6194
0.7	0.5042	0.5694	0.1074	-0.3672
0.8	-0.6833	-0.6282	-0.2865	-0.8337
0.9	-1.0585	-0.9983	-0.6803	-0.8532

The graphs of the exact solution and approximate solutions of integral equation (46) obtained by CAS wavelet, Legendre wavelet and Haar wavelet method are shown in the Fig.(2).

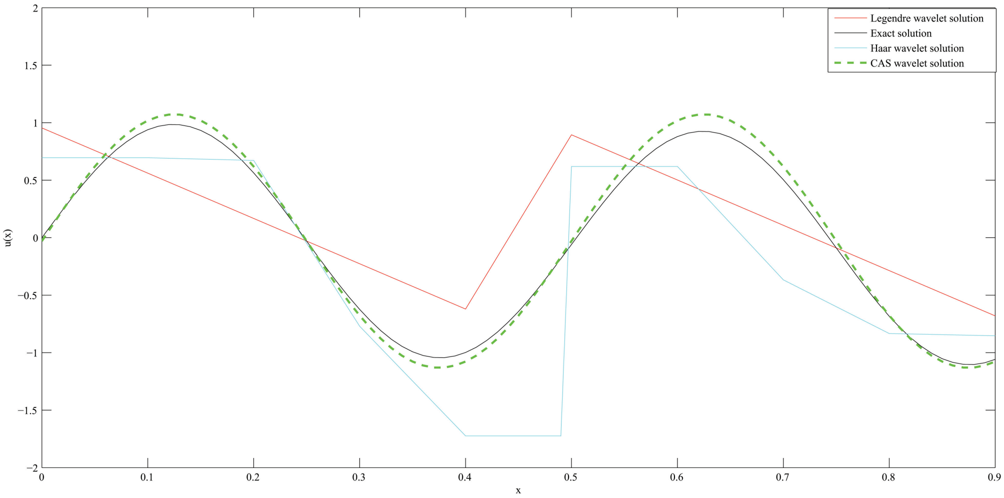

Fig.(2)

By numerical comparison in Table(2) and graphs shown in Fig.(2), it is observed that the solution of Fredholm integral equation (46) by CAS wavelet method is more accurate than solutions obtained by Legendre wavelet and Haar wavelet methods.

Note: The solutions of Fredholm integral equations in examples (1) and (2) by CAS wavelet method propoesd in this research paper and their numerical comparison with Legendre wavelet and Haar wavelet methods show the advantages of CAS wavelet method than Legendre wavelet and Haar wavelet methods.

8 Remarks

1. CAS wavelet approximation of Theorem (5.1) is given by
$E^{(1)}_{2^k,2M+1}(f) = O(\frac{1}{\sqrt{M+1}\ 2^{k(\alpha+1)}})$. $E^{(1)}_{2^k,2M+1}(f) \to 0$ as $M \to \infty,\ k \to \infty$.

CAS wavelet approximation of Theorem (5.2) is given by
$E^{(2)}_{2^k,2M+1}(f) = O(\frac{1}{(M+1)^{\frac{3}{2}}\ 2^{k(\alpha+2)}})$. $E^{(2)}_{2^k,2M+1}(f) \to 0$ as $M \to \infty,\ k \to \infty$.

Therefore, estimators $E^{(1)}_{2^k,2M+1}(f)$ and $E^{(2)}_{2^k,2M+1}(f)$ are best possible in wavelet analysis (Zygmund [5]).

2.
$$\begin{aligned} \because (M+1)^{\frac{3}{2}}2^{k(\alpha+2)} &\geqslant (M+1)^{\frac{1}{2}}2^{k(\alpha+1)},\ M\geqslant 1, k\geqslant 1 \\ \therefore \frac{1}{(M+1)^{\frac{3}{2}}2^{k(\alpha+2)}} &\leqslant \frac{1}{(M+1)^{\frac{1}{2}}2^{k(\alpha+1)}} \\ ie.\ E^{(2)}_{2^k,2M+1}(f) &\leqslant E^{(1)}_{2^k,2M+1}(f). \end{aligned}$$

Hence, estimator $E^{(2)}_{2^k,2M+1}(f)$ is sharper than estimator $E^{(1)}_{2^k,2M+1}(f)$. This shows that the estimator of a function f having $f'' \in H^\alpha[0,1)$ is sharper than the estimator of f having $f' \in H^\alpha[0,1)$.

3. CAS wavelet method is more effective than Legendre wavelet and Haar wavelet method in finding the solution of Fredholm integral equations (31) and (46).

4. Fredholm integral equation of first kind,

$$\int_0^1 K(x,t)y(t)dt = f(x)$$

can be solved by CAS wavelet method as follows:

$$\int_0^1 \Psi(x)\mathbf{K}\Psi^T(t)\Psi(t)Y = \Psi(x)F$$

Using orthonormality of CAS wavelet, we get $\mathbf{K}Y = F$. By finding the matrix $\mathbf{K}$ and F as in the case of Fredholm integral of second kind, we can find Y and hence the solution y(x).

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