

On Homomorphisms from \mathbb{C}^n to \mathbb{C}^m

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Abstract

In this paper, using elementary algebra and analysis, we characterize and compute all continuous homomorphism from \mathbb{C}^n to \mathbb{C}^m . Also we prove that the cardinality of the set of all non-continuous group homomorphism from \mathbb{C}^n to \mathbb{C}^m is at least the cardinality of the continuum.

Keywords: homomorphism; continuous function; Hamel basis;

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1 Introduction

Hamel [1905] introduced the concept of basis for real numbers and proved its existence in 1905 by exploring functions which satisfy Cauchy's functional equation $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Using the existence of such a basis, he described all solutions of Cauchy's functional equation and established the existence of discontinuous solutions. Cauchy demonstrated that any additive function is rationally homogeneous. He also proved that the only continuous additive functions are real homogeneous and thus linear, and that an additive function with a discontinuity is discontinuous throughout. Further restrictions were placed on a non-linear additive function by Darboux [1875] who showed in 1875 that an additive function bounded above or below on some interval is continuous, hence linear. A survey of the research concerning additive functions can be found in Green and Gustin [1950]

The continuous ring homomorphisms from \mathbb{C} to \mathbb{C} are trivial map, identity map and complex conjugation. Since \mathbb{C} is a field, all non-trivial ring homomorphisms are automorphisms on \mathbb{C} . Thus identity map and complex conjugation are the only continuous automorphisms on \mathbb{C} . Any automorphisms on \mathbb{C} other than identity and complex conjugation is called a "wild" automorphism on \mathbb{C} . Kestelman [1951] proved the existence of so-called wild automorphism on \mathbb{C} and the showed that the set of such automorphisms on \mathbb{C} has cardinality 2^c . Many properties of wild automorphism on \mathbb{C} are still open.

Calculating the number of homomorphisms between two groups or two rings is a fundamental problem in abstract algebra. It is not easy to determine the number of distinct homomorphism between any two given groups or rings. Most of the current results in this area are limited to groups or specific types of rings. For example, Chigira et al. [2000] studied the number of homomorphisms from a finite group to a general linear group over a finite field. In a later study Bate [2007] furnished the upper and lower limits for the number of completely reducible homomorphisms from a finite group Γ to general linear and unitary groups over arbitrary finite fields and to orthogonal and symplectic groups over finite fields of odd characteristics. Matei and Suciu [2005] discusses a method for calculating the number of epimorphisms from a finitely presented group G to a finite solvable group Γ . Further discussion on homomorphisms on certain finite groups can be found in Mal'cev [1983], Riley [1971], Hyers and Rassias [1992], but the solution to the general problem is still elusive. Hence the purpose of the paper is to characterize and compute all continuous group homomorphisms from \mathbb{C}^n to \mathbb{C}^m .

2 Notations and Basic Results

Most of the notations, functions and terms we mentioned in this paper can be find in Jacobson [2013], Gallian [1994] and Kestelman [1951].

We can interpret Hamel's concept as follows. The set \mathbb{R} of real numbers is a linear space over the field \mathbb{Q} of rational numbers. This linear space has a basis. Namely, there exists a subset $H \subset \mathbb{R}$ such that every non-zero $x \in \mathbb{R}$ can uniquely be written as a linear combination of the elements of H with rational coefficients. That is, there exist distinct elements h_1, h_2, \dots, h_k of H and non-zero rational numbers $w_{h_1}(x), w_{h_2}(x), \dots, w_{h_k}(x)$ such that

$$x = \sum_{i=1}^k w_{h_i}(x)h_i \quad (1)$$

Thus for $x \in \mathbb{R}$, by adding the terms of the form $0 \cdot h_j$ in the representation (1), we can write

$$x = \sum_{h \in H} w_h(x)h \quad (2)$$

where $w_h(x) \in \mathbb{Q}$ and $w_h(x) = 0$ for all h except for a finite number of values of h . Hamel based his argument on Zermelo's fundamental result which states that every set can be well ordered. Hamel's argument is valid for an arbitrary linear space $L \neq \{0\}$ over a field. For this reason, recently such a basis is called a Hamel basis(see also Cohn and Cohn [1981], Halpern [1966], Jacobson [2013], Kharazishvili [2017]).

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is additive, then it is easy to derive

$$f(rx) = rf(x)$$

for every $r \in \mathbb{Q}$ and $x \in \mathbb{R}$. Thus, if $H \subset \mathbb{R}$ is a Hamel basis and x is a real number, we obtain

$$f(x) = f\left(\sum_{h \in H} w_h(x)h\right) = \sum_{h \in H} f\left(w_h(x)h\right) = \sum_{h \in H} w_h(x)f(h). \quad (3)$$

Observing that the Hamel bases of a linear space L coincide with the maximal linearly independent subsets of L the existence of a Hamel basis is established with the aid of Zorn's maximum principle.

Theorem 2.1. *Let L be a vector space over the field F . Then L has a Hamel basis.*

Theorem 2.2. *Any continuous function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ which assume only rational values is constant.*

Halbeisen and Hungerbühler [2000] showed that in an infinite dimensional Banach space, every Hamel base has the cardinality of the Banach space, which is at least the cardinality of the continuum.

Theorem 2.3. *If $K \subset \mathbb{C}$ is a field and E is a Banach space over K such that $\dim(E) = \infty$, then every Hamel base of E has cardinality $|E|$.*

3 Homomorphisms from \mathbb{C}^n to \mathbb{C}^m

First we will characterize all continuous group homomorphisms from \mathbb{C}^n to \mathbb{C}^m .

Theorem 3.1. *The cardinality of the set of continuous group homomorphisms from \mathbb{C}^n to \mathbb{C}^m is equal to the cardinality of the continuum.*

Proof. Let $\phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a continuous group homomorphism. For $1 \leq j \leq n$; denote e_j for the n -tuple whose j^{th} component is 1 and 0's elsewhere, and denote \hat{e}_j for the n -tuple whose j^{th} component is i and 0's elsewhere.

We will complete the proof by the following steps.

Step 1: $\phi(ne_j) = n\phi(e_j)$ and $\phi(n\hat{e}_j) = n\phi(\hat{e}_j)$ for all $n \in \mathbb{Z}$ and for all j ($1 \leq j \leq n$).

For $n \in \mathbb{N}$, the argument is clear since ϕ is a group homomorphism.

Since ϕ is a group homomorphis,

$$\phi(-ne_j) = -\phi(ne_j) = -n\phi(e_j) \quad \text{and} \quad \phi(0e_j) = 0\phi(e_j)$$

Therefore $\phi(ne_j) = n\phi(e_j)$ for all $n \in \mathbb{Z}$ and for all j ($1 \leq j \leq n$). Similarly we can prove $\phi(n\hat{e}_j) = n\phi(\hat{e}_j)$ for all $n \in \mathbb{Z}$ and for all j ($1 \leq j \leq n$).

Step 2: $\phi(re_j) = r\phi(e_j)$ and $\phi(r\hat{e}_j) = r\phi(\hat{e}_j)$ for all $r \in \mathbb{Q}$ and for all j ($1 \leq j \leq n$).

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Let $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$. Then $rq = p$ and hence $rqe_j = pe_j$. So

$$\begin{aligned}\phi(rqe_j) &= \phi(pe_j) \\ \implies q\phi(re_j) &= p\phi(e_j) \\ \implies \phi(re_j) &= \frac{p}{q}\phi(e_j) \\ \implies \phi(re_j) &= r\phi(e_j), \text{ for all } r \in \mathbb{Q} \text{ and for all } j (1 \leq j \leq n).\end{aligned}$$

Similarly, $\phi(r\hat{e}_j) = r\phi(\hat{e}_j)$ for all $r \in \mathbb{Q}$ and for all $j (1 \leq j \leq n)$.

Step 3: $\phi(xe_j) = x\phi(e_j)$ and $\phi(x\hat{e}_j) = x\phi(\hat{e}_j)$ for all $x \in \mathbb{R}$ and for all $j (1 \leq j \leq n)$.

Let $x \in \mathbb{R}$ and $1 \leq j \leq n$. Then there is a sequence (r_m) of rational numbers such that $r_m \rightarrow x$ in \mathbb{R} . Then $r_me_j \rightarrow xe_j$ as $m \rightarrow \infty$. Since ϕ is continuous at xe_j , we have

$$\begin{aligned}\phi(xe_j) &= \lim_{m \rightarrow \infty} \phi(r_me_j) \\ &= \left(\lim_{m \rightarrow \infty} r_m \right) \phi(e_j) \quad ; \text{ by step 2} \\ &= x\phi(e_j)\end{aligned}$$

Similarly, $\phi(x\hat{e}_j) = x\phi(\hat{e}_j)$ for all $x \in \mathbb{R}$ and for all $j (1 \leq j \leq n)$.

Step 4: Characterization of continuous homomorphisms from \mathbb{C}^n to \mathbb{C}^m .

Let $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$. For $1 \leq j \leq n$, let $x_j = \text{Re}(z_j)$ and $y_j = \text{Im}(z_j)$. Then

$$\begin{aligned}z &= (x_1, x_2, \dots, x_n) + (iy_1, iy_2, \dots, iy_n) \\ &= \sum_{j=1}^n x_j e_j + \sum_{j=1}^n y_j \hat{e}_j\end{aligned}$$

So

$$\begin{aligned}
 \phi(z) &= \phi\left(\sum_{j=1}^n x_j e_j + \sum_{j=1}^n y_j \hat{e}_j\right) \\
 &= \sum_{j=1}^n \phi(x_j e_j) + \sum_{j=1}^n \phi(y_j \hat{e}_j) \\
 &= \sum_{j=1}^n x_j \phi(e_j) + \sum_{j=1}^n y_j \phi(\hat{e}_j) \\
 &= \sum_{j=1}^n \operatorname{Re}(z_j) \phi(e_j) + \sum_{j=1}^n \operatorname{Im}(z_j) \phi(\hat{e}_j).
 \end{aligned}$$

Conversly, if $a_j(1 \leq j \leq n)$ and $b_j(1 \leq j \leq n)$ be $2n$ complex numbers , then the map ϕ given by

$$\phi(z_1, z_2, \dots, z_n) = \sum_{j=1}^n \operatorname{Re}(z_j) a_j + \sum_{j=1}^n \operatorname{Im}(z_j) b_j$$

is a continuous group homomorphism from \mathbb{C}^n to \mathbb{C}^m . Hence the cardinality of the set of continuous group homomorphisms from \mathbb{C}^n to \mathbb{C}^m is same as the cardinality of \mathbb{C}^{2nm} , which is the cardinality of the continuum. \square

Now, we provide a proof to the existence of non-continuous group homomorphism from \mathbb{C}^n to \mathbb{C}^m .

Theorem 3.2. *The cardinality of the set of all non-continous group homomorphism from \mathbb{C}^n to \mathbb{C}^m is at least the cardinality of the continuum.*

Proof. Consider \mathbb{C}^n as a vector space over the field \mathbb{Q} of rational numbers and H be a Hamel basis of \mathbb{C}^n over \mathbb{Q} . Then every vector $z \in \mathbb{C}^n$ can be uniquely expressed

$$z = \sum_{h \in H} w_h(z) h \tag{4}$$

where $w_h(z) \in \mathbb{Q}$ and $w_h(x) = 0$ for all h except for a finite number of values of h . Let e_0 and \hat{e}_0 are the zero elements in \mathbb{C}^n and \mathbb{C}^m respectively. Let e_1 and \hat{e}_1 are the n -tuple and m -tuple respectively such that first component is 1 and all other components are 0. Let h' be a fixed element in H . Define a map $\psi_{h'} : \mathbb{C}^n \rightarrow \mathbb{C}^m$ by

$$\psi_{h'}(z) = \psi_{h'}\left(\sum_{h \in H} w_h(z) h\right) = w_{h'}(z) \hat{e}_1.$$

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Let $z = \sum_{h \in H} w_h(z)h$ and $z' = \sum_{h \in H} w_h(z')h$ be two elements in \mathbb{C}^n . Then

$$\begin{aligned} \psi_{h'}(z + z') &= \psi_{h'}\left(\sum_{h \in H} w_h(z)h + \sum_{h \in H} w_h(z')h\right) \\ &= \psi_{h'}\left(\sum_{h \in H} [w_h(z) + w_h(z')]h\right) \\ &= [w_{h'}(z) + w_{h'}(z')]\hat{e}_1 \\ &= w_{h'}(z)\hat{e}_1 + w_{h'}(z')\hat{e}_1 \\ &= \psi_{h'}(z) + \psi_{h'}(z'). \end{aligned}$$

Hence $\psi_{h'} : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is a group homomorphism.

For $z = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m$, define $\phi : \mathbb{C}^m \rightarrow \mathbb{C}$ by $\phi(z) = z_1$. Then ϕ is a continuous function. Define $g : \mathbb{C}^n \rightarrow \mathbb{C}$ by $g(z) = \phi \circ \psi_{h'}(z)$ for all $z \in \mathbb{C}^n$.

Then for $z = \sum_{h \in H} w_h(z)h \in \mathbb{C}^n$,

$$\begin{aligned} g(z) &= \phi \circ \psi_{h'}\left(\sum_{h \in H} w_h(z)h\right) = \phi(w_{h'}(z)\hat{e}_1) = w_{h'}(z) \in \mathbb{Q}, \\ g(h') &= g\left(1 \cdot h' + \sum_{h \in H, h \neq h'} 0h\right) = \phi \circ \psi_{h'}\left(1 \cdot h' + \sum_{h \in H, h \neq h'} 0h\right) = \phi(1 \cdot \hat{e}_1) = 1 \end{aligned}$$

and

$$g(e_0) = \phi \circ \psi_{h'}(e_0) = \phi \circ \psi_{h'}\left(0 \cdot h' + \sum_{h \in H, h \neq h'} 0h\right) = \phi(0 \cdot \hat{e}_1) = 0.$$

Hence g is a non-constant function from \mathbb{C}^n to \mathbb{C} which assumes only rational values. Therefore g is not continuous and which gives the function $\psi_{h'}$ is discontinuous.

Let h' and h'' be two distinct elements in H . Then

$$\psi_{h'}(h') = \psi_{h'}(1 \cdot h') = 1 \cdot \hat{e}_1 = \hat{e}_1$$

and

$$\psi_{h''}(h') = \psi_{h''}(0 \cdot h'' + 1 \cdot h') = 0 \cdot \hat{e}_1 = \hat{e}_0.$$

Therefore $\psi_{h'}$ and $\psi_{h''}$ are distinct. Then the cardinality of set of all non-continuous group homomorphism from \mathbb{C}^n to \mathbb{C}^m is at least $|H| = |\mathbb{C}^n| =$ the cardinality of the continuum. \square

4 Conclusions

In this paper, we characterized all continuous group homomorphisms from \mathbb{C}^n to \mathbb{C}^m . Also we proved that the cardinality of the set of all non-continuous group homomorphism from \mathbb{C}^n to \mathbb{C}^m is at least the cardinality of the continuum.

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