Ratio Mathematica Volume 39, 2020, pp. 261-268

Thermodynamic behavior of the polytropic gas in cosmology

Prasanta Das[*]
Kangujam Priyokumar Singh[†]

Abstract

In this paper, we investigate on the thermodynamic behavior of Polytropic gas as a candidate for dark energy by considering the relation $P = K\rho^{1+\frac{1}{n}}$, where K and n are the Polytropic constant and Polytropic index respectively. Furthermore, P indicates the pressure and ρ is the energy density of the fluid such that $\rho = \frac{U}{V}$ where U and V represent the internal energy and volume, respectively. At first, we find an exact expression for the energy density of the Polytropic gas using thermodynamics and later on, discuss different physical parameters. Finally our study shows that the Polytropic gas may be used to describe the expansion history of the universe from the dust dominated era to the current accelerated era and it is thermodynamically stable.

Keywords: Cosmology; Dark energy; Polytropic gas; Thermodynamics.
2010 AMS subject classification: 83F05, 37D35, 82B30.[‡]

[*]Department of Mathematical Sciences, Bodoland University, Kokrajhar, BTR, Assam-783370, India; prasantadasp4@gmail.com.
[†]Department of Mathematical Sciences, Bodoland University, Kokrajhar, BTR, Assam-783370, India; pkangujam18@gmail.com.
[‡]Received on October 4th, 2020. Accepted on December 17th, 2020. Published on December 31st, 2020. doi: 10.23755/rm.v39i0.564. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

Cosmologists suggest that our universe expands under an accelerated expansion [1]-[7]. In the standard Friedman Lemaitre Robertson Walker (FLRW) cosmology, a new energy with negative pressure, called dark energy (DE) is responsible for this expansion [8]. The nature of the DE is still unknown and various problems have been proposed by the researchers in this field**.** About 70% of the present energy of the universe is contained in the DE. The cosmological constant with the time independent equation of state is the earliest, simplest and most traditional candidate for the dark energy which can be taken into account as a perfect fluid satisfying the relation $\rho + P = 0$. But it has some problems like fine-tuning and cosmic coincidence puzzles [9], [10]. Besides the cosmological constant, the other dark energy models are quintessence [11], phantom [12], tachyon [13], holographic dark energy [14] [15], K-essence [16] and Chaplygin gas models with various equation of state. Polytropic gas is one of the dynamical dark energy models [17].

In the present study, we want to investigate the thermodynamic behavior of the Polytropic gas. K. Karami et al. investigated the interaction between the Polytropic gas and cold dark matter and found that the Polytropic gas behaves as the phantom dark energy [18]. K. Karami and S. Ghaffari showed that the generalized second law of thermodynamics is always satisfied by a universe filled with a Polytropic gas and a cold dark matter [19]. K. Kleidis and N.K. Spyron used the first law of thermodynamics in the Polytropic gas model and they show that the Polytropic gas behaves as dark energy and this model leads to a suitable fitting with the observational data about the current expanding era [20]. H. Moradpour, A. Abri and H. Ebadi, investigated the thermo dynamical behavior and stability of the Polytropic gas [21]. M. Salti et al. discussed validity of the first and generalized second law of thermodynamics in locally rotationally symmetric Bianchi-type II space time which is dominated by a combination of Polytropic gas and baryonic matter[22]. Moreover, Muzaffer Askin et al. studied the cosmological scenarios of the Polytropic gas dark matter-energy proposal in a Friedmann-Robertson- Walker universe and they found an exact expression for the energy density of the Polytropic gas model according to the thermo dynamical point of views and a relationship between a homogeneous minimally coupled scalar field and the Polytropic gas [23].This paper is organized as follows: in section 2 we construct the basic thermodynamic formalism of the Polytropic gas model and discuss the thermodynamic behavior of this model. Finally in section 3 we provide a brief discussion.

2. Basic Formalism

In this work, we consider the following equation of state which is well known as Polytropic gas equation of state

$$P = K\rho^{1+\frac{1}{n}} \tag{1}$$

Here $K(>0)$ and $n(<0)$ are Polytropic constant and Polytropic index respectively. Moreover, P is the pressure and ρ is the energy density of the fluid such that

$$\rho = \frac{U}{V} \tag{2}$$

Where U and V are the internal energy and volume filled by the fluid respectively.

First of all, we try to find the internal energy U and energy density ρ of the polytropic gas as a function of its volume V and entropy S.

From the general thermodynamics, we have

$$\left(\frac{\partial U}{\partial V}\right)_S = -P \tag{3}$$

From the equations (1), (2) and (3), we get

$$\left(\frac{\partial U}{\partial V}\right)_S = -K\left(\frac{U}{V}\right)^{1+\frac{1}{n}} \tag{4}$$

Integrating the equation (4), we get

$$U = (-1)^{-n}\left(KV^{-\frac{1}{n}} + \xi\right)^{-n} \tag{5}$$

Where the parameter ξ is the constant of integration which may be a universal constant or a function of entropy S only

The equation (5) also can rewrite in the following form

$$U = (-1)^{-n}K^{-n}V\left(1 + \left(\frac{V}{\varepsilon}\right)^{\frac{1}{n}}\right)^{-n} \tag{6}$$

Where $$\varepsilon = \left(\frac{K}{\xi}\right)^{n} \tag{7}$$

And it has a dimension of volume.

Therefore, the energy density ρ of the Polytropic gas is

$$\rho = \frac{U}{V} = (-1)^{-n}K^{-n}\left(1 + \left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}\right)^{-n} \tag{8}$$

When $n < 0$ then equation (8) gives

$$\rho \sim (-1)^{-n}K^{-n}\frac{\varepsilon}{V} \tag{9}$$

Now we will use these equations to discuss different physical parameters.

a) Pressure:

Using the equation (8) in the equation (1) we get the pressure of the Polytropic gas as a function of entropy S and volume V in the following form

$$P = (-1)^{n+1} K^{-n} \left(1 + \left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}\right)^{-(n+1)} \tag{10}$$

We can rewrite the equation (10) in the following form

$$P = -\frac{\rho}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{11}$$

When $n < 0$ and ε does not diverge then for small volume i.e. at early stage of universe, $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$, we get
$\mathrm{P} \simeq 0$, which represents a dust dominated universe. When $n < 0$ and ε does not diverge then for large volume i.e. at late stage of universe, $V \gg \varepsilon$ ie $\frac{V}{\varepsilon} \gg 1$, we get $\mathrm{P} \simeq -\rho$, which indicates an accelerated expansion of the universe.

b) Caloric equation of state:

Now from the equations (8) and (10) we get the caloric equation of state parameter as

$$\omega = \frac{P}{\rho} = -\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{12}$$

When $n < 0$ and ε does not diverge then for small volume $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$, we get $\omega \simeq 0$ (Dust dominated)
When $n < 0$ and ε does not diverge then for large volume $V \gg \varepsilon$ ie $\frac{V}{\varepsilon} \gg 1$, we get $\omega \simeq -1$ (Cosmological constant)
Thus the equation of state parameter (ω) of the Polytropic gas with $n < 0$ is decreased from $\omega \simeq 0$ (for small volume) to $\omega \simeq -1$ (for large volume). It indicates that the universe expands from the dust dominated era to the current accelerating era.

c) Deceleration parameter:

We get the deceleration parameter of the Polytropic gas with the help of equation (12)

$$q = \frac{1}{2} + \frac{3}{2}\frac{P}{\rho} = \frac{1}{2} - \frac{3}{2}\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{13}$$

When $n < 0$ and ε does not diverge then for small volume $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$, we get $q > 0$, which correspond to the deceleration universe.
When $n < 0$ and ε does not diverge then for large volume $V \gg \varepsilon$ ie $\frac{V}{\varepsilon} \gg 1$, we get $q < 0$, which correspond to the accelerated universe.

d) Square velocity of sound:

From the equation (11) we get the velocity of sound (V_s) as

$$V_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = -\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{14}$$

When $n < 0$ and ε does not diverge then for small volume $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$, we get $V_s^2 \simeq 0$ .Since velocity of sound is zero in vacuum. Therefore the Polytropic gas behaves like a pressure less fluid at the early stage of the universe. When $n < 0$ and ε does not diverge then for large volume $V \gg \varepsilon$ ie $\frac{V}{\varepsilon} \gg 1$, we get $V_s^2 \simeq -1$, which gives an imaginary speed of sound leading to a perturbation cosmology.

e) Thermodynamic stability:

The conditions of the thermodynamic stability of a fluid are

$$\left(\frac{\partial P}{\partial V}\right)_S < 0 \tag{15}$$

And $$C_V > 0 \tag{16}$$

Here C_V is the thermal capacity at constant volume. From the equation (10) we have

$$\left(\frac{\partial P}{\partial V}\right)_S = -\left(1+\frac{1}{n}\right)\frac{P}{V}\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{-\frac{1}{n}}} \tag{17}$$

If $-1 < n < 0$ and $\varepsilon < 0$ then from (17), we have

$$\left(\frac{\partial P}{\partial V}\right)_S < 0$$

Thus the stability condition (15) of thermodynamics is satisfied.

Now we have to verify the positivity of the thermal capacity at constant volume C_V where $C_V = T\left(\frac{\partial S}{\partial T}\right)_V$ (18)

Now we determine the temperature T of the Polytropic gas as a function of its entropy S and its volume V. The temperature T of the Polytropic gas is determined from the relation

$$T = \left(\frac{\partial U}{\partial S}\right)_V \qquad (19)$$

Using (6) in (19) we get

$$T = (-1)^{n+1} V^{1+\frac{1}{n}} \left(K + \xi V^{\frac{1}{n}}\right)^{-(n+1)} \frac{d\xi}{dS} \qquad (20)$$

This gives the temperature of the Polytropic gas.
We can rewrite the equation (20) in the following form

$$T = -n \frac{\rho V^{1+\frac{1}{n}}}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \frac{d\xi}{dS} \qquad (21)$$

From (5) we have

$$[\xi]^{-n} = [U] \qquad (22)$$

Since $\quad [U] = [TS] \qquad (23)$

Therefore from the equations (22) & (23) we get

$$\xi = [U]^{-\frac{1}{n}} = [T_* S]^{-\frac{1}{n}} \qquad (24)$$

Where T_* (> 0) is a universal constant with temperature dimension.
Differentiating (24) with respect to 'S' we get

$$\frac{d\xi}{dS} = -\frac{1}{n} T_*^{-\frac{1}{n}} S^{-\frac{1}{n}-1} \qquad (25)$$

Using (8) & (24) in (25) we get

$$T = (-1)^n V^{1+\frac{1}{n}} \left(T_*^{-\frac{1}{n}} S^{-\frac{1}{n}-1}\right) \left[K + T_*^{-\frac{1}{n}} S^{-\frac{1}{n}} V^{\frac{1}{n}}\right]^{-(n+1)} \qquad (26)$$

This leads to the entropy of the Polytropic gas as

$$S = \left[(-1)^{\frac{n}{n+1}} \left(\frac{T_*}{T}\right)^{\frac{1}{n+1}} - 1\right]^n \quad \frac{V}{K^n T_*} \qquad (27)$$

We know that entropy (S) of a thermo dynamical system should be positive ie $S > 0$ [24]
Here $S > 0$ if $K^n T_* > 0$
Now the thermal capacity at constant volume is

$$C_V = T \left(\frac{\partial S}{\partial T}\right)_V$$

$$= (-1)^{\frac{2n+1}{n+1}} \left(\frac{n}{n+1}\right) \frac{S}{\left[(-1)^{\frac{n}{n+1}}\left(\frac{T_*}{T}\right)^{\frac{n}{n+1}} - 1\right]} \left(\frac{T_*}{T}\right)^{\frac{1}{n+1}} \qquad (28)$$

Therefore, the condition $C_V > 0$ is satisfied if $K^n T_* > 0$. Thus both the conditions of thermo dynamic stability are satisfied. So the Polytropic gas is thermo dynamically stable.

3. Discussion

We have studied the thermo dynamical behavior of the Polytropic gas. Here, we have considered the value of $n < 0$ to study the whole work done in this article. Some important results are given below:

(i) As we have considered $n < 0$, the pressure goes more and more negative as volume increases.

(ii) The equation of state parameter (ω) of the Polytropic gas is $\omega \simeq 0$ at early stage of the universe and $\omega \simeq -1$ at late stage of the universe. This indicates that the universe expands from the dust dominated era to the present accelerated era.

(iii) The deceleration parameter (q) is investigated in the context of thermodynamics as well as Polytropic gas and our analysis shows that universe is decelerated ($q > 0$) at early stage of the universe and accelerated ($q < 0$) at late stage of the universe.

Both the conditions of the thermo dynamical stability of the Polytropic gas are studied for $K^n T_* > 0$ and our analysis shows that the Polytropic gas is thermodynamically stable.

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