Ratio Mathematica Volume 40, 2021, pp. 27-46

Integrity of generalized transformation graphs

Bommanahal Basavanagoud*
Shruti Policepatil†

Abstract

The values of vulnerability helps the network designers to construct such a communication network which remains stable after some of its nodes or communication links are damaged. The transformation graphs considered in this paper are taken as model of the network system and it reveals that, how network can be made more stable and strong. For this purpose the new nodes are inserted in the network. This construction of new network is done by using the definition of generalized transformation graphs of a graphs. Integrity is one of the best vulnerability parameter. In this paper, we investigate the integrity of generalized transformation graphs and their complements. Also, we find integrity of semitotal point graph of combinations of basic graphs. Finally, we characterize few graphs having equal integrity values as that of generalized transformation graphs of same structured graphs.

Keywords: Vulnerability; connectivity; integrity; generalized transformation graphs; semitotal point graph.
2020 AMS subject classifications: 05C40, 90C35.[1]

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*Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India; b.basavanagoud@gmail.com.
†Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India; shrutipatil300@gmail.com.
[1]Received on February 25th, 2021. Accepted on June 23th, 2021. Published on June 30th, 2021. doi: 10.23755/rm.v40i1.576. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

The stability of a communication network composed of processing nodes and communication links are of prime importance to network designers. As the network begins losing links or nodes, eventually there will be a decrease to certain extent in its effectiveness. Thus, communication networks must be constructed as stable as possible; not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. In the analysis of vulnerability of a communication network we often consider the following quantities: 1. the number of members of the largest remaining group within mutual communication can still occur, 2. the number of elements that are not functioning.

The communication network can be represented as an undirected graph. Consequently, a number of other parameters have recently been introduced in order to attempt to cope up with this difficulty. Tree, mesh, hypercube and star graphs are popular communication networks. If we think of graph as modeling a network, there are many graph theoretical parameters used in the past to describe the stability of communication networks. Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The best known measure of reliability of a graph is its vertex-connectivity. The difficulty with these parameters is that they do not take into account what remains after the graph has been disconnected. To estimate these quantities, the concept of integrity was introduced by Barefoot et al. in [6] as a measure of the stability of a graph.
The integrity of a graph G is defined in [6] as

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G - S)\},$$

where $m(G - S)$ denotes the order of the largest component of $G - S$. In [6], the authors have compared integrity, connectivity, toughness and binding number for several classes of graphs. In 1987, Barefoot et al. [7] have investigated the integrity of trees and powers of cycles. In 1988, Goddard et al. [14] have obtained integrity of the join, union, product and composition of two graphs. The integrity of a small class of regular graphs was studied by Atici et al. [1].The authors in [3, 20] have studied the integrity of cubic graphs. For more details on integrity of a graph refer to [2, 4, 5, 11–13, 15].

In this paper, we are concerned with nontrivial, simple, finite, undirected graphs. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex $d_G(v)$ is the number of edges incident to it in G. The symbol $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x and $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x. For undefined graph theoretic terminologies and notations refer to [16] or [17].

2 Preliminaries

2.1 Basic results on integrity

In this subsection, we review some of the known results about integrity of graphs.

Theorem 2.1. *[5] The integrity of*

(i) complete graph K_n, $I(K_n) = n$,

(ii) null graph $\overline{K_n}$, $I(\overline{K_n}) = 1$,

(iii) star $K_{1,b}$, $I(K_{1,b}) = 2$,

(iv) path P_n, $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$,

(v) cycle C_n, $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$,

(vi) complete bipartite graph $K_{a,b}$, $I(K_{a,b}) = 1 + min\{a, b\}$,

(vii) wheel W_n, $I(W_n) = \lceil 2\sqrt{n-1} \rceil$.

3 Generalized transformation graphs

Sampathkumar and Chikkodimath [19] defined the semitotal-point graph $T_2(G)$ as the graph whose vertex set is $V(G) \cup E(G)$, and where two vertices are adjacent if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. Inspired by this definition, Basavanagoud et al. [9] introduced some new graphical transformations. These generalize the concept of semitotal-point graph.

Let $G = (V, E)$ be a graph, and let α, β be two elements of $V(G) \cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G, otherwise is $-$. Let xy be a 2-permutation of the set $\{+, \}$. We say that α and β correspond to the first term x of xy if both α and β are in $V(G)$, whereas α and β correspond to the second term y of xy if one of α and β is in $V(G)$ and the other is in $E(G)$. The generalized transformation graph G^{xy} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{xy} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of xy.

We denote the complement of the generalized transformation graph G^{xy} by $\overline{G^{xy}}$.

In view of above, one can obtain four graphical transformations of graphs, since there are four distinct 2-permutations of $\{+, -\}$. Note that G^{++} is just the semitotalpoint graph $T_2(G)$ of G, whereas the other generalized transformation graphs are G^{+-}, G^{-+} and G^{--}.

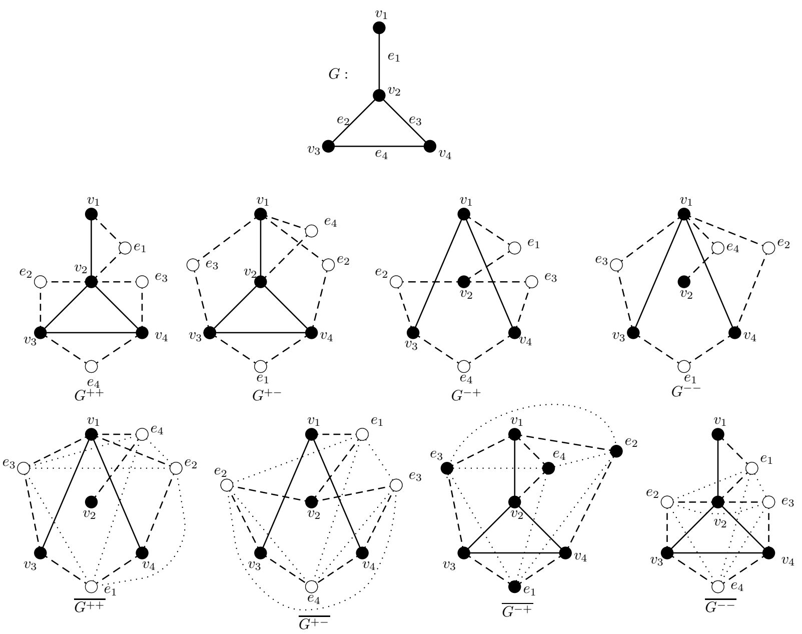

Figure 1: Graph G, its generalized transformation G^{xy} and their complements $\overline{G^{xy}}$

The *generalized transformation graph* G^{xy}, introduced by Basavanagoud et al. [9], is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:

(a) $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$ and α, β are nonadjacent in G if $x = -$

(b) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are nonincident in G if $y = -$

An example of generalized transformation graphs and their complements are shown in Figure 1. The vertex v of G^{xy} corresponding to a vertex v of G is referred to as a *point vertex*. The vertex e of G^{xy} corresponding to an edge e of G is referred to as a *line vertex*.
For more details on generalized transformation graphs, refer to [8–10, 17–19].

4 Main results

In this section, we determine the integrity of semitotal point graph(G^{++}) of some standard families of graphs. Also, the integrity of generalized transformation graphs G^{+-}, G^{-+}, G^{--}, $\overline{G^{++}}$, $\overline{G^{+-}}$, $\overline{G^{-+}}$ and $\overline{G^{--}}$ are obtained. Then, we calculate integrity of semitotal point graph of cartesian product and composition of some graphs

4.1 *Integrity of generalized transformation graphs*

Theorem 4.1. *For a graph P_n $(n \geq 4)$,*

$$I(P_n^{++}) = \begin{cases} \lceil 2\sqrt{2n} \rceil - 2, & if\ n\ is\ odd, \\ \lceil 2\sqrt{2n-1} \rceil - 1, & if\ n\ is\ even. \end{cases}$$

Proof. Let S be a subset of $V(P_n^{++})$. The number of remaining components after removing $|S| = r$ vertices is given in Table 1 and Table 2.
Case 1. Suppose n is even. The number of vertices in P_n^{++} is $2n-1$. If r vertices are removed from graph P_n^{++}, then one of the connected components has at least $\frac{2n-1-r}{r}$ vertices. So, the order of the largest component is $m(P_n^{++} - S) \geq \frac{2n-1-r}{r}$. So

$$I(P_n^{++}) \geq min\left\{r + \frac{2n-1-r}{r}\right\}.$$

The function $r + \frac{2n-1-r}{r}$ takes its minimum value at $r = \sqrt{2n-1}$. If we substitute the minimum value in the function, then we have $I(P_n^{++}) = 2\sqrt{2n-1} - 1$. Since the integrity is integer valued, we round this up to get a lower bound. So the integrity of P_n^{++} is, $I(P_n^{++}) = \lceil 2\sqrt{2n-1} \rceil - 1$.

Number of removing vertices	1	2	3	...	r
Number of remaining components	1	2	3	...	r

Table 1: n is even

Number of removing vertices	1	2	3	...	r
Number of remaining components	2	3	4	...	$r+1$

Table 2: n is odd

Case 2. Suppose n is odd. Since the number of vertices in P_n^{++} is $2n-1$ and $m(P_n^{++} - S) \geq \frac{2n-1-r}{r+1}$, we have $I(P_n^{++}) \geq min\{r + \frac{2n-1-r}{r+1}\}$. After the required elementary arithmetical operations, we get

$$I(P_n^{++}) = \lceil 2\sqrt{2n} \rceil - 2.$$

□

Example 4.1. *Consider a graph P_5 and its semitotal point graph P_5^{++}. Let $S = \{a, b\} \subset V(P_5^{++})$ (see Figure 2) such that $|S| = 2$ and $m(P_5^{++} - S) = 3$. So, $I(P_5^{++}) = 5$.*

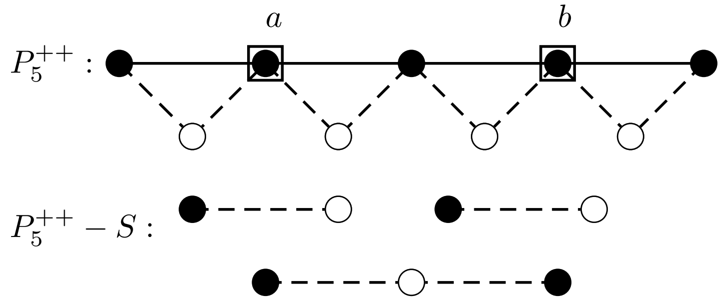

Figure 2: Graph $P_5^{++} - S$.

Theorem 4.2. *For a cycle C_n of length $n \geq 4$,*

$$I(C_n^{++}) = \begin{cases} \lceil \frac{n}{2} \rceil + 3, & if\ n(\leq 7)\ is\ odd\ and\ n(\leq 16)\ is\ even, \\ \lceil \frac{n}{3} \rceil + 5, & if\ n(\geq 9)\ is\ odd\ and\ n(\geq 18)\ is\ even. \end{cases}$$

Proof. C_n^{++} has $2n$ vertices and $3n$ edges. Let $S \subset V(C_n^{++})$.
Case 1. Suppose $n(\leq 7)$ is odd and $n(\geq 16)$ is even.
Choose a set S in such a way that it is an independent set of vertices of C_n. It is clear that $|S| = \lceil \frac{n}{2} \rceil = \beta_0(C_n)$ and $m(C_n^{++} - S) = 3$. So, $I(C_n^{++}) = \lceil \frac{n}{2} \rceil + 3$.
Case 2. Suppose $n(\geq 9)$ is odd and $n(\leq 18)$ is even.
Choose a set S in such a way that it is an independent set of vertices of C_n having distance 3 in between them. It is clear that $|S| = \lceil \frac{n}{3} \rceil$ and $m(C_n^{++} - S) = 5$. So, $I(C_n^{++}) = \lceil \frac{n}{3} \rceil + 5$. □

Example 4.2. *Consider a graph C_6 and its semitotal point graph C_6^{++}. Let $S = \{a, b, c\} \subset V(C_6^{++})$ (see Figure 3) such that $|S| = 3$ and $m(C_6^{++} - S) = 3$. So, $I(C_6^{++}) = 6$.*

Theorem 4.3. *For a complete graph K_n of order $n \geq 2$,*

$$I(K_n^{++}) = n + 1.$$

Proof. K_n^{++} has $\frac{n(n+1)}{2}$ vertices and $\frac{3n(n-1)}{2}$ edges. Let $S \subset V(K_n^{++})$ be a set containing all the vertices of K_n. So, $|S| = n$. The removal of vertices from K_n^{++} leaves a totally disconnected graph with $\frac{n(n-1)}{2}$ vertices. Hence, $m(K_n^{++} - S) = 1$. Therefore, $|S| + m(K_n^{++} - S) = n + 1$ is minimum for above set S. Then it is clear that, $I(K_n^{++}) = n + 1$. □

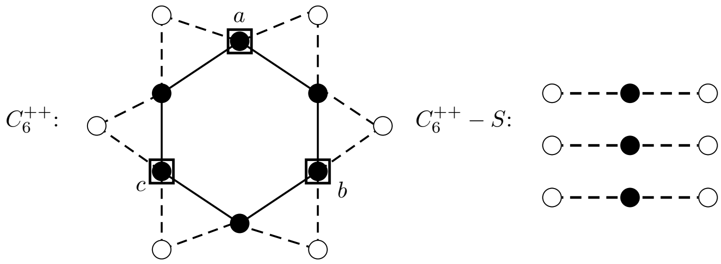

Figure 3: Graph $C_6^{++} - S$.

Example 4.3. *Consider a graph K_4 and its semitotal point graph K_4^{++}. Let $S = \{a, b, c, d\} \subset V(K_4^{++})$ (see Figure 4). It is clear that $m(K_4^{++} - S) = 1$. So, $I(K_4^{++}) = 5$.*

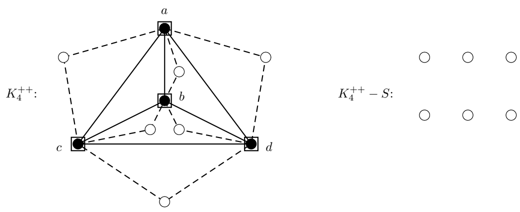

Figure 4: Graph $K_4^{++} - S$.

Corolary 4.1. *$I(K_p) = I(K_q^{++})$ if and only if $p = q + 1$.*

Theorem 4.4. *For a complete bipartite graph $K_{a,b}$ of order $a + b$,*

$$I(K_{a,b}^{++}) = 2min\{a, b\} + 1.$$

Proof. $K_{a,b}^{++}$ has $2ab$ vertices and $3ab$ edges. Let us select S in such a way that it should contain minimum number of vertices among two partite sets of $K_{a,b}$. So, $|S| = min\{a, b\}$. The deletion of vertices of S from $K_{a,b}^{++}$ results in union of stars $K_{1,min\{a,b\}}$. Hence, $m(K_{a,b}^{++} - S) = min\{a, b\}+1$. The value of $|S|+m(K_{a,b}^{++} - S)$ whose sum is minimum for chosen S. Therefore, $I(K_{a,b}^{++}) = 2min\{a, b\}+1$. □

Example 4.4. *Consider a graph $K_{2,3}$. Let $S = \{a, b\} \subset V(K_{2,3}^{++})$ (see Figure 5). It is clear to write $|S| = 2$ and $m(K_{2,3}^{++} - S) = 3$. So, $I(K_{2,3}^{++}) = 5$.*

Corolary 4.2. *$I(K_{a_1,b_1}) = I(K_{a_2,b_2}^{++})$ if and only if $min\{a_1, b_1\} = 2min\{a_2, b_2\}$. $K_{2,2}$ and $K_{1,2}$ are the smallest graphs satisfying above condition such that $I(K_{2,2}) = I(K_{1,2}^{++})$.*

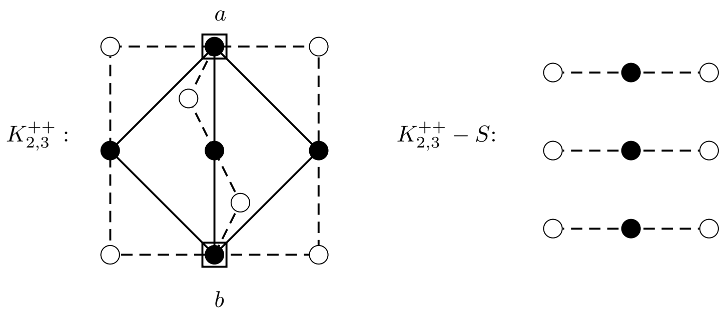

Figure 5: Graph $K_{2,3}^{++} - S$.

Theorem 4.5. *For a star $K_{1,b}$ of order $b + 1$,*

$$I(K_{1,b}^{++}) = 3.$$

Proof. $K_{1,b}^{++}$ has $2b + 1$ vertices and $3b$ edges. Let $S \subset V(K_{1,b}^{++})$ containing a central vertex of $K_{1,b}$. So, $|S| = 1$. The removal of a vertex of set S from $K_{1,b}^{++}$ results in graph bK_2. Hence, $m(K_{1,b}^{++} - S) = 2$. The value $|S| + m(K_{1,b}^{++} - S)$ is minimum for the chosen S. Therefore, $I(K_{1,b}^{++}) = 3$. $\square$

Remark 4.1. *The values of integrity of star graph and integrity of semitotal point graph of star graph are never same, since $I(K_{1,b}) = 2$ and $I(K_{1,b}^{++}) = 3$.*

Example 4.5. *Consider a graph $K_{1,3}$. Let $S = \{a\} \subset V(K_{1,3}^{++})$ (see Figure 6). It is clear to write $|S| = 1$ and $m(K_{1,3}^{++} - S) = 2$ So, $I(K_{1,3}^{++}) = 3$*

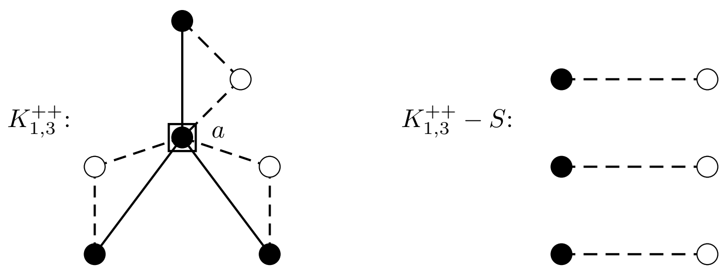

Figure 6: Graph $K_{1,3}^{++} - S$.

Theorem 4.6. *For a wheel W_n of order $n \geq 5$,*

$$I(W_n^{++}) = \left\lceil \frac{n-1}{2} \right\rceil + 5.$$

Proof. W_n^{++} has $3n-2$ vertices and $6(n-1)$ edges. Let $S \subset V(W_n^{++})$.
Case 1. Suppose n is odd
Clearly, the order of an outer cycle of wheel is $n-1$, which is even. Choose a set S_1 in such a way that it is an independent set of vertices of C_{n-1}. It is clear that $|S_1| = \frac{n-1}{2} = \beta_0(C_{n-1})$.
Case 2. Suppose n is even
Clearly, the order of an outer cycle of wheel is $n-1$, which is odd. Let S_2 be an independent set of vertices of C_{n-1} such that $|S_2| = \frac{n-2}{2}$. Let v_1 be a vertex of $V(C_{n-1}) \setminus S_2$ such that v_1 is adjacent to a vertex of S_2 as well as to a vertex $V(C_n) \setminus S_2$. Let us take $S_1 = S_2 \cup \{v\}$ and hence $|S_1| = \frac{n}{2}$.
Combining the above two cases we get, $|S_1| = \left\lceil \frac{n-1}{2} \right\rceil$, for all n, Let v_2 be a central vertex of W_n. Let us define a set S in such a manner that $S = S_1 \cup \{v_2\}$. It is to be noted that $|S| = \left\lceil \frac{n-1}{2} \right\rceil + 1$. The deletion of vertices of set S from W_n^{++} gives a graph whose components are P_4's and K_1's. Hence, $m(W_n^{++} - S) = 4$. The set S defined in this manner gives minimum value of $|S| + m(W_n^{++} - S)$. Therefore, $I(W_n^{++}) = \left\lceil \frac{n-1}{2} \right\rceil + 5$. □

Corolary 4.3. *$I(W_p) = I(W_q^{++})$ if and only if $\lceil 2\sqrt{p-1} \rceil = \left\lceil \frac{q-1}{2} \right\rceil + 5$. W_{11} and W_5 are the smallest graphs which satisfy the above condition such that $I(W_{11}) = I(W_5^{++})$.*

Example 4.6. *Consider a graph W_7. Let $S = \{a, b, c, d\} \subset V(W_7^{++})$ (see Figure 7). It is clear to write $|S| = 4$ and $m(W_7^{++} - S) = 4$. So, $I(W_7^{++}) = 8$*

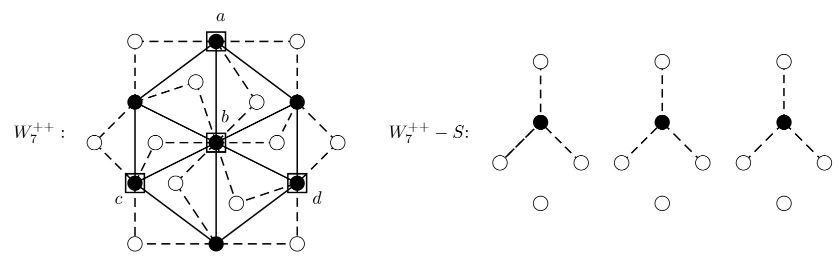

Figure 7: Graph $W_7^{++} - S$.

Theorem 4.7. *For a connected graph $G \ncong K_{1,b}$ of order n and size m,*

$$I(G^{+-}) = n + 1.$$

Proof. For an (n, m) graph G, G^{+-} has $n+m$ vertices and $m(n-1)$ edges. The n vertices have degree m and m vertices have $n-2$ in G^{+-}. Let $S \subset V(G^{+-})$. Consider a set S consisting a vertices of G^{+-} which corresponds to the vertices of a graph G. Then it is clear that $|S| = n$. The removal of the vertices of set S from G^{+-} results in a null graph $\overline{K_m}$. Hence, $m(G^{+-} - S) = 1$. $|S| + m(G^{+-} - S)$ is minimum for above chosen S. Therefore, $I(G^{+-}) = n + 1$.. □

Theorem 4.8. *For a star $K_{1,b}$ $(b \geq 3)$,*

$$I(K_{1,b}^{+-}) = b + 1.$$

Proof. For a star $K_{1,b}$ of order $b+1$ and size b, the graph $K_{1,b}^{+-}$ has $2b+1$ vertices and b^2 edges. Let $S \subset V(K_{1,b}^{+-})$. Choose as set S in such a way that it should contain the pendant vertices of $K_{1,b}$. So, $|S| = b$. The deletion of the vertices of set S from $K_{1,b}^{+-}$ results in null graph $\overline{K_{b+1}}$. So, $m(K_{1,b}^{+-} - S) = 1$. $|S| + m(K_{1,b}^{+-} - S)$ is minimum for above chosen S. Therefore, $I(K_{1,b}^{+-}) = b + 1$. □

The column 2 and 4 of Table Table 3 shows integrity of basic graphs and integrity of transformation graph G^{+-} of graphs with same structure.

G	$I(G)$	G^{+-}	$I(G^{+-})$	G^{-+}	$I(G^{-+})$	G^{--}	$I(G^{--})$
P_6	4	P_3^{+-}	4	P_3^{-+}	4	P_3^{--}	4
P_{10}	5	P_{10}^{+-}	11	P_{10}^{-+}	8	P_{10}^{--}	11
C_5	4	C_3^{+-}	4	C_3^{-+}	4	C_3^{--}	4
C_7	5	C_7^{+-}	8	C_7^{-+}	8	C_7^{--}	8
K_n	n	K_n^{+-}	$n+1$	K_n^{-+}	$n+1$	K_n^{--}	$n+1$
$K_{5,5}$	6	$K_{2,3}^{+-}$	6	$K_{2,3}^{-+}$	6	$K_{2,3}^{--}$	6
$K_{5,6}$	6	$K_{5,6}^{+-}$	12	$K_{5,6}^{-+}$	12	$K_{5,6}^{--}$	12
$K_{1,b}$	2	$K_{1,b}^{+-}$	$b+1$	$K_{1,b}^{-+}$	$b+2$	$K_{1,b}^{--}$	$b+1$
W_8	6	W_5^{+-}	6	W_5^{-+}	6	W_5^{--}	6
W_9	6	W_9^{+-}	10	W_9^{-+}	10	W_9^{--}	10

Table 3:

Theorem 4.9. *For a connected graph $G \ncong K_{1,b}$ of order n and size m,*

$$I(G^{-+}) = n + 1.$$

Proof. For an (n, m) graph G, G^{-+} has $n+m$ vertices and $\frac{n(n-1)}{2} + m$ edges. The n vertices have degree 2 in G^{-+}. Let $S \subset V(G^{-+})$. Consider a set S consisting a vertices of G^{-+} which corresponds to the vertices of a graph G. Then it is clear that $|S| = n$. The removal of the vertices of set S from G^{-+} results in a null graph $\overline{K_m}$. Hence, $m(G^{-+} - S) = 1$. The value $|S| + m(G^{-+} - S)$ is minimum for above chosen S. Therefore, $I(G^{-+}) = n + 1$. □

The column 2 and 6 of Table 3 shows integrity of basic graphs and integrity of transformation graph G^{-+} of graphs with same structure.

Theorem 4.10. *For a star $K_{1,b}$ $(b \geq 3)$,*

$$I(K_{1,b}^{-+}) = b + 1.$$

Proof. The proof is similar to that of Theorem 4.8. $\square$

Theorem 4.11. *For a connected graph $G \ncong K_{1,b}$ of order n and size m,*

$$I(G^{--}) = n + 1.$$

Proof. For an (n, m) graph G, G^{--} has $n + m$ vertices and $\frac{n(n-1)}{2} + m(n-3)$ edges. Let $S \subset V(G^{--})$. Consider a set S consisting a vertices of G^{--} which corresponds to the vertices of a graph G. Then it is clear that $|S| = n$. Deleting the vertices of set S from G^{--} results in a null graph $\overline{K_m}$. Hence, $m(G^{--} - S) = 1$. $|S| + m(G^{--} - S)$ is minimum for above chosen S. Therefore, $I(G^{--}) = n + 1$. $\square$

Theorem 4.12. *For a star $K_{1,b}(b \geq 3)$,*

$$I(K_{1,b}^{--}) = b + 1.$$

Proof. For a star $K_{1,b}$ of order $b+1$ and size b, the graph $K_{1,b}^{--}$ has $2b+1$ vertices. Let $S \subset V(K_{1,b}^{--})$. Choose as set S in such a way that it should contain the pendant vertices of $K_{1,b}$. So, $|S| = b$. The deletion of the vertices of set S from $K_{1,b}^{--}$ results in null graph $\overline{K_{b+1}}$. So, $m(K_{1,b}^{--} - S) = 1$. $|S| + m(K_{1,b}^{--} - S)$ is minimum for above chosen S. Therefore, $I(K_{1,b}^{--}) = b + 1$. $\square$

The column 2 and 8 of Table 3 shows integrity of basic graphs and integrity of transformation graph G^{--} of graphs with same structure.

Theorem 4.13. *For any connected graph G of order n and size $m \geq 2$,*

$$I(\overline{G^{++}}) = n + m - 2.$$

Proof. For a connected graph G of order n and size $m \geq 2$, $\overline{G^{++}}$ has order $n + m$. Let $S \subset V(\overline{G^{++}})$. Consider a set S containing all the vertices and edges of G except one edge and its incident vertices. So, $|S| = n + m - 3$. The removal of the vertices of set S from $\overline{G^{++}}$ gives $3K_1$. Hence, $m(\overline{G^{++}} - S) = 1$. The value $|S| + m(\overline{G^{++}} - S)$ is minimum for the selected subset S. Therefore, $I(\overline{G^{++}}) = n + m - 2.$. $\square$

Theorem 4.14. *For any connected graph G of order n and size $m \geq 2$,*

$$I(\overline{G^{+-}}) = n + m - 2.$$

Proof. For a connected graph G of order n and size $m \geq 2$, $\overline{G^{+-}}$ has order $n + m$. Let $S \subset V(\overline{G^{+-}})$. Consider a set S containing all the vertices and edges of G except one edge and two nonincident vertices(adjacent vertices). So, $|S| = n + m - 3$. The removal of the vertices of set S from $\overline{G^{+-}}$ gives $3K_1$. Hence, $m(\overline{G^{+-}} - S) = 1$. The value of $|S| + m(\overline{G^{+-}} - S)$ is minimum for the selected subset S. Therefore, $I(\overline{G^{+-}}) = n + m - 2$.. □

Corolary 4.4. *The integrity of*

(i) path P_n, $I(\overline{P_n^{++}}) = I(\overline{P_n^{+-}}) = 2n - 3$,

(ii) cycle C_n, $I(\overline{C_n^{++}}) = I(\overline{C_n^{+-}}) = 2n - 2$,

(iii) complete graph K_n, $I(\overline{K_n^{++}}) = I(\overline{K_n^{+-}}) = \frac{n(n+1)}{2} - 2$,

(iv) complete bipartite graph $K_{a,b}$, $I(\overline{K_{a,b}^{++}}) = I(\overline{K_{a,b}^{+-}}) = a + b + ab - 2$,

(v) star $K_{1,b}$, $I(\overline{K_{1,b}^{++}}) = I(\overline{K_{1,b}^{+-}}) = 2b - 1$,

(vi) wheel W_n, $I(\overline{W_n^{++}}) = I(\overline{W_n^{+-}}) = 3n - 4$.

The Table 4 shows integrity of basic graphs and integrity of transformation graphs $\overline{G^{++}}$ and $\overline{G^{+-}}$ of graphs with same structure.

G	$I(G)$	$\overline{G^{++}}$ and $\overline{G^{+-}}$	$I(\overline{G^{++}}) = I(\overline{G^{+-}})$
P_4	3	$\overline{P_3^{++}}$ and $\overline{P_3^{+-}}$	3
P_5	3	$\overline{P_5^{++}}$ and $\overline{P_5^{+-}}$	7
C_5	4	$\overline{C_3^{++}}$ and $\overline{C_3^{+-}}$	4
C_6	4	$\overline{C_6^{++}}$ and $\overline{C_6^{+-}}$	10
K_n	n	$\overline{K_n^{++}}$ and $\overline{K_n^{+-}}$	$\frac{n(n+1)}{2} - 2$
$K_{8,8}$	9	$\overline{K_{2,3}^{++}}$ and $\overline{K_{2,3}^{+-}}$	9
$K_{8,9}$	9	$\overline{K_{8,9}^{++}}$ and $\overline{K_{8,9}^{+-}}$	87
$K_{1,b}$	2	$\overline{K_{1,b}^{++}}$ and $\overline{K_{1,b}^{+-}}$	$2b - 1$
W_{27}	11	$\overline{W_5^{++}}$ and $\overline{W_5^{+-}}$	11
W_{28}	11	$\overline{W_{28}^{++}}$ and $\overline{W_{28}^{+-}}$	80

Table 4:

Theorem 4.15. *For any connected graph G of order n and size m,*

$$I(\overline{G^{-+}}) = min\{n+m-1, m+I(G)\}.$$

Proof. For a connected graph G of order n and size m, $\overline{G^{-+}}$ has order $n+m$. Let $S_1 \subset V(\overline{G^{-+}})$. Choose a set S_1 containing the edges of G. So, $|S_1| = |E(G)| = m$. The removal of elements of set S_1 from a graph $\overline{G^{-+}}$ gives a graph G. Consider the value $|S_1| + I(G) = m + I(G)$.
Choose a set $S_2 \subset V(\overline{G^{-+}})$ consisting of all the elements of $\overline{G^{-+}}$ except an edge and two incident vertices. So, $|S_2| = n+m-3$. The removal of elements of set S_2 from $\overline{G^{-+}}$ gives $K_2 \cup K_1$. Hence, $m(\overline{G^{-+}} - S_2) = 2$. Consider, $|S_2| + m(\overline{G^{-+}} - S_2) = n+m-1$.
The minimum value among $m+I(G)$ and $n+m-1$ gives integrity of $\overline{G^{-+}}$. Therefore, $I(\overline{G^{-+}}) = min\{n+m-1, m+I(G)\}$. □

Corolary 4.5. *The integrity of*

(i) path $P_n (n \geq 3)$, $I(\overline{P_n^{-+}}) = n + \lceil 2\sqrt{n+1}\rceil - 3$,

(ii) cycle $C_n (n \geq 4)$, $I(\overline{C_n^{-+}}) = n + 2\lceil n \rceil - 1$,

(iii) complete graph K_n, $I(\overline{K_n^{-+}}) = \frac{n(n+1)}{2} - 1$,

(iv) complete bipartite graph $K_{a,b} (a, b \geq 2)$, $I(\overline{K_{a,b}^{-+}}) = ab + 1 + min\{a, b\}$,

(v) star $K_{1,b} (b \geq 2)$, $I(\overline{K_{1,b}^{-+}}) = b + 2$,

(vi) wheel $W_n (n \geq 5)$, $I(\overline{W_n^{-+}}) = 2n + \lceil 2\sqrt{n-1}\rceil - 2$.

The column 2 and 4 Table 5 shows integrity of basic graphs and integrity of transformation graphs $\overline{G^{-+}}$ of graphs with same structure.

Theorem 4.16. *For any connected graph G of order n and size m,*

$$I(\overline{G^{--}}) = m + I(G).$$

Proof. Let G be an (n, m) graph. Then $\overline{G^{--}}$ is a graph of order $n+m$. Let $S_1 \subset V(\overline{G^{--}})$. Consider a set S_1 containing all the edges of G. So, $|S| = |E| = m$. The removal of the vertices of set S_1 from $\overline{G^{--}}$ gives a graph G. Therefore, $I(\overline{G^{--}}) = m + I(G)$. □

Corolary 4.6. *The integrity of*

(i) path P_n, $I(\overline{P_n^{--}}) = n - 3 + \lceil 2\sqrt{n+1}\rceil$,

G	$I(G)$	$\overline{G^{-+}}$	$I(\overline{G^{-+}})$	$\overline{G^{--}}$	$I(\overline{G^{--}})$
P_6	4	$\overline{P_3^{-+}}$	4	$\overline{P_3^{--}}$	4
P_7	4	$\overline{P_7^{-+}}$	10	$\overline{P_7^{--}}$	10
C_{13}	7	$\overline{C_4^{-+}}$	6	$\overline{C_4^{--}}$	7
C_{14}	7	$\overline{C_{14}^{-+}}$	21	$\overline{C_{14}^{--}}$	21
K_n	n	$\overline{K_n^{-+}}$	$\frac{n(n+1)}{2} - 1$	$\overline{K_n^{--}}$	$\frac{n(n+1)}{2}$
$K_{8,9}$	9	$\overline{K_{2,3}^{-+}}$	9	$\overline{K_{2,3}^{--}}$	9
$K_{9,9}$	9	$\overline{K_{9,9}^{-+}}$	91	$\overline{K_{9,9}^{--}}$	91
$K_{1,b}$	2	$\overline{K_{1,b}^{-+}}$	$b+2$	$\overline{K_{1,b}^{--}}$	$2b+1$
W_{32}	12	$\overline{W_5^{-+}}$	12	$\overline{W_5^{--}}$	12
W_{33}	12	$\overline{W_{33}^{-+}}$	76	$\overline{W_{33}^{--}}$	76

Table 5:

(ii) *cycle* C_n, $I(\overline{C_n^{--}}) = n - 1 + \lceil 2\sqrt{n} \rceil$,

(iii) *complete graph* K_n, $I(\overline{K_n^{--}}) = \frac{n(n+1)}{2}$,

(iv) *complete bipartite graph* $K_{a,b}$, $I(\overline{K_{a,b}^{--}}) = ab + 1 + min\{a, b\}$,

(v) *star* $K_{1,b}$, $I(\overline{K_{1,b}^{--}}) = 2b + 1$,

(vi) *wheel* W_n, $I(\overline{W_n^{--}}) = 2n - 2 + \lceil 2\sqrt{n-1} \rceil$.

The column 2 and 6 of Table 5 shows integrity of basic graphs and integrity of transformation graphs $\overline{G^{--}}$ of graphs with same structure.

4.2 *Integrity of semitotal point graph of combination of basic graphs*

Definition 4.1. *[16] The product $G \times H$ of two graphs G and H is defined as follows:*
Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = (V_1, V_2)$. Then u and v are adjacent in $G \times H$ whenever [$u_1 = v_1$ and u_2 adj v_2] or [$u_2 = v_2$ and u_1 adj v_1].
If G and H are (n_1, m_1) and (n_2, m_2) graphs respectively. Then, $G \times H$ is $(n_1n_2, n_1m_2 + n_2m_1)$ graph.

Theorem 4.17. *For a graph $K_2 \times P_n$ $(n \geq 3)$,*

$$I\big((K_2 \times P_n)^{++}\big) = \begin{cases} 7, & if\ n = 3, \\ 11, & if\ n = 5, \\ \frac{5n-7}{2}, & if\ n\ is\ odd\ and\ n \geq 7, \\ \frac{5n-2}{2}, & if\ n\ is\ even. \end{cases}$$

Proof. The graph $(K_2 \times P_n)^{++}$ has $5n - 2$ vertices and $3(3n - 2)$ edges.
Let $S \subset V\big((K_2 \times P_n)^{++}\big)$. The proof includes the following cases.
Case 1. Suppose n is odd and $n \geq 7$.
Choose a set S containing the two internal vertices adjacent to corresponding central vertices of each of two P_n's in $K_2 \times P_n$. So, $|S| = 4$. The deletion of vertices of set S from $(K_2 \times P_n)^{++}$ results in a graph with components of orders 1, 7, $\frac{5(n-3)}{2}$. Hence, $m\big((K_2 \times P_n)^{++} - S\big) = \frac{5(n-3)}{2}$, since $n \geq 7$. The value of $|S| + m\big((K_2 \times P_n)^{++} - S\big)$ for this set S is $\frac{5n-7}{2}$ and it is minimum.
Therefore, $I\big((K_2 \times P_n)^{++}\big) = \frac{5n-7}{2}$.
Case 2. Suppose n is even.
Let S be a set containing the two internal vertices which are central vertices of each of two P_n's in $K_2 \times P_n$. So, $|S| = 4$. The removal of vertices of set S from $(K_2 \times P_n)^{++}$ results in a graph with components of orders 1, $\frac{5(n-2)}{2}$. Hence, we can write $m\big((K_2 \times P_n)^{++} - S\big) = \frac{5(n-2)}{2}$. The value of $|S| + m\big((K_2 \times P_n)^{++} - S\big)$ for the above set S is minimum.
Therefore, $I\big((K_2 \times P_n)^{++}\big) = \frac{5n-2}{2}$.
Case 3. Suppose $n = 3, 5$.
By direct calculation using the definition of integrity, the result follows. □

Theorem 4.18. *For a graph $K_2 \times C_n$ $(n \geq 4)$,*

$$I\big((K_2 \times C_n)^{++}\big) = \begin{cases} 2\left\lceil \frac{n}{2} \right\rceil + 7, & if\ n\ is\ even, \\ 2\left\lceil \frac{n}{2} \right\rceil + 7, & if\ n\ is\ odd\ and\ n \leq 7, \\ 2\left\lceil \frac{n}{3} \right\rceil + 12, & if\ n\ is\ odd\ and\ n \geq 9. \end{cases}$$

Proof. The graph $(K_2 \times C_n)^{++}$ has $5n$ vertices and $9n$ edges.
Let $S \subset V\big((K_2 \times C_n)^{++}\big)$.
Case 1. Suppose n is even.
Let S_1 be an independent set of vertices of C_n such that $|S_1| = \beta_0(C_n) = \frac{n}{2}$.
Case 2. Suppose n is odd.
Let S' be an independent set C_n such that $|S'| = \beta_0(C_n) = \frac{n-1}{2}$. Let v_1 be a vertex of $V(C_n) \setminus S'$ such that v_1 is adjacent to a vertex of S' as well as to a vertex of $V(C_n) \setminus S'$. Let $S_1 = S' \cup \{v_1\}$.
Combining the above two cases we get, $S_1 = \left\lceil \frac{n}{2} \right\rceil$. Choose a set S consisting of vertices of two C_n's of $K_2 \times C_n$ such that $|S| = 2|S_1| = 2\left\lceil \frac{n}{2} \right\rceil$. The removal of

vertices of set S from $(K_2 \times C_n)^{++}$ results in a graph with components of orders $1, 7$. Hence, $m\big((K_2 \times C_n)^{++} - S\big) = 7$. The value of $|S| + m\big((K_2 \times C_n)^{++} - S\big)$ for the above set S is minimum.
Therefore, $I((K_2 \times C_n)^{++}) = 2\left\lceil \frac{n}{2} \right\rceil + 7$.
Case 3. Suppose $n(\geq 9)$ is odd.
Let S be an independent set of vertices of two C_n's in a manner that the distance between the selected vertices is 3. Then, $|S| = 2\left\lceil \frac{n}{3} \right\rceil$. The removal of vertices of set S from $(K_2 \times C_n)^{++}$ results in a disconnected graph with components of orders $1, 12$. Hence, $m\big((K_2 \times C_n)^{++} - S\big) = 12$. Therefore, $I\big((K_2 \times C_n)^{++}\big) = 2\left\lceil \frac{n}{3} \right\rceil + 12$. □

Theorem 4.19. *For a graph $K_2 \times K_{1,b}$ $(b \geq 2)$,*

$$I\big((K_2 \times K_{1,b})^{++}\big) = 7.$$

Proof. The semitotal point graph $(K_2 \times K_{1,b})^{++}$ has $5b+3$ vertices and $3(3b+1)$ edges.
Let S be a subset of $V\big((K_2 \times K_{1,b})^{++}\big)$. Choose S such that it contains the two vertices corresponding to central vertices of each of two stars of $K_2 \times K_{1,b}$. Clearly, $|S| = 2$. The removal of vertices of S from $(K_2 \times K_{1,b})^{++}$ results in a graph with components of orders $1, 5$. Hence, $m\big((K_2 \times K_{1,b})^{++} - S\big) = 5$. This set S gives least value of $|S| + m\big((K_2 \times K_{1,b})^{++} - S\big)$. Therefore, $I\big((K_2 \times K_{1,b})^{++}\big) = 7$. □

Theorem 4.20. *For a graph $K_p \times K_q$ $(p = q \geq 2)$,*

$$I\big((K_p \times K_q)^{++}\big) = pq + 1.$$

Proof. The semitotal point graph $(K_p \times K_q)^{++}$ has $\frac{pq(p+q)}{2}$ vertices and $\frac{3pq(p+q-2)}{2}$ edges.
Select a set S such that the elements of it correspond to all the vertices of $K_p \times K_q$. So, $|S| = pq$. The removal of vertices of S from $(K_p \times K_q)^{++}$ results in a totally disconnected graph with $\frac{3pq(p+q-2)}{2}$ vertices. Clearly, $m\big((K_p \times K_q)^{++} - S\big) = 1$. Therefore, $I\big((K_p \times K_q)^{++}\big) = pq + 1$. □

Definition 4.2. *[16] The corona $G \circ H$ of graphs G and H is a graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then joining by an edge each vertex of the i'th copy of H is named (H, i) with the i'th vertex of G.*

If G and H are (n_1, m_1) and (n_2, m_2) graphs respectively. Then, $G \circ H$ is $(n_1(1 + n_2), m_1 + n_1 m_2 + n_1 n_2)$ graph.

Theorem 4.21. *For a graph $K_2 \circ P_n$ $(n \geq 3)$,*

$$I\big((K_2 \circ P_n)^{++}\big) = \begin{cases} 7, & if\ n = 3, \\ 10, & if\ n = 5, \\ \frac{3(n+1)}{2}, & if\ n\ is\ odd\ and\ n \geq 7, \\ 3(\frac{n}{2}+1), & if\ n\ is\ even. \end{cases}$$

Proof. The graph $(K_2 \circ P_n)^{++}$ has $6n+1$ vertices and $3(4n-1)$ edges.
Let $S \subset V\big((K_2 \circ P_n)^{++}\big)$. The proof includes the following cases.
Case 1. Suppose n is odd and $n \geq 7$.
Choose a set S containing the two internal vertices adjacent to corresponding central vertices of each of two P_n's and the two vertices of K_2 in $K_2 \circ P_n$. So, $|S| = 6$. The removal of vertices of S results in a graph with components of orders 1, 4, $\frac{3(n-1)}{2}$. Hence, $m\big((K_2 \circ P_n)^{++} - S\big) = \frac{3(n-1)}{2}$, since $n \geq 7$. The value of $|S| + m\big((K_2 \circ P_n)^{++} - S\big)$ for this set S is $\frac{5n-7}{2}$ is least. Therefore, $I\big((K_2 \circ P_n)^{++}\big) = \frac{5n-7}{2}$.
Case 2. Suppose n is even.
Choose a set S containing the two internal vertices which are central vertices of each of two P_n's and the two vertices of K_2 in $K_2 \circ P_n$. So, $|S| = 6$. The removal of vertices of S from $(K_2 \circ P_n)^{++}$ results in a graph with components of orders 1, $\frac{3(n-2)}{2}$. Clearly, we can write $m\big((K_2 \circ P_n)^{++} - S\big) = \frac{3(n-2)}{2}$. The value of $|S| + m\big((K_2 \circ P_n)^{++} - S\big)$ for the above set S is minimum. Therefore, $I\big((K_2 \circ P_n)^{++}\big) = 3(\frac{n}{2}+1)$.
Case 3. Suppose $n = 3, 5$.
By direct calculation using the definition of integrity, we can obtain the result. □

Theorem 4.22. *For a graph $K_2 \circ C_n$ $(n \geq 4)$,*

$$I\big((K_2 \circ C_n)^{++}\big) = 2\left\lceil \frac{n}{2} \right\rceil + 6.$$

Proof. The graph $(K_2 \circ C_n)^{++}$ has $7n+2$ vertices and $15n$ edges.
Let $S \subset V\big((K_2 \circ C_n)^{++}\big)$.
Case 1. Suppose n is even.
Let S_1 be an independent set of vertices of C_n such that $|S_1| = \beta_0(C_n) = \frac{n}{2}$.
Case 2. Suppose n is odd.
Let S' be an independent set of vertices of C_n such that $|S'| = \beta_0(C_n) = \frac{n-1}{2}$. Let v_1 be a vertex of $V(C_n) \setminus S'$ such that v_1 is adjacent to a vertex of S' as well as to a vertex of $V(C_n) \setminus S'$. Let $S_1 = S' \cup \{v_1\}$ and $|S_1| = \frac{n+1}{2}$.
Combining the above two cases we get, $S_1 = \lceil \frac{n}{2} \rceil$. Choose a set S_2 consisting of vertices of two C_n's of $K_2 \circ C_n$ such that $|S_2| = 2|S_1| = 2\lceil \frac{n}{2} \rceil$. Select a set S_3 consisting of the vertices of K_2 of $K_2 \circ C_n$. So, $S = S_2 \cup S_3$. Hence,

$|S| = 2\left\lceil \frac{n}{2} \right\rceil + 2$. The removal of vertices of set S from $(K_2 \circ C_n)^{++}$ results in a graph with components of orders 1, 4. Hence, $m\big((K_2 \circ C_n)^{++} - S\big) = 4$. The value of $|S| + m\big((K_2 \circ C_n)^{++} - S\big)$ for the above set S is minimum. Therefore, $I\big((K_2 \circ C_n)^{++}\big) = 2\left\lceil \frac{n}{2} \right\rceil + 6$. $\square$

Theorem 4.23. *For a graph $K_p \circ K_q$,*

$$I\big((K_p \circ K_q)^{++}\big) = pq + p + 1.$$

Proof. The semitotal point graph of $K_p \circ K_q$ has $p(q+1)$ vertices and $\frac{p}{2}[p+q(q+1)-1]$ edges. Let S be a subset of $V\big((K_p \circ K_q)^{++}\big)$. Choose S such that it contains vertices of K_p and vertices of p copies of K_q of $K_p \circ K_q$. So $|S| = p(q+1)$. The removal of vertices of set S from $(K_p \circ K_q)^{++}$ results in a totally disconnected graph with $\frac{p}{2}[p + q(q+1) - 1]$ vertices. Clearly, $m\big((K_p \circ K_q)^{++} - S\big) = 1$. Therefore, $I\big((K_p \circ K_q)^{++}\big) = pq + p + 1$. $\square$

5 Conclusion

In this paper, we have computed the integrity of generalized transformation graphs in terms of elements of a graph G. Also, integrity of semitotal point graph of combinations of basic graphs are obtained. Finally, we have established the relation between integrity of basic graphs and integrity of their generalized transformation graphs. We conclude that integrity of generalized transformation graphs are greater than or equal to integrity of graphs that have same structure.

6 Acknowledgement

*This work is partially supported by the University Grants Commission (UGC), New Delhi, through UGC-SAP DRS-III for 2016-2021: F.510/3/DRS-III/2016(SAP-I).

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