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$gp\alpha$- Kuratowski closure operators in topological spaces

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Abstract

In this paper, we introduce and study topological properties of gpα-limit points, $gp\alpha$-derived sets, $gp\alpha$-interior and $gp\alpha$-closure using the concept of $gp\alpha$-open set. Further, the relationships between these concepts are investigated. Also, Kuratowski axioms are discussed.
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1 Introduction

Topology is an indispensable object of study in Mathematics with open sets as well as closed sets being the most fundamental concepts in topological spaces. General topology plays an important role in many branches of mathematics as well as many fields of applied sciences.

Introduction to the concept of pre-open sets and pre-closed sets were made by Mashhour et al. (7) and the idea of α-open sets was introduced by Njastad (6).The concepts and characterizations of semi open and semi pre open sets are studied in (4) and (1) repectively. The concept, generalized closed sets of Levine (5) opened the flood gates of research in generalizations of closed sets in general topology. Many researchers (3), (8), (10), (11) worked on weaker forms of closed sets. Recently, Benchalli et al.(2) and Patil et al. (9) introduced and studied the concept of $\omega\alpha$-open sets and $gp\alpha$-open sets in topological spaces.

The present authors continued the study of $gp\alpha$-closed sets and their properties. We study the $gp\alpha$-closure, $gp\alpha$-interior, $gp\alpha$-neighbourhood, $gp\alpha$-limit points and $gp\alpha$-derived sets by using the concept of $gp\alpha$-open sets and their topological properties. We provide the relationship between $gp\alpha$-derived set (resp. $gp\alpha$-limit points, $gp\alpha$-interior) and pre-derived set (resp. pre-limit points and pre-interior). Also, we studied Kuratowski closure axioms with respect to $gp\alpha$-open sets.

2 Preliminaries

Throught the paper, let X and Y (resp. (X, τ) and (Y, σ)) always denotes non-empty topological spaces on which no separation axioms are assumed unless explicitly mentioned.

The following definitions are useful in the sequel:

Definition 2.1. *A subset A of X is said to be a*
(a) $\omega\alpha$-closed (2) if α-cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is ω-open in X.
(b) $gp\alpha$-closed (9) if p-cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is α-open in X.

3 $gp\alpha$-Interior and $gp\alpha$-closure in topological spaces

This section deals with $gp\alpha$-interior and $gp\alpha$-closure and some of their properties.

Definition 3.1. *(9) Let $A \subset X$, then $gp\alpha$-interior of A is denoted by $gp\alpha$-int(A), and is defined as $gp\alpha$-int(A) = $\cup\{G : A \subseteq G :$ G is $gp\alpha$-open in X $\}$.*

Definition 3.2. *(9) Let $A \subset X$, then $gp\alpha$-closure of A is denoted by $gp\alpha$-cl(A), and is defined as $gp\alpha$-cl(A) = $\cap\{G : A \subseteq G :$ G is $gp\alpha$-closed in X $\}$.*

Theorem 3.1. *If A is $gp\alpha$-closed then $gp\alpha$-cl(A) = A.*
Proof: *Let A be $gp\alpha$-closed in X. Since $A \subseteq A$, and A is $gp\alpha$-closed in X. Then $A \in \{G : A \subseteq G$ and G is $gp\alpha$-closed in X $\}$, that is $A = \cap\{G : A \subseteq G$ and G is $gp\alpha$-closed $\}$. Hence $gp\alpha$-cl(A) $\subseteq A$. But $A \subseteq gp\alpha$-cl(A) is always true. Therefore $gp\alpha$-cl(A) = A.*

In general the converse of Theorem 3.1 is not true.

Example 3.1. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. Let $A = \{a, b\}$. Then we can observe that $gp\alpha$-cl(A) = $\{a, b\}$. therefore $gp\alpha$-cl(A) = A. But A is not $gp\alpha$-closed in X.*

Remark 3.1. *(9) Let $A \subseteq X$ and A is $gp\alpha$-closed in X. Then $gp\alpha$-cl(A) is the smallest $gp\alpha$-closed set containing A.*

However, the converse of the Remark 3.1 is not true in general.

Example 3.2. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$. Consider $A = \{a, c\}$, then $gp\alpha$-cl(A) = X , which is the smallest $gp\alpha$-closed set containing A. But A is not $gp\alpha$-closed in X.*

Remark 3.2. *(9) For subsets A,B of X, then $gp\alpha$-cl$(A \cap B)$$\subseteq$$gp\alpha$-cl(A) $\cap$ $gp\alpha$-cl(B).*

Remark 3.3. *(9) For subsets A,B of X, then $gp\alpha$-cl$(A \cup B)$ = $gp\alpha$-cl(A) $\cup$$gp\alpha$-cl(B).*

Remark 3.4. *For any $A, B \subseteq X$ then we have the following properties:*
(i) $gp\alpha$-int(X) = X and $gp\alpha$-int$(\phi) = \phi$. (9)
(ii) $gp\alpha$-int(A) $\subseteq A$.(9)
(iii) If B is any $gp\alpha$-open set contained in A then $B \subseteq gp\alpha$-int(A).(9)
(iv) if $A \subseteq B$, $gp\alpha$-int(A) $\subseteq$ $gp\alpha$-int(B).
(v) $gp\alpha$-int($gp\alpha$-int(A)) = $gp\alpha$-int(A).

Remark 3.5. *For any $A, B \subseteq X$, $gp\alpha$-int$(A \cup B) =$ $gp\alpha$-int(A) $\cup$ $gp\alpha$-int(B).*

4 $gp\alpha$-Neighbourhood points in topological spaces

This section deals with the properties of gpα-neighbourhood points in topological spaces.

Definition 4.1. *(9) A subset N of X is said to be gpα-neighbourhood of a point $x \in X$, if there exists an gpα-open set G containing x such that $x \in G \subseteq N$.*

Theorem 4.1. *Every neighbourhood N of X is a gpα-neighbourhood of X.*
Proof: *Follows from the definition 4.1 and every open set is gpα-open (9).*

From the following example converse of the Theorem 4.1is not true.

Example 4.1. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a, b\}, \{a, b, c\}\}$.*
Let $a \in X$. Consider $A = \{a, d\}$. Since A is gpα-neighbourhood of the point a, but A is not a neighbourhood of the point a.

Remark 4.1. *If $N \subseteq X$ is gpα-open then, N is gpα-neighbourhood of each of its points.*

Converse of the Theorem 4.1 need not true in general, as seen from the following example.

Example 4.2. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. Here the gp$\alpha$-open sets are: X, ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{c, d\}$, $\{a, c, d\}$. Then the set $A = \{a, b\}$ is gpα-neighbourhod of the points a and b, but the set $A = \{a, b\}$ is not gpα-open in X.*

Theorem 4.2. *Let A be gpα-closed set in X and $x \in A^c$. Then there exists gpα-neighbourhood N of x such that $N \cap A = \phi$.*
Proof: *Let A be gpα-closed in X and $x \in A^c$. Then $A^c = X \setminus A$ which is gpα-open in X. Then from Remark 4.1, A^c is gpα-neighbourhood of each of its points. Hence, for every point $x \in A^c$, there exists gpα-neighbourhood N of X such that $N \subseteq A^c$. Hence $N \cap A = \phi$.*

Theorem 4.3. *Let $x \in X$ and gp$\alpha N(x)$ be the collection of all gpα-neighbourhoods of X. Then the following results holds:*
(i) gp$\alpha N(x) \neq \phi$, $\forall\, x \in X$.
(ii) $N \in$ gp$\alpha N(x)$ implies $x \in N$.
(iii) Let $N \in$ gpαN(x) and $N \subseteq M$, then $M \in$ gpαN(x).
(iv) $N \in$ gp$\alpha N(x)$ and $M \in$ gp$\alpha N(x)$ then $N \cap M \in$ gp$\alpha N(x)$.
Proof: *(i) We have X is always gpα-open in X. Hence X is in gpαN(x) of every point $x \in X$. Therefore gpα-N(X) $\neq \phi$ for every point $x \in X$.*
(ii) From definition of $N \in$ gp$\alpha N(x)$, it follows that $x \in N$.

(iii) Let $N \in gp\alpha N(x)$ and $N \subseteq M$. Then there exists $gp\alpha$-open set G such that $x \in G \subseteq N$. Since $N \subseteq M$, then $x \in G \subseteq M$. So by definition of 4.1, M is a $gp\alpha$-neighbourhood point of x. Hence $M \in gp\alpha N(x)$.
(iv) Let $N \in gp\alpha N(x)$ and $M \in gp\alpha N(x)$. Then, there exist $gp\alpha$-open sets U and V such that $x \in U \subseteq N$ and $x \in V \subseteq M$. That is $x \in U \cap V \subseteq N \cap M$. Therefore, for every point $x \in X$, there exists $gp\alpha$-open set $U \cap V$ such that $x \in U \cap V \subseteq N \cap M$. So, $N \cap M$ is a $gp\alpha$-neighbourhood of a point x. Hence, intersection of two $gp\alpha$-neighbourhood of a point is again a $gp\alpha$-neighbourhood of point.

Corollary 4.1. *For any subset A of X, every α-interior point of A is $gp\alpha$-interior point of A.*
Proof: *It follows from the fact that every α-open set is $gp\alpha$-open in X (9).*

Theorem 4.4. *For any subset A of X, every pre-interior point of A is $gp\alpha$-interior point of A.*
Proof: *For any pre-interior point x of A. Then there exists pre-open set G containing x such that $G \subseteq A$. Since every pre-open set is $gp\alpha$-open (9), then G is $gp\alpha$-open in X. Hence x is a $gp\alpha$-interior point of A.*

5 $gp\alpha$-Kuratowski closure operators in topological spaces

Theorem 5.1. *If P-C(X, τ) is closed under finite union, then $gp\alpha$-C(X, τ) is closed under finite union, where P-C(X, τ) and $gp\alpha$-C(X, τ) are the families of pre-closed sets and $gp\alpha$-closed sets in (X, τ) respectively.*
Proof: *Let A and B are $gp\alpha$-closed sets in X and $A \cup B \subseteq G$, where G is $\omega\alpha$-open in X. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are $gp\alpha$-closed, then $pcl(A) \subseteq G$ and $pcl(B) \subseteq G$. Then $pcl(A) \cup pcl(B) = pcl(A \cup B) \subseteq G$ from (7). Thus, from hypothesis, pcl$(A \cup B) \subseteq G$. Hence $A \cup B$ is $gp\alpha$-closed in X.*

Definition 5.1. *Let $\tau^*_{gp\alpha}$ be the topology on X generated by $gp\alpha$-closure in the usual manner, $\tau^*_{gp\alpha} = \{G \subset X : gp\alpha\text{-}cl(X \setminus G) = X \setminus G\}$.*

Definition 5.2. *Let $\tau^*_{g^*p}$ be the topology on X generated by g^*p-closure in the usual manner, that is*
$\tau^*_{g^*p} = \{G \subset X : g^*p\text{-}cl(X \setminus G) = X \setminus G\}$

Theorem 5.2. *Let $A \subseteq X$. Then the following statements holds:*
(i) $\tau \subseteq \tau^*_{gp\alpha}$
(ii) $\tau \subseteq \tau^p \subseteq \tau^*_{gp\alpha}$ (τ^p is family of pre-open sets.(12))

Proof: *(i) Let $A \in \tau$, then A^c is closed in X. We have $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq cl(A^c)$. Since, A^c is closed, then $cl(A^c)$ is also closed in X. Hence $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq A^c$. Therefore $gp\alpha$-cl$(A^c) \subseteq A^c$. But $A^c \subseteq gp\alpha$-cl(A^c) is always true. Thus $gp\alpha$-cl$(A^c) = A^c$. Hence $A \in \tau^*_{gp\alpha}$.*
(ii) Since, every pre-closed set is $gp\alpha$-closed in X, proof follows.

Theorem 5.3. *For any subset A of a topological space X, $\tau \subseteq \tau^*_{gp\alpha} \subseteq \tau^*_{g^*p}$.*
Proof: *Let us consider $A \in \tau$, then A^c is closed in X. Then $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq cl(A^c)$. Since, A^c is closed, then $cl(A^c) = A^c$. Therefore $A^c \subseteq gp\alpha$-cl$(A^c) \subseteq A^c$. Hence, $gp\alpha$-cl$(A^c) \subseteq A^c$. But $(A^c) \subseteq gp\alpha$-cl(A^c) is always true. Hence $(A^c) = gp\alpha$-cl(A^c). Thus $A \in \tau^*_{gp\alpha}$ and hence $\tau \subseteq \tau^*_{gp\alpha}$.*
Let $A \in \tau^*_{gp\alpha}$. Then $gp\alpha$-cl$(A^c) = A^c$. But $A^c \subseteq g^*p$-cl$(A^c) \subseteq gp\alpha$-cl$(A^c) = A^c$ from (9). Hence g^*p-cl$(A^c) = A^c$. Hence $A \in \tau^*_{g^*p}$. Thus $A \in \tau^*_{gp\alpha}$ implies that $A \in \tau^*_{g^*p}$. Hence $\tau \subseteq \tau^*_{gp\alpha} \subseteq \tau^*_{g^*p}$.

Theorem 5.4. *The following statements are equal for the space X:*
(i) Every $gp\alpha$-closed set is pre-closed.
(ii) $\tau^p = \tau^*_{gp\alpha}$.
(iii) For each $x \in X$, $\{x\}$ is $\omega\alpha$-open or pre-open.
Proof: *$(i) \to (ii)$ Let $G \in \tau^*_{gp\alpha}$. Then from (9) and by the Theorem 3.1, we have $gp\alpha$-cl(A) = A. Hence $X \setminus G = gp\alpha$-cl$(X \setminus G)$ = p-cl$(X \setminus G)$. Therefore $X \setminus G$ is pre-closed and so G is pre-open. Therefore $\tau^*_{gp\alpha} \subseteq \tau^p$ and from (9), $\tau^p \subseteq \tau^*_{gp\alpha}$. Hence $\tau^p = \tau^*_{gp\alpha}$.*
$(ii) \to (iii)$ Let $\{x\} \in X$. By (9), we have $X \setminus \{x\} = gp\alpha$-cl$(X \setminus \{x\})$ is true only when $\{x\}$ is not $\omega\alpha$-closed. Hence $\{x\} \in \omega\alpha$-C(X, τ) or $x \in \tau^p$.
$(iii) \to (i)$ Let A be $gp\alpha$-closed in X and $x \in p - cl(A)$. Then, we have $x \in A$.
case I: If $\{x\}$ is $\omega\alpha$-closed. Suppose $x \notin A$, then $p - cl(A) \setminus A$ contains $\omega\alpha$-closed set $\{x\}$, which is contradiction. Hence $x \in A$.
case II: If $\{x\}$ is pre-open. Since $x \in pcl(A)$, then $\{x\} \cap A = \phi$. Hence, we have pcl(A) = A and thus A is pre-closed. Therefore $gp\alpha$-C$(X, \tau) \subset p$-C(X, τ).

Theorem 5.5. *Every $gp\alpha$-closed set closed if and only if $\tau = \tau^*_{gp\alpha}$.*
Proof: *Suppose every $gp\alpha$-closed set is closed. Let A be $gp\alpha$-closed then, $gp\alpha$-cl(A) = cl(A). Thus $\tau = \tau^*_{gp\alpha}$.*
Conversely, let A be $gp\alpha$-closed then from (9), $A = gp\alpha$-cl(A). Hence $X \setminus A \in \tau^*_{gp\alpha}$. Hence, A is closed in X.

Theorem 5.6. *Every $gp\alpha$-closed set is pre-closed if and only if $\tau^p = \tau^*_{gp\alpha}$.*
Proof: *Suppose that every $gp\alpha$-closed set is pre-closed. Let A be $gp\alpha$-closed in X. Then from hypothesis, $gp\alpha$-cl(A) = pcl(A). Thus $\tau^p = \tau^*_{gp\alpha}$.*
Conversely, let A be $gp\alpha$-closed in X. Then $A = gp\alpha$-cl(A). Thus $X \setminus A \in \tau^*_{gp\alpha}$. Hence, A is pre-closed in X.

Remark 5.1. *Let A be any subset of X. Then $gp\alpha$-int(A) is the largest $gp\alpha$-open set contained in A if A is $gp\alpha$-open.*

Theorem 5.7. *$gp\alpha$-closure is a Kuratowski closure operator on X.*
Proof:*Follows from the Definition 3.2 and (9)*

6 Characterizations of gpα-closed sets in topological spaces

Definition 6.1. *(9) A point $x \in X$ is a $gp\alpha$-limit point of a subset A of X, if and only if every $gp\alpha$-neighbourhood of x contains a point of A distinct from x. That is, $[N \setminus \{x\}] \cap A \neq \phi$ for each $gp\alpha$-neighbourhood N of x.*

Definition 6.2. *(9) The set of all $gp\alpha$-limit points of A is a $gp\alpha$-derived set of A and is denoted by $gp\alpha$-d(A).*

Example 6.1. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$.*
Let $A = \{c\}$. Then the only limit point with respect to the set $A = c$ is point b.
Therefore $d(A) = \{b\}$.
But $gp\alpha$-limit point with respect to the set A is ϕ. Therefore $gp\alpha$-d(A) = ϕ.

Theorem 6.1. *Let A be any subset of X, Then, A is $gp\alpha$-closed if and only if $gp\alpha$-d(A)$\subseteq A$.*
Proof: *Let A be $gp\alpha$-closed in X, then A^c is $gp\alpha$-open in X such that $x \in A^c$. Then for each point $x \in X$ and from Definition 4.1 , there exist $gp\alpha$-open set G such that $x \in G \subseteq X \setminus A$. Then $A \cap (X \setminus A) = \phi$. Therefore, $gp\alpha$-neighbourhood of G contains no points of A. Hence, x is not a $gp\alpha$-limit point of A. Thus, no point of $X \setminus A$ is a $gp\alpha$-limit point of A, that is A contains all the $gp\alpha$-limit points. Therefore A contains the $gp\alpha$-derived points. Hence $gp\alpha$-d(A) $\subseteq A$.*
Conversely, suppose $gp\alpha$-d(A)$\subseteq A$ and let $x \in A^c$. So $x \notin A$. Hence $x \notin gp\alpha$-d(A). Therefore x is not a limit point of A. Then, there exists $gp\alpha$-open set G such that $G \cap (A \setminus \{x\}) = \phi$, that is $G \subseteq X \setminus A$. Therefore for each $x \in X \setminus A$, there exists $gp\alpha$-open set G such that $x \in G \subseteq X \setminus A$. Therefore $X \setminus A$ is $gp\alpha$-open in X and hence A is $gp\alpha$-closed.

Theorem 6.2. *Let τ_1 and τ_2 be any two topologies on a set X such that $gp\alpha$-O$(X, \tau_1) \subseteq gp\alpha$-O$(X, \tau_2)$. Then for every subset A of X, every $gp\alpha$-limit point of A with respect to τ_2 is $gp\alpha$-limit point of A with respect to τ_1.*
Proof: *Let x be a $gp\alpha$-limit point of A with respect to τ_2. Then by definition of $gp\alpha$-limit point $(G \cap A) \setminus \{x\} \neq \phi$, this is true for every $G \in gp\alpha$-O(X, τ_2) and $x \in G$.*

But by hypothesis, gpα-O$(X, \tau_1) \subseteq$ gpα-O(X, τ_2). Hence $(G \cap A) \setminus \{x\} \neq \phi$ for every $G \in$ gpα-O(X, τ_1) such that $x \in G$. Hence x is a gpα-limit point of A with respect to the topology τ_1.

Theorem 6.3. *Let A and B be any two subsets of (X, τ). Then the following assertions are valid:*
(i) gpα-d(A) $\subseteq d_p(A)$, where d_p is a pre-derived set (12).
(ii) gpα-d($A \cup$ gpα-d(A)) $\subseteq A \cup$ gpα-d(A).
Proof: *(i) It clearly observed from the fact that every pre-open set is gpα-open in X.*
Then $y \in G$ and $y \in$ gpα-d(A) $\setminus\{x\}$. That is $y \in G$ and $y \in$ gpα-d(A). Hence $G \cap (A \setminus \{y\}) \neq \phi$. Let $z \in G \cap (A \setminus \{y\})$, then $x \neq z$ as $x \notin A$. Thus $G \cap (A \setminus \{x\}) \neq \phi$.
(ii) Let $x \in$ gpα-d($A \cup$ gpα-d(A)). If $x \in A$, then $x \in$ gpα-d(A). Therefore $x \in A \cup$ gpα-d(A). On the contrary assume that $x \notin A$. Then $G \cap (A \cup$ gpα-d(A)) $\setminus\{x\}) \neq \phi$, is true for all $G \in$ gpα-d(A) and $x \in G$. Therefore $(G \cap A) \setminus \{x\} \neq \phi$ or $G \cap$ (gpα-d(A)) $\setminus\{x\}) \neq \phi$. Thus $x \in$ gpα-d(A).
If $G \cap$ (gpα-d(A)) $\setminus\{x\} \neq \phi$, then will get $x \in$ gpα-d(gpα-d(A)). Since $x \notin A$, then $x \in$ gpα-d(gpα-d(A)) $\setminus A$. Therefore gpα-d($A \cup$ gpα-d(A)) $\subseteq A \cup$ gpα-d(A).

Remark 6.1. *We can see the following implification with respect to gpα-open sets.*

Example 6.2. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$. In (X, τ) we have,*
pre-open sets are: X, ϕ, $\{a\}, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{c, d\}, \{d, e\}, \{a, c, d\}$, $\{a, c, e\}, \{a, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}$.
gpα-open sets are: X, ϕ, $\{a\}, \{c\}, \{d\}, \{a, b\}, \{b, d\}, \{c, d\}, \{a, c\}$, $\{b, c\}, \{d, e\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c\}, \{a, c, e\}, \{a, d, e\}$, $\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}$.
(i)Let $A = \{b, c, d\}$.
Then pre-limit point of the set A is $\{b, e\}$ and
gpα-limit point of A is $\{e\}$.
Hence $d_p(A) = \{b, e\}$ and gpα-d(A) = $\{e\}$.
Hence, gpα-d(A) $\subseteq d_p(A)$.

(ii)Let $A = \{a, c, d\}$, then gpα-d(A) = $\{b, e\}$.
Consider $A \cup$ gpα-d(A) = $\{a, c, d\} \cup \{b, e\} = X$.
But gpα-d(X) = $\{b, e\}$.
Therefore gpα-d($A \cup$ gpα-d(A)) = gpα-d(X) = $\{b, e\}$.
Now consider $A \cup$ gpα-d(A) = $\{a, c, d\} \cup \{b, e\} = X$.
Hence gpα-d($A \cup$ gpα-d(A)) $\neq A \cup$ gpα-d(A), that is $\{b, e\} \neq X$, but gpα-d($A \cup$ gpα-d(A) $\subset A \cup$ gpα-d(A).

Remark 6.2. *if $A \subseteq B$, then gpα-d(A) $\subseteq$ gpα-d(B).*

Example 6.3. *Consider the Example 6.2,*
Let $A = \{b, c, d\}$. Then we have gpα-d(A) = $\{e\}$.
Let $B = \{a, e\}$. Then gpα-limit point of the set B is ϕ, that is gpα-d(B) = ϕ.
Thus we can observe that gpα-d(B) $\subset$ gpα-d(A) but $B \nsubseteq A$.

Remark 6.3. *gpα-d$(A \cap B) \subseteq$ gpα-d(A) $\cap$ gpα-d(B).*

Example 6.4. *From the Example 6.2,*
Let $A = \{b, c, d\}$ and $B = \{b, c\}$ are any two subsets of X.
Then gpα-d(A) = $\{e\}$ and gpα-d(B) = ϕ.
Also gpα-d$(A \cap B)$ = ϕ.
Therefore gpα-d$(A \cap B) \subseteq$ gpα-d(A) $\cap$ gpα-d(B)

Theorem 6.4. *Let A be any subset of X and $x \in X$. Then the following statements are equal:*
(i) For each $x \in X$, $A \cap G \neq \phi$ where G is gpα-open in X.
(ii) $x \in$ gpα-cl(A).
Proof: *Let A be any subset of X.*
$(i) \rightarrow (ii)$: On the contrary assume that $x \notin$ gpα-cl(A). Then there exists gpα-closed set F such that $A \subseteq F$ and $x \notin F$. Then $X \setminus F$ is gpα-open in X containing a point x. Hence $A \cap (X \setminus F) \subseteq A \cap (X \setminus A) = \phi$, which is contradiction to the assumption. Hence $x \in$ gpα-cl(A).
$(ii) \rightarrow (i)$: Follows from the Definition 3.2.

Corollary 6.1. *For any subset A of a space X, gpα-d(A) $\subseteq$ gpα-cl(A).*

Theorem 6.5. *Let A be any subset of X, then gpα-cl(A) = $A \cup$ gpα-d(A).*
Proof: *Let $x \in$ gpα-cl(A). On the contrary assume that $x \notin A$. Let G be any gpα-open set containing a point x. Then $(G \setminus \{x\}) \cap A \neq \phi$. Therefore x is gp$\alpha$-limit point of A and hence x is gpα-derived set of A, that is $x \in$ gpα-d(A).*
Hence gpα-cl(A) $\subseteq A \cup$ gpα-d(A).
From the Corollary 6.1, we have gpα-d(A) $\subseteq$ gpα-cl(A) and $A \subseteq$ gpα-cl(A) is always true. Hence $A \cup$ gpα-d(A) $\subseteq$ gpα-cl(A).
Therefore gpα-cl(A) = $A \cup$ gpα-d(A).

Theorem 6.6. *Let A be gpα-open set in X and B be any subset of X. Then $A \cap$gpα-cl(B) $\subseteq$ gpα-cl$(A \cap B)$.*
Proof: *Let $x \in A \cap$gpα-cl(B). Then $x \in A$ and $x \in$ gpα-cl(B). From the Theorem 6.5, we have gpα-cl(B) = $B \cup$ gpα-d(B).*
If $x \in B$ then $x \in A \cap B$. Then $A \cap B \subseteq$ gpα-cl$(A \cap B)$.
If $x \notin B$ then $x \in$ gpα-d(B). From the definition of gpα-limit point, we have

$G \cap B \neq \phi$ for every $gp\alpha$-open set G containing x. Therefore $G \cap (A \cap B) = (G \cap A) \cap B \neq \phi$. Hence $x \in gp\alpha$-d$(A \cap B) \subseteq gp\alpha$-cl$(A \cap B)$. Therefore $A \cap gp\alpha$-cl(B) $\subseteq gp\alpha$-cl$(A \cap B)$.

However the equality does not holds in general

Example 6.5. *Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}\}$.*
Let $A = \{a, b\}$ and $B = \{a, c\}$ are two subsets of X.
Then $A \cap gp\alpha$-cl(B) = $\{a, b\}$
and $gp\alpha$-cl$(A \cap B) = X$
This implies $A \cap gp\alpha$-cl(B) $\neq gp\alpha$-cl$(A \cap B)$. But, $A \cap gp\alpha$-cl(B) $\subset gp\alpha$-cl$(A \cap B)$.

7 Conclusions

In this present work, we have analyzed the notion of generalized pre α-closed sets in topological spaces. We have established the results of $gp\alpha$-closure, $gp\alpha$-interior, $gp\alpha$-neighbourhood and $gp\alpha$-limit points. Moreover, we have characterized these concepts with suitable examples. Finally, we apply $gp\alpha$-open sets for Kuratowski closure axioms. There is a scope to study and extend these newly defined concepts.

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