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Certain results on metric and norm in fuzzy multiset setting

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Abstract

Fuzzy multiset is an extension of fuzzy set in multiset framework. In this paper, we review the concept of fuzzy multisets and study the notions of metric and norm on fuzzy multiset. Some results on metric and norm are established in fuzzy multiset context.

Keywords: fuzzy set; fuzzy multiset; metric; norm.

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1. Introduction

The theory of set proposed by Cantor in 1915, is a collection of well-defined objects of thought and intuition. The limitation of set theory is its inability to deal with the vague properties of its member or element, and likewise its distinctness property which does not allows repetition in the collection. In other to handle the vague property of a set, Zadeh [19] proposed a mathematical model that deal with vagueness of a set known as fuzzy sets. The distinct property of crisp set has been violated by allowing repetition of an element in a collection. This gave birth to a set called multiset. The term multiset was first suggested by De Bruijn to Knuth in a private correspondence as noted in [13]. The theory of multisets has been studied [3, 4, 10, 11, 12, 16]. Lake [14] presented an abridge account on sets, fuzzy sets, multisets and functions.

By synthesizing the concepts of fuzzy sets and multisets, Yager [18] introduced the concept of fuzzy multiset (FMS) that deal with vagueness property of a set and allowed the repetition of its membership function. In fact, fuzzy bag or fuzzy multiset generalizes fuzzy sets in such a way that the membership degree of a fuzzy set is allowed to repeat. Some fundamentals properties of fuzzy bags have been studied [6, 15]. The concept of fuzzy bags has been applied in multi-criteria decision-making [1, 2], sequences [5] and computational science [17].

Metric is a function that defines a concept of distance between any members of the set, which are usually called points. The notions of metric and norm have been extended to the environment of fuzzy sets [7, 8, 9]. In this work, we present the notions of norm and metric in fuzzy multiset context.

2. Preliminaries

In this section, we review some definitions and result that are important for the main work.

Definition 2.1 [18]. Assume X is a set of elements. Then, a fuzzy bag/multiset, A drawn from X can be characterized by a count membership function CM_A such that $CM_A : X \to Q$, where Q is the set of all crisp bags or multisets from the unit interval, $I = [0,1]$.

According to Syropoulos [17], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0,1] \to N,$$

where $I = [0,1]$ and $N = \mathbb{N} \cup \{0\}$.

The count membership degrees, $CM_A(x)$ for $x \in X$ is given as

$CM_A(x) = \{ \mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^n(x), \ldots \}$,

where $\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^n(x), \ldots \in [0,1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq \mu_A^3(x)$, $\geq \ldots \geq \mu_A^n(x) \geq \ldots$, whereas in a finite case, we write

$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^n(x)\}$ for $\mu_A^1(x) \geq \mu_A^2(x) \geq \ldots \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{< \frac{CM_A(x)}{x} > \mid x \in X\} \text{ or } A = \{< x, CM_A(x) > \mid x \in X\}.$$

In a simple term, a fuzzy multiset, A of X is characterized by the count membership function, $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0,1]$. We denote the set of all fuzzy multisets by $FMS(X)$.

Example 2.2. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{0.7, 0.6, 0.1\}, CM_A(b) = \{0.9, 0.7, 0.5\}, CM_A(c) = \{0.5, 0.4, 0.2\}$, A is a fuzzy multiset of X written as

$$A = \{< \frac{0.7,0.6,0.1}{a} >, < \frac{0.9,0.7,0.5}{b} >, < \frac{0.5,0.4,0.2}{c} >\}.$$

Definition 2.3 [15]. Let $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)\ \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.4 [15]. Let $A, B \in FMS(X)$. Then, A and B are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x)\ \forall x \in X$.

Definition 2.5 [17]. Let $A, B \in FMS(X)$.Then, the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, are defined by

(i) $CM_{A\cap B}(x) = CM_A(x) \wedge CM_B(x)\ \forall x \in X.$
(ii) $CM_{A\cup B}(x) = CM_A(x) \vee CM_B(x)\ \forall x \in X.$

Definition 2.6 [17]. Let $A, B \in FMS(X)$. Then, the sum of A and B, denoted by $A \oplus B$, is defined by the addition operation in $X \times [0,1]$ for crisp multiset. That is,
$CM_{A\oplus B}(x) = CM_A(x) + CM_B(x)\ \forall x \in X$

The addition operation is carry out by merging the membership degree in a decreasing order.
Definition 2.7 [6]. Let $A, B \in FMS(X)$. Then, the difference of B from A is a fuzzy multiset $A \ominus B$ such that $\forall x \in X,\ CM_{A\ominus B}(x) = CM_A(x) - CM_B(x) \vee 0.$

Definition 2.8 [6]. Let $A, B \in FMS(X)$. Then, the complement of A is a fuzzy multiset A' such that $\forall x \in X,\ CM_{A'}(x) = 1 - CM_A(x)$. **Metric and Norm defined over Fuzzy Multisets**

3. Metric and norm defined over fuzzy multisets

In this section, we present metrics and norm defined over fuzzy multiset.
Definition 3.1. Let X be an arbitrary non-empty set and let $A, B \in FMS(X)$. A metric or distance function between A and B on X is a function $d: X \times X \rightarrow [0,1]$ with the following properties:

(i) $d\,(CM_A(x), CM_B(x)) \geq 0\ \forall x \in X.$

(ii) $d\,(CM_A(X), CM_B(x)) = 0$ iff $CM_A(x) = CM_B(x)\ \forall x \in X.$

(iii) $d\,(CM_A(x), CM_B(x)) = d\,(CM_B(x), CM_A(x))\ \forall x \in X.$

(iv)
$$d\,(CM_A(x), CM_C(x)) \leq d\,(CM_A(x), CM_B(x)) + d\,(CM_B(x), CM_C(x))$$

$\forall\ x \in X$ if $C \in FMS(X)$.

Note:
(i) The distance is a non-negative function and only zero at a single point.
(ii) The distance is a symmetric function.
(iii) The distance satisfy triangle.
Proposition 3.2. Let $A, B, C\ \in FMS(X)$. Then $d(A, B) = \ |A - B|$ is a metric defined on $\mathrm{FMS(X)}$.

Proof. We use Definition 3.1:
Axiom (i)

$$d(A, B) = d\big(CM_A(x), CM_B(x)\big) = \ |A - B| = \ |CM_A(x) - CM_B(x)|$$

$$= V\{CM_A(x) - CM_B(x), 0\ \} \geq 0.$$

Axiom (ii)
If
$d(A, B) = 0\ \Longrightarrow |CM_A(x) - CM_B(x)| = 0 \Longrightarrow CM_A(x) - CM_B(x) = 0 \Longrightarrow$
$CM_A(x) = CM_B(x)$

.
Conversely, if $A = B \Longrightarrow d(A,A) = \ |A - A| = \ |0|$.

Axiom (iii)
$d(A, B) = \ |A - B| = \ |-1||B - A| = |B - A| = d\ (B, A)$.

Axiom (iv)

$d(A, B) = \ |A - B| = \ |A - C + C - B| \geq |A - C| + \ |C - B| \geq d(A, C) + d(C, B)$.

The following are distances between fuzzy multisets:
Hamming distance;

$$d(A, B) = \ \sum_{i=1}^{n} |CM_A(x) - CM_B\ (x)|.$$

Euclidean distance;

$$d\ (A, B) = \ \sqrt{\sum_{i=1}^{n} (CM_A(x) - CM_B(x))^2}$$

Normalized Hamming distance;

$$d(A, B) = \frac{1}{n} \sum_{i=1}^{1} |CM_A(x) - CM_B(x)|.$$

Normalized Euclidean distance;

$$d(A, B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(CM_A(x) - CM_B(x))^2}.$$

Theorem 3.3. Let X be non-empty set and $A, B \in FMS(X)$, then

$$d(A, B) = d(A', B').$$

Proof. We show that $d(A, B) = d(A', B')$ or

$d\big(CM_A(x), CM_B(x)\big) = d\big(CM_{A'}(x), CM_{,}(x)\big)$. But $CM_{A'}(x) = 1 - CM_A(x)$ and $CM_{B'}(x) = 1 - CM_B(x)$.

Thus,
$$\begin{aligned} d\big(CM_A(x), CM_B(x)\big) &= |CM_A(x) - CM_B(x)| \\ &= |[1 - CM_{A'}(x)] - [1 - CM_{B'}(x)]| \\ &= |-CM_{A'}(x) + CM_{B'}(x)| \\ &= |-1||CM_{A'}(x) - CM_{B'}(x)| \\ &= |CM_{A'}(x) - CM_{B'}(x)| \\ &= d(CM_{A'}(x), CM_{B'}(x)) \end{aligned}$$

Hence $d\big(CM_A(x), CM_B(x)\big) = d\big(CM_{A'}(x), CM_{B'}(x)\big)$.

Corollary 3.4. If $d\big(CM_A(x), CM_B(x)\big)$ is a distance of fuzzy multiset of A and B, then

$d^*\big(CM_A(x), CM_B(x)\big) = \frac{1}{2}\big[d\big(CM_A(x), CM_B(x)\big) + d\big(CM_A(x), CM_B(x)\big)\big]$**Proof.** Clearly,

$$\begin{aligned} d^*\big(CM_A(x), CM_B(x)\big) &= \frac{1}{2}\big[d\big(CM_A(x), CM_B(x)\big) + d\big(CM_A(x), CM_B(x)\big)\big] \\ &= d\big(CM_A(x), CM_B(x)\big). \end{aligned}$$

Proposition 3.5. If $d\big(CM_A(x), CM_B(x)\big)$ is a metric of fuzzy multiset A and B, then $d\big(CM_A(x), CM_B(x)\big) - d\big(CM_B(x), CM_A(x)\big) = 0$.

Proof. By Definition 3.1, if $d\big(CM_A(x), CM_B(x)\big) = d\big(CM_B(x), CM_A(x)\big)$, so it follows that $d\big(CM_A(x), CM_B(x)\big) - d\big(CM_B(x), CM_A(x)\big) = 0$.

Proposition 3.6. Let $\lambda \in \mathcal{R}$ and $d\big(CM_A(x), CM_B(x)\big)$ is a metric defined on $FMS(x)$. Then $\lambda d\big(CM_A(x), CM_B(x)\big)$ is also a metric.

Proof. The proof is obvious, since $\lambda d\left(CM_A(x), CM_B(x)\right) = d(\lambda(CM_A(x), CM_B(x))) =$

$d\left(\lambda CM_A(x), \lambda CM_B(x)\right)$.

Hence $\lambda d\left(CM_A(x), CM_B(x)\right)$ is a metric.

Corollary 3.7. If $\lambda > 1$, then $\lambda d\left(CM_A(x), CM_B(x)\right) \geq d\left(CM_A(x), CM_B(x)\right)$.

Proof. The proof is straightforward.

Corollary 3.8. If $\lambda < 1$ then $\lambda d\left(CM_A(x), CM_B(x)\right) < d\left(CM_A(x), CM_B(x)\right)$.

Proof. The proof is straightforward.

Corollary 3.9. If $\lambda < 0$ then $\lambda d\left(CM_A(x), CM_B(x)\right) = d\left(CM_B(x), CM_A(x)\right)$

Proof. The proof is straightforward.

Definition 3.10. Let X be a non-empty set and A be a fuzzy multiset of X. A non-negative real-valued function $\|.\|$ defined on A is called a norm if the following properties are satisfied:

(i) $\|A\| = 0$ iff $A = 0$ that is, $\|CM_A(x)\| = 0$ iff $CM_A(x) = 0$.

(ii) $\|\alpha A\| = |\alpha|\|A\|$ which implies that

$\|CM_{\alpha A}(x)\| = |\alpha|\,\|CM_A(x)\|\ \forall\, \alpha \in \mathcal{R}$.

(iii) $\|A + B\| \leq \|A\| + \|B\|$ which implies

that $\|CM_{A+B}(x)\| \leq \|CM_A(x)\| + \|CM_B(x)\|$.

The Fuzzy multiset equipped with a norm is called Normed Fuzzy multiset.

Proposition 3.11. Let $A, B \in FMS(x)$, then $\|A + B\| = \|A - B\|$.

Proof. We show that $\|A + B\| = \|A - B\|$. Now,

$$\|A - B\| = \|CM_{A-B}(x)\| = \left\|CM_{A+(-B)}(x)\right\|$$
$$= \|CM_A(x)\| + |-1|\|CM_B(x)\| = \|CM_A(x)\| + \|CM_B(x)\|$$
$$= \|CM_{A+B}(x)\| = \|A + B\|.$$

Proposition 3.12. Let $A \in FMS(x)$ and a norm $\|.\|$ define over $A\ as\ \|A\| = |A|$.

Proof.

(i) $\|A\| = |CM_A(x)| = CM_A(x) > 0$.

(ii) $\|\alpha A\| = |\alpha CM_A(x)| = |\alpha|\|CM_A(x)\| = |\alpha||CM_A(x)|$.

(iii) $\|A+B\| = \|CM_{A+B}(x)\| = |CM_{A+B}(x)| \leq |CM_A(x)| + |CM_B(x)|$

$= \|A\| + \|B\|$.

Hence $\|A\| = |A|$ is a norm defined over fuzzy multiset A.

Corollary 3.13. If $\alpha > 0$, then $\|\alpha A\| > \|A\|$ and if $\alpha \in [0,1]$, then $\|\alpha A\| < \|A\|$.

Proof. The proof is obvious.

4 Conclusions

We have presented a brief review on the concept of fuzzy multisets and explored metric and norm in fuzzy multiset context. A number of results on metric and norm were established, respectively.

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