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EL–hyperstructures: an overview

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Abstract

This paper gives a current overview of theoretical background of a special class of hyperstructures constructed from quasi / partially ordered (semi) groups using a construction known as the "Ends lemma". The paper is a collection of both older and new results presented at AHA 2011.

Key words: group, hyperstructure, partial ordering, quasi ordering, semigroup.

MSC2010: 20N20.

1 Introduction

This paper is a written version of a lecture given at the *11th International Conference on Algebraic Hyperstructures and Applications* in October 2011. Together with results new at that time I presented some older results published in journals or proceedings of rather local impact thus unknown to the international hyperstructure community. The new results presented at the conference were meanwhile published as [20]. Therefore, this article instead of presenting new unpublished results gives an overview of what has so far been achieved in the area of *EL*–hyperstructures. For proofs of respective theorems as well as for examples explaining their meaning cf either [20] or works indicated throughout the paper. Where not stated otherwise all definitions of hyperstructure concepts or properties of hyperstructures are used in the sense of the standard book [3].

2 The "Ends lemma"

The EL–hyperstructures are hyperstructures constructed from quasi / partially ordered (semi)groups using the "Ends lemma", which has the form of the following theorems from [4]. The notation $[a)_\leq$ used below stands for the set $\{x \in H; a \leq x\}$, where the properties of H are specified in the respective theorems (typically $(H, \cdot, \leq)$ is a quasi-ordered / partially ordered set / semigroup / group).

Lemma 2.1 *([4], Theorem 1.3, p. 146) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $* : S \times S \to \mathcal{P}'(S)$ defined by*

$$a * b = [a \cdot b)_\leq \tag{1}$$

is associative. The semi-hypergroup $(S, *)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.

In accordance with other papers regarding this topic, the hyperstructure $(S, *)$ constructed in this way will further on be called the *associated* hyperstructure to the single-valued structure $(S, \cdot)$ or *an "Ends lemma"–based hyperstructure*, or an *EL–hyperstructure* for short. Instead of S the carrier set will from Lemma 2.3 onward be denoted by H.

Lemma 2.2 *([4], Theorem 1.4, p. 147) Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*

1^0 *For any pair $(a, b) \in S^2$ there exists a pair $(c, c') \in S^2$ such that $b \cdot c \leq a$ and $c' \cdot b \leq a$*

2^0 *The associated semi-hypergroup $(S, *)$ is a hypergroup.*

Remark 2.1 *If $(S, \cdot, \leq)$ is a partially ordered group, then if we take $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$, then condition 1^0 is valid. Therefore, if $(S, \cdot, \leq)$ is a partially ordered group, then its associated hyperstructure is a hypergroup.*

Remark 2.2 *The wording of the above lemmas is the exact translation of theorems from [4]. The respective proofs, however, do not change in any way, if we regard* quasi-ordered *structures instead of* partially ordered *ones as the anti-symmetry of the relation $\leq$ is not needed (with the exception of the $\Leftarrow$ implication of the part on commutativity, which does not hold in this case). The often quoted version of the "Ends lemma" is therefore the version assuming quasi–ordered structures.*

The "Ends lemma" was later extended (cf e.g. [23]). Notice that if $(H, \cdot)$ is commutative, then $(H, *)$ is a join space. Also notice that unlike in the original "Ends lemma" the underlying single-valued structure in the following theorem is a group (not a semigroup).

Lemma 2.3 *([23], Theorem 4) Let $(H, \cdot, \leq)$ be a quasi-ordered group and $(H, *)$ be the associated hypergroupoid. Then $(H, *)$ is the transposition hypergroup.*

Initially, the typical use of the "Ends lemma" was creating hyperstructures and proving or deriving their properties at random without any (or with a very limited) theoretical background. This model is used in e.g. [6, 8, 13, 21, 23]. In order to overcome this inconvenience, theoretical background of the "Ends lemma" is being developed.

3 Extending the lemma, identities and inverses

After the "Ends lemma" extension, i.e. Lemma 2.3, was proved, there arose a question of whether one can go any further to stronger hyperstructures such as canonical hypergroups, strongly canonical hypergroups, etc. A positive answer to this question would mean that numerous ring-like analogies of *EL*–hyperstructures could be studied extensively. Unfortunately, the answer – obtained in [18] – turned out to be negative.

Theorem 3.1 *Let $(H, \cdot, \leq)$ be a non-trivial quasi-ordered group, where the relation $\leq$ is not the identity relation, and let $(H, *)$ be its associated transposition hypergroup. Then:*

1. *$(H, *)$ does not have a scalar identity.*
2. *Regardless of commutativity, $(H, *)$ cannot be a canonical hypergoup.*

Naturally, if we regard the definition of a canonical hypergroup, 2 immediately follows from 1.

Once it was established that looking for scalar identities in *EL*–hyperstructures based on groups is of no point, at least the issue of identities was explored. In [18] the following simple results concerning identities were obtained.

Theorem 3.2 *Let $(H, *)$ be the semi-hypergroup associated to a quasi-ordered semigroup $(H, \cdot, \leq)$ with the identity u.*

1. *An element $e \in H$ is an identity of $(H, *)$ if and only if $e \leq u$.*
2. *If $(H, \cdot)$ is a group, then the identity of $(H, \cdot)$ is an identity of $(H, *)$.*

Again, 2 is naturally an immediate corollary of 1 yet we set it aside due to uniqueness of the single-valued group identity.

Lemma 3.1 *Let $(H, *)$ be the join space associated to a quasi-ordered commutative group $(H, \cdot, \leq)$. If an element $e \in H$ is an identity of $(H, *)$, then $e \leq e^{-1}$.*

Since the concept of an *inverse* in a hyperstructure is defined using the concept of an identity, the issue of inverses was touched upon in the same paper and the following result was obtained for the set $i(a)$ of inverses of an arbitrary element $a \in H$ in $(H, *)$.

Theorem 3.3 *Let $(H, *)$ be the transposition hypergroup associated to a quasi-ordered group $(H, \cdot, \leq)$. Then for an arbitrary $a \in H$ there holds*

$$i(a) = \{a' \in H; a' \leq a^{-1}\} = (a^{-1}]_{\leq},$$

where a^{-1} is the inverse of a in $(H, \cdot)$.

Corollary 3.1 *Let $(H, *)$ be the transposition hypergroup associated to a quasi-ordered group $(H, \cdot, \leq)$. Then $(H, *)$ is regular.*

4 Ring-like hyperstructures

Since it turned out that once we start with groups the "Ends lemma" cannot effectively be used to construct canonical hypergroups, the scope for use of this idea in the area of ring-like hyperstructures narrowed. Recall that there is a great variety of definitions of ring-like hyperstructures, yet many of them including the most often used one – that of *Krasner hyperring* – is built on canonical hypergroups. However, the idea of limits of the "Ends lemma" in the area of hyperstructures with two (hyper)operations is still worth exploring. In [19] (published before the author was able to get [12]) three possible extensions are suggested and explored:

1. Let $(H, +)$ and $(H, \cdot)$ be two single-valued structures. We can define a hyperoperation using one of the operations $+$ or $\cdot$ by e.g. $a * b = [a+b)_{\leq}$ – thus we get an *EL*–hyperstructure $(H, *)$. The hyperstructure will then be a triplet $(H, *, \cdot)$ where $*$ is a hyperoperation based on the single-valued operation $+$.

2. Let $(H,+)$ and $(H,\cdot)$ be two single-valued structures. We can define two hyperoperations, each based on one single-valued operation, i.e. for an arbitrary pair $(a,b) \in H^2$ we can define $a * b = [a+b)_{\leq}$ and $a \circ b = [a \cdot b)_{\leq}$. Thus we get a triplet $(H,*,\circ)$, where $*$ and $\circ$ are hyperoperations.

3. However, we can also start with a single single-valued structure $(H,\cdot)$ and using it define a hyperoperation $*$ by $a * b = [a \cdot b)_{\leq}$. The hyperstructure will then be a triplet $(H,*,\cdot)$ where $*$ is a hyperoperation based on the single-valued operation $\cdot$.

If we have a triplet $(H,+,\cdot)$, where symbols $+$ and $\cdot$ may stand for both single-valued and multivalued operation, then for an arbitrary triplet $(a,b,c) \in H^3$ we may either ask that $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$ or we may ask that inclusions holds instead of equalities.[1] Each of the three ways to create ring-like hyperstructures was explored with respect to both of these types of distributivity and the following results were obtained in [19]. Notice the variety of conditions imposed on the respective structures (group / semigroup, quasi-ordered / partially ordered, single-valued / multivalued).

Definition 4.1 [cf [28], p. 21, included as plain text] $(R,+,.)$ *is a hyperring in the general sense if* $(R,+)$ *is a hypergroup*[2], $(\cdot)$ *is associative hyperoperation and the distributive law*[3] $x(y+z) \subseteq xy + xz$, $(x+y)z \subseteq xz + yz$ *is satisfied for every* x,y,z *of* R. *Additive hyperring is the one of which only* $(+)$ *is a hyperoperation, multiplicative hyperring is the one of which only* $(\cdot)$ *is a hyperoperation.* [...] $(R,+,\cdot)$ *will be called semihyperring if* $(+),(\cdot)$ *are associative hyperoperations, where* $(\cdot)$ *is distributive with respect to* $(+)$. *The rest of definitions are analogous. If the equality in the distributive law is valid, then the hyperring is called strong or good.*

Theorem 4.1 *Let* $(H,+,\cdot)$ *be a ring such that* $(H,+)$ *is a group,* $(H,\cdot)$ *a semigroup and* $\leq$ *quasi-ordering on* H *such that* $(H,+,\leq)$ *is a quasi-ordered group and* $(H,\cdot,\leq)$ *is a quasi-ordered semigroup. Further, for an arbitrary pair of elements* $(a,b) \in H^2$ *define* $a * b = [a+b)_{\leq}$ *and* $a \circ b = [a \cdot b)_{\leq}$. *Then* $(H,*,\circ)$ *is a hyperring in the general sense.*

[1] The former distributivity is sometimes called *good* distributivity while the latter is often called *weak* distributivity.

[2] Vougiouklis uses the term *hypergroup of Marty*.

[3] Vougiouklis uses the sign $\subset$ in the sense of $\subseteq$.

Theorem 4.2 *Let $(H,+)$ be a semigroup, $(H,\cdot)$ a group and $\leq$ quasi-ordering on H such that $(H,+,\leq)$ is a quasi-ordered semigroup and $(H,\cdot,\leq)$ is a quasi-ordered group. Further, for an arbitrary pair of elements $(a,b) \in H^2$ define $a * b = [a+b)_{\leq}$ and $a \circ b = [a \cdot b)_{\leq}$. Finally, let $\cdot$ distribute over $+$ from both left and right. Then $(H,*,\circ)$ is a good semihyperring in the sense of Definition 4.1.*

Theorem 4.3 *Let $(H,+,\cdot)$ be a ring such that $(H,+)$ is a group with neutral element 0, $(H\setminus\{0\},\cdot)$ a group and $\leq$ quasi-ordering on H such that $(H,+,\leq)$ and $(H,\cdot,\leq)$ are quasi-ordered groups. Further, for an arbitrary pair of elements $(a,b) \in H^2$ define $a*b = [a+b)_{\leq}$ and $a \circ b = [a \cdot b)_{\leq}$. Then $(H,*,\circ)$ is a good hyperring in the general sense.*

Theorem 4.4 *Let $(H,+)$ be a group and $(H,\cdot,\leq)$ a quasi-ordered semigroup and for an arbitrary pair of elements $(a,b) \in H^2$ define $a \circ b = [a \cdot b)_{\leq}$. Further, let $(H,+,\cdot)$ be such that the operation $\cdot$ distributes over the operation $+$ from both left and right. Then $(H,+,\circ)$ is a good multiplicative hyperring.*

Definition 4.2 [cf [14], Definition 2.1 and Remark] *A hyperalgebra $(R,+,\cdot)$ is called a semihyperring if and only if*

(i) *$(R,+)$ is a semihypergroup;*

(ii) *$(R,\cdot)$ is a semigroup;*

(iii) *$\forall a,b,c \in R$, $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$*

If we replace (iii) by

$$\forall a,b,c \in R, a \cdot (b+c) \subseteq a \cdot b + a \cdot c \text{ and } (b+c) \cdot a \subseteq b \cdot a + c \cdot a$$

we say that R is a weak distributive semihyperring. A semihyperring is called with zero element, if there exists a unique element $0 \in R$ such that $0 + x = x = x + 0$ and $0 \cdot x = 0 = x \cdot 0$ for all $x \in R$. [...] A semihyperring is called a hyperring provided $(R,+)$ is a canonical hypergroup.

Theorem 4.5 *Let $(H,\cdot,\leq)$ be a quasi-ordered semigroup such that $\cdot$ is a commutative idempotent operation. Further, for an arbitrary pair of elements $(a,b) \in H^2$ define $a \circ b = [a \cdot b)_{\leq}$. Then $(H,\circ,\cdot)$ is a weak distributive semihyperring.*

Unfortunately, from Theorem 3.1 there follows that Krasner hyperrings cannot be constructed using the "Ends lemma" if the underlying single-valued structure $(H,+)$ is a group. However, there are weaker structures such as e.g. *hyperringoids*, which are defined as semihyperrings in the sense of Definition 4.2 where $(R, +)$ is a join space[4], for the construction of which the "Ends lemma" might still be used.

The assumptions of the following theorem seem rather complicated. The reason is simple: the requirement "$(H, +)$ is a group" results in trivialities. Notice that condition 1 is the condition used in Lemma 2.1 – the one which secures that $(H, *)$ is a hypergroup.

Theorem 4.6 *Let $(H, +)$ be a commutative semigroup, $(H, \cdot)$ a group and $\leq$ quasi-ordering on H such that*

1. *to every pair of elements $(a, b) \in H^2$ such that $a \leq b$ there exists a pair of elements $(c, c') \in H^2$ such that $b + c \leq a$, $c' + b \leq a$,*
2. *$(H, +, \leq)$ is a quasi-ordered semigroup and*
3. *$(H, \cdot, \leq)$ is a quasi-ordered group.*

Moreover, for an arbitrary pair of elements $(a, b) \in H^2$ define $a * b = [a + b)_{\leq}$. Finally, suppose that $\cdot$ distributes over $+$ from both left and right. Then

if $(H, *)$ satisfies the transposition axiom, then $(H, *, \cdot)$ is a hyperringoid.

Corollary 4.1 *If in Theorem 4.6 we suppose that $(H, +, \leq)$ is a quasi-ordered semigroup without any further assumptions, then $(H, *, \cdot)$ is a semi-hyperring in the sense of Definition 4.2.*

Theorem 4.7 *Let $(H, +, \leq)$ be a non-trivial quasi-ordered group with neutral element 0 such that $\leq$ is not the identity relation, $(H \setminus \{0\}, \cdot)$ a group. Moreover, for an arbitrary pair of elements $(a, b) \in H^2$ define $a * b = [a + b)_{\leq}$. Finally, suppose that $\cdot$ distributes over $+$ from both left and right. Then $(H, *, \cdot)$ is a weak distributive hyperringoid.*

5 The issue of a subhyperstructure

In order to proceed to the study of properties of *EL*–hyperstructures, one must clarify the concept of a subhyperstructure of an *EL*–hyperstructure

[4] This definition is used in [3], Chapter 6. On contrary Massouros brothers in [17] call this hyperstructure a *join* hyperringoid, while they call hyperringoid a hyperstructure such that $(R, +)$ is a hypergroup only. Further on in Theorem 4.6 the definition used in [3] is regarded.

since there are two possible approaches to it. If we regard a quasi-ordered semigroup $(H, \cdot, \leq)$ and define a hyperoperation $*$ on H by

$$a * b = [a \cdot b)_{\leq} = \{x \in H; a \cdot b \leq x\} \tag{2}$$

for an arbitrary pair of elements $(a, b) \in H^2$, we may in a subsemigroup $(G, \cdot)$ of the semigroup $(H, \cdot)$ set either

$$a *_G b = [a \cdot b)_{\leq_G} = \{x \in G; a \cdot b \leq x\} \tag{3}$$

or

$$a *_H b = [a \cdot b)_{\leq_H} = \{x \in H; a \cdot b \leq x\} \tag{4}$$

and thus create either $(G, *_G)$ or $(G, *_H)$. None of these concepts is obviously the only possible and "correct" one since the definition of both can be justified. In a way, the idea of $*_H$ conforms to the idea of the "Ends lemma" better. Thus in [22], which discussed the issue of subhyperstructures of *EL*–hyperstructures, it was this concept that was favoured. The concept of the *upper set* was introduced.[5]

Definition 5.1 *Let $(H, \cdot, \leq)$ be a partially ordered semigroup and let G be a nonempty subset of H.*

1. *If for an arbitrary element $g \in G$ there holds $[g)_{\leq} \subseteq G$, we call G an upper end of H.*

2. *If there exists an element $g \in G$ such that there exists an element $x \in H \setminus G$ such that $g \leq x$ (i.e. $x \in [g)_{\leq}$)[6], we say that G is not an upper end of H because of the element x.*

Among other results concerning hyperoperation $*_H$ (4) the following was proved.

Theorem 5.1 *Let $(H, *)$ be the semihypergroup associated to a quasi-ordered semigroup $(H, \cdot, \leq)$. Suppose that G is an upper end of H. If $(G, \cdot)$ is a subgroup of $(H, \cdot)$, then $(G, *)$ is a subhypergroup of $(H, *)$.*

[5]The concept itself is naturally not a new invention; the definition was only tailored for use in the "Ends lemma" context.

[6]We could – probably more properly since $x \notin G$ – write $g < x$ and $x \in [g)_{\leq} \setminus \{g\}$ yet in the definition we keep the $\leq$ notation of the Ends lemma for consistency reasons.

Proposition 5.1 *Let $(H,*)$ be the semihypergroup associated to a partially ordered semigroup $(H,\cdot,\leq)$ and $G\subseteq H$ nonempty. If $(G,*)$ is a subhypergroup of $(H,*)$, then $(G,\cdot)$ is a subsemigroup of $(H,\cdot)$ and G is an upper end of H such that for any pair $(a,b)\in G^2$ there exists a pair $(c,c')\in G^2$ such that $b\cdot c\leq a$ and $c'\cdot b\leq a$.*

Theorem 5.2 *Let $(H,*)$ be the semihypergroup associated to a partially ordered semigroup $(H,\cdot,\leq)$. Further, let $G\subseteq H$ be non-empty and such that $(G,\cdot)$ is a subgroupoid of $(H,\cdot)$, and let the relation $\leq_G$ be a restriction of $\leq$ on G, i.e. $\leq_G=\leq\cap(G\times G)$.[7] Finally – if it exists – denote u the identity of $(H,\cdot)$ and define a new hyperoperation $*_G: G\times G\to P^*(G)$ for arbitrary elements $a,b\in G$ by (3), i.e. by*

$$a *_G b=[a\cdot b)_{\leq_G}=\{x\in G; a\cdot b\leq_G x\}.$$

Then

1. *$(G,\cdot)$ is a semigroup if and only if $(G,*_G)$ is a semihypergroup.*
2. *$(G,\cdot)$ is a monoid if and only if $(G,*_G)$ is a semihypergroup and $u\in G$.*
3. *If $(G,\cdot)$ is a group, then $(G,*)$ is a transposition hypergroup.*
4. *If $(G,*)$ is a hypergroup, then $(G,\cdot)$ is a semigroup such that for any pair $(a,b)\in G^2$ there exists a pair $(c,c')\in G^2$ such that $b\cdot c\leq a$ and $c'\cdot b\leq a$.*

6 Properties of *EL*–hyperstructures and their subhyperstructures

Results in this section were presented at AHA 2011 and later included in [20].

Theorem 6.1 *Let $(H,*)$ be the hypergroup associated to a quasi-ordered group $(H,\cdot,\leq)$ and $(G,\cdot)$ its subgroup such that G is an upper end of H. Then $(G,*)$, where $*$ is defined for an arbitrary pair $(a,b)\in H^2$ as $a*b=[a\cdot b)_{\leq_H}$, is invertible and closed in H.*

[7]The exact quote from [22] at this place reads "for arbitrary elements $a,b\in G$ let $a\leq b\Rightarrow a\leq_G b$".

Theorem 6.2 *Let $(H, *)$ be the hypergroup associated to a quasi-ordered group $(H, \cdot, \leq)$ and $(G, *)$ its arbitrary subhypergroup associated to a subgroup $(G, \cdot)$ of $(H, \cdot)$, where G is an upper end of H (i.e. as defined in Theorem 5.1 using hyperoperation $*_H$). Denote u the identity of $(H, \cdot)$. Then*

1. *G is ultraclosed if and only if for any $h \in H$ such that $h \leq u$ it follows that $h \in G$.*
2. *If $G \neq H$ and if $(H, \cdot, \leq)$ has the smallest element, then $(G, *)$ is not ultraclosed.*
3. *If $(H, \cdot)$ or $(H, *)$ is commutative, then G is a complete part of H if and only if for every $h \in H$ such that $h \leq u$ there is $h \in G$.*

Theorem 6.3 *Let $(H, *)$ be the associated hypergroup of a quasi-ordered group $(H, \cdot, \leq)$ and $(G, *)$ its arbitrary subsemihypergroup associated to a subsemigroup $(G, \cdot)$ of $(H, \cdot)$.*[8] *If for arbitrary $x \in H$ and $g \in G$ there holds $x \cdot g \cdot x^{-1} \in G$, then $(G, *)$ is normal.*

Corollary 6.1 *Let $(H, *)$ be the hypergroup associated to a quasi-ordered group $(H, \cdot, \leq)$ and $(G, \cdot)$ its normal subgroup such that G is an upper end of H. Then $(G, *)$, where $*$ is defined as $a * b = [a \cdot b)_{\leq_H}$, is reflexive.*

Theorem 6.4 *Let $(H, *)$ be the hypergroup associated to a quasi-ordered group $(H, \cdot, \leq)$. Then $(H, *)$ is reversible.*

As far as *regularity* of *EL*–hyperstructures is concerned, cf Theorem 3.3 and its corollary. In the following theorem notice that by a subhypergroup we mean a subhyperstructure defined by hyperoperation $*_H$ (4). This is important to consider since the definition of inner irreducibility relies on subhyperstructures as a commutative hypergroup $(H, *)$ is called *inner irreducible* if for any pair of its subhypergroups G_1, G_2 such that $G_1 * G_2 = H$ there holds $G_1 \cap G_2 \neq \emptyset$.

Theorem 6.5 *Let $(H, *)$ be the associated hypergroup of a partially ordered commutative group $(H, \cdot, \leq)$.*

1. *If for every $x \in H$ such that x, x^{-1} are incomparable with respect to $\leq$ there is either $[x)_{\leq} \cap [x^{-1})_{\leq} \neq \emptyset$ or $(x]_{\leq} \cap (x^{-1}]_{\leq} \neq \emptyset$, then $(H, *)$ is inner irreducible.*

[8] In this context the fact whether we define a subhyperstructure by means of $*_H$ (4) or $*_G$ (3) is not important.

2. *If $(H, \leq)$ is a linear ordered set or if $(H, \leq)$ has the smallest or the greatest element, then $(H, *)$ is inner irreducible.*

Naturally, 2 is a corollary of 1 since in linear ordered sets all elements are comparable.

7 The issue of origins of a hypergroup

The "Ends lemma" describes a way to construct semihypergroups from quasi-ordered semigroups and hypergroups from semigroups with a special property. We know that groups are such structures that this property holds trivially. This means that we know that using the "Ends lemma" we may create a hypergroup from a group.

Thus one can ask: *if have a hypergroup (itself or as a subhypergroup of a larger structure) created in the "Ends lemma" fashion, is there a way to determine whether its underlying single-valued structure is a group or a semigroup?* Answering this question is not academic only as proofs of some of the above theorems have to answer this question in a rather complicated way. This issue is also connected to the issue of the converse of the "Ends lemma", which was already necessary to complete some proofs of theorems on subhyperstructures. The proof of the following theorem can be found in [22].

Theorem 7.1 *Let $(H, \cdot)$ be a non-trivial groupoid and $\leq$ a binary partial ordering on H such that for an arbitrary pair of elements $(a, b) \in H^2$, $a \leq b$, and for arbitrary $c \in H$ there holds $c \cdot a \leq c \cdot b$ and $a \cdot c \leq b \cdot c$. Further define a hyperoperation $* : H \times H \to \mathcal{P}^*(H)$ for an arbitrary pair of elements $(a, b) \in H^2$ by $a * b = [a \cdot b)_{\leq} = \{x \in H; a \cdot b \leq x\}$. Then if the hyperoperation $*$ is associative, then the single-valued operation $\cdot$ is associative too. Furthermore, if there exists an element $e \in H$ such that for every $a \in H$ there holds $a * e = e * a = [a)_{\leq}$, then this element e is the identity of the semigroup $(H, \cdot)$.*

Notice that if the relation $\leq$ is not antisymmetric, the above theorem is not true. This is caused by the fact that only for antisymmetric relations $\leq$ there holds that $[a)_{\leq} = [b)_{\leq}$ implies that $a = b$. Indeed, suppose a simple two element set $M = \{a, b\}$ where the relation $\leq$ is defined as $a \leq a, a \leq b, b \leq a, b \leq b$. This reflexive and transitive relation $\leq$ is obviously not antisymmetric and there holds $[a)_{\leq} = [b)_{\leq}$ yet $a \neq b$.

A simple way of distinguishing between semigroups and groups is the study of idempotent elements, i.e. elements which (if we ignore the identity) exist in semigroups only. In [20] a few basic results concerning idempotent elements are included.

Theorem 7.2 *Let $(H, *)$ be the semihypergroup associated to a quasi–ordered semigroup $(H, \cdot, \leq)$. For an arbitrary element $a \in H$ there holds*

$a * a = \{a\} \Leftrightarrow$ *a is an idempotent and simultaneously a maximal element of* $(H, \cdot, \leq)$.

Corollary 7.1 *Let $(H, *)$ be the associated hypergroup of such a quasi-ordered semigroup $(H, \cdot, \leq)$ that at least two distinct elements $a, b \in H$ are in relation $\leq$ (i.e. $\leq$ is not trivial). If there exists an element $a \in H$ such that $a * a = \{a\}$, then $(H, \cdot)$ is not a group.*

Corollary 7.2 *Let $(H, *)$ be the hypergroup associated to a quasi-ordered semigroup $(H, \cdot, \leq)$ and $(G, *)$ a subhypergroup of $(H, *)$.*[9]

1. *Denote u the identity of $(H, \cdot)$. If u is the maximal element of $(G, \leq)$ and at least two distinct elements $a, b \in G$ are in relation $\leq$, then $(G, \cdot)$ is a subsemigroup of $(H, \cdot)$ yet not a subgroup of $(H, \cdot)$.*
2. *If for two distinct elements $a, b \in H$ there holds $a * a = \{a\}$, $b * b = \{b\}$, then H does not have the greatest element. Also – obviously – $(H, \cdot)$ is not a group.*

8 "Ends lemma" in a broader context

Naturally, the "Ends lemma" is not a revolutionary stand alone concept. The study of relation of hyperstructures and ordered sets or binary relations is included in [3] as chapter 3. This part of the "canonical" book on hyperstructures was inspired by works of Chvalina (especially [5]) and Rosenberg (especially [24]). Results concerning the relation of hyperstructures and ordered sets have also been included in [12], another "canonical" book on hyperstructure theory. Among older works concerning relation of ordered sets and hyperstructures there is e.g. [2], which studies the relation in general giving a number of possible ways to create hyperstructures from ordered sets, and [29], in which a concept in a way similar to the "Ends lemma" may be found. Recent works related to the concept discussed in this article include e.g. works of Cristea and Ştefănescu which deal with n–ary relations on hypergroups (e.g. [9, 10]) or study of fundamental relations on hypergroupoids associated with binary relations (such as [11]), or works of e.g. Spartalis or Massouros (such as [16, 25, 26]).

[9] The statement is valid for subhypergroups based on both $*_H(4)$ or $*_G(3)$.

9 Open issues

There are many loose ends that wait to be tied. Most importantly, the full potential of the property included in Lemma 2.1 must be explored. This is closely connected to the problem of reversing the "Ends lemma", which has been partly answered by Theorem 7.1, and to the problem of telling the origins of the hypergroup for which the idea of idempotent elements is only a first and insufficient attempt. Clarifying this issue would also help in the study of ring-like *EL*–hyperstructures and in the study of such properties of *EL*–hyperstructures that rely on the concept of a subhypergroup.

Naturally, it might be very useful to set the issue of *EL*–hyperstructures in a broader perspective of hyperstructures constructed from binary operations of single-valued (semi)groups.

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