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Some studies on products of fuzzy soft graphs

Shashikala S[*]
Anil P N[†]

Abstract

In this paper, alpha, beta and gamma product of two fuzzy soft graphs are defined. The degree of a vertex in these product fuzzy soft graphs are determined and its regular properties are studied.

Keywords: alpha, beta and gamma product of fuzzy soft graphs, degree of a vertex, regular properties.

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1. Introduction

Graph theory is used to model various types of relations that exist in different fields of physics, chemistry, medicine, electrical network, computer science such as networking, image processing etc. Whenever the information provided is imprecise, uncertainty exists. Molodtsov [1] initiated the concept of soft sets to deal with uncertainty. A. Rosenfeld [2] developed the theory of fuzzy graphs in 1975 based on fuzzy sets which were initiated by Zadeh [3] in 1965. Maji et al. [5,7] presented the definition of fuzzy soft sets and applied it in decision making problems. Later many researchers progressively worked on these concepts and developed it. Operations on fuzzy graphs were demonstrated by J. N. Mordeson and C. S. Peng [6]. Akram and Saira Nawaz [8,11] introduced fuzzy soft graphs, studied some of its properties and applied these concepts in social network and road network. Shashikala S and Anil P N [12,15] discussed connectivity in fuzzy soft graphs and studied hamiltonian fuzzy soft cycles. A. Pouhassani and H. Doostie [13] studied degree, total degree, regularity and total regularity of fuzzy soft graph and its properties.

[*] Shashikala S (Global Academy of Technology, Bangalore, India); shashikala.s@gat.ac.in
[†] Anil P N (Global Academy of Technology, Bangalore, India); anilpn@gat.ac.in
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Regular fuzzy soft graphs and its related properties are studied by B Akhilandeswari [16]. Shovan Dogra [10] studied some types of fuzzy graph products such as modular product, homomorphic product and determined its degree of vertices. Union and intersection of fuzzy soft graphs and some of its properties are studied by Mohinta and Samanta [9]. Fuzzy soft theory provides a clear picture of the problems that allows parameterization finds applications in many areas. Recently, it is used to represent the oligopolistic competition among the wireless internet connection providers in Malaysia by Akram and Saba Nawaz [17].

In this paper, some products of fuzzy soft graphs namely alpha, beta and gamma products are defined and degree of a vertex in these products are determined and its regular properties are studied.

2. Preliminaries

Definition 2.1: [11] A fuzzy soft graph $\tilde{G}$ over a graph $G^*:(V,E)$ is a triple $(\tilde{F},\tilde{K},A)$ where:

a) A is a nonempty set of parameters

b) $(\tilde{F},A)$ is a fuzzy soft set over V

c) $(\tilde{K},A)$ is a fuzzy soft set over E

d) $\left(\tilde{F}(e_i),\tilde{K}(e_i)\right)$ is a fuzzy graph on G^* $\forall e_i \in A$

i.e. $\tilde{K}(e_i)(xy) \le \min\{\tilde{F}(e_i)(x),\tilde{F}(e_i)(y)\}$ for all $e_i \in A$ and $x,y \in V$.

Definition 2.2: [14] The underlying crisp graph of a fuzzy soft graph $\tilde{G}$ is denoted by $G^* = \left(F^*, K^*\right)$ where $F^* = \left\{x \in V : \tilde{F}(e_i)(x) > 0 \ \textit{for some } e_i \in A\right\}$, $K^* = \left\{(x,y) \in V \times V : \tilde{K}(e_i)(x,y) > 0 \ \textit{for some } e_i \in A\right\}$.

Definition 2.3: [8] Let $\tilde{G}$ be a fuzzy soft graph on G^*. The degree of a vertex x is defined as $\deg_{\tilde{G}}(x) = \sum_{e_i \in A} \sum_{y \in V, y \neq x} \tilde{K}(e_i)(xy)$.

Definition 2.4: [8]] $\tilde{G}$ is said to be a regular fuzzy soft graph if $\left(\tilde{F}(e_i),\tilde{K}(e_i)\right)$ is regular fuzzy graph for all $e_i \in A$. If $\left(\tilde{F}(e_i),\tilde{K}(e_i)\right)$ is a regular fuzzy graph of degree k for all $e_i \in A$ then $\tilde{G}$ is a k-regular fuzzy soft graph.

Definition 2.5: [4] The degree $d_{G^*}(x)$ of a vertex v in G^* is the number of edges incident with x.

In this paper, we assume that $G^*:(V,E)$ of any fuzzy soft graph $\tilde{G}$ is finite and simple.

Notation: Let $\tilde{G}_1:(\tilde{F}_1,\tilde{K}_1,A_1)$ and $\tilde{G}_2:(\tilde{F}_2,\tilde{K}_2,A_2)$ be two fuzzy soft graphs. The relation $\tilde{F}_1(e_i)\geq\tilde{K}_2(e_j)$ for all $e_i\in A_1$, $e_j\in A_2$ means that $\tilde{F}_1(e_i)(x)\geq\tilde{K}_2(e_j)(e)\ \ \forall x\in V_1, e\in E_2$ where $\tilde{F}_1$ is a fuzzy soft subset of V_1 and $\tilde{K}_2$ is a fuzzy soft subset of E_2.

3. Alpha product $(\alpha-product)$, Beta product $(\beta-product)$ and Gamma product $(\gamma-product)$ of Fuzzy soft graph

Definition 3.1: Let $\tilde{G}_1:(\tilde{F}_1,\tilde{K}_1,A_1)$ and $\tilde{G}_2:(\tilde{F}_2,\tilde{K}_2,A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. The $\alpha-product$ $\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2:(\tilde{F}_1\underset{\alpha}{\times}\tilde{F}_2,\tilde{K}_1\underset{\alpha}{\times}\tilde{K}_2,A_1\times A_2)$ is defined as follows:

$(\tilde{F}_1\underset{\alpha}{\times}\tilde{F}_2):A_1\times A_2\rightarrow FS(V_1\times V_2)$ by

$(\tilde{F}_1\underset{\alpha}{\times}\tilde{F}_2)(e_i,e_j)(x_ky_l)=\tilde{F}_1(e_i)(x_k)\wedge\tilde{F}_2(e_j)(y_l)\ \forall e_i\in A_1, e_j\in A_2, x_ky_l\in V_1\times V_2$ and

$(\tilde{K}_1\underset{\alpha}{\times}\tilde{K}_2):A_1\times A_2\rightarrow FS(E_1\times E_2)$ by

$$(\tilde{K}_1\underset{\alpha}{\times}\tilde{K}_2)(e_i,e_j)(x_ky_l)(x_my_n)=\begin{cases}\tilde{F}_1(e_i)(x_k)\wedge\tilde{K}_2(e_j)(y_ly_n) & if\ x_k=x_m, y_ly_n\in E_2\\ \tilde{F}_2(e_j)(y_l)\wedge\tilde{K}_1(e_i)(x_kx_m) & if\ y_l=y_n, x_kx_m\in E_1\\ \tilde{F}_2(e_j)(y_l)\wedge\tilde{F}_2(e_j)(y_n)\wedge\tilde{K}_1(e_i)(x_kx_m) & if\ x_kx_m\in E_1,\ y_ly_n\notin E_2\\ \tilde{F}_1(e_i)(x_k)\wedge\tilde{F}_1(e_i)(x_m)\wedge\tilde{K}_2(e_j)(y_ly_n) & if\ x_kx_m\notin E_1,\ y_ly_n\in E_2\end{cases}$$

Definition 3.2: Let $\tilde{G}_1:(\tilde{F}_1,\tilde{K}_1,A_1)$ and $\tilde{G}_2:(\tilde{F}_2,\tilde{K}_2,A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. The $\beta-product$ $\tilde{G}_1\underset{\beta}{\times}\tilde{G}_2:(\tilde{F}_1\underset{\beta}{\times}\tilde{F}_2,\tilde{K}_1\underset{\beta}{\times}\tilde{K}_2,A_1\times A_2)$ is defined as follows:

$(\tilde{F}_1\underset{\beta}{\times}\tilde{F}_2):A_1\times A_2\rightarrow FS(V_1\times V_2)$ by

$(\tilde{F}_1\underset{\beta}{\times}\tilde{F}_2)(e_i,e_j)(x_ky_l)=\tilde{F}_1(e_i)(x_k)\wedge\tilde{F}_2(e_j)(y_l)\ \forall e_i\in A_1, e_j\in A_2, x_ky_l\in V_1\times V_2$ and

$(\tilde{K}_1\underset{\beta}{\times}\tilde{K}_2):A_1\times A_2\rightarrow FS(E_1\times E_2)$ by

$$(\tilde{K}_1 \underset{\beta}{\times} \tilde{K}_2)(e_i, e_j)(x_k y_l)(x_m y_n) = \begin{cases} \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_1(e_i)(x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & if\ x_k \neq x_m, \\ & y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{F}_2(e_j)(y_n) \wedge \tilde{K}_1(e_i)(x_k x_m) & if\ y_l \neq y_n, \\ & x_k x_m \in E_1 \\ \tilde{K}_1(e_i)(x_k x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & if\ x_k x_m \in E_1, y_l y_n \in E_2 \end{cases}$$

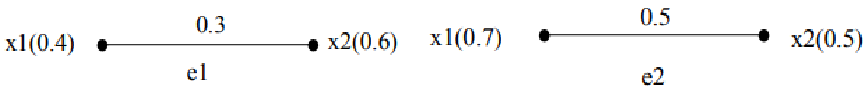
Definition 3.3: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. The $\gamma - product$ $\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2 : (\tilde{F}_1 \underset{\gamma}{\times} \tilde{F}_2, \tilde{K}_1 \underset{\gamma}{\times} \tilde{K}_2, A_1 \times A_2)$ is defined as follows: $(\tilde{F}_1 \underset{\gamma}{\times} \tilde{F}_2) : A_1 \times A_2 \rightarrow FS(V_1 \times V_2)$ by $(\tilde{F}_1 \underset{\gamma}{\times} \tilde{F}_2)(e_i, e_j)(x_k y_l) = \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_2(e_j)(y_l)\ \forall e_i \in A_1, e_j \in A_2, x_k y_l \in V_1 \times V_2$ and $(\tilde{K}_1 \underset{\gamma}{\times} \tilde{K}_2) : A_1 \times A_2 \rightarrow FS(E_1 \times E_2)$ by

$$(\tilde{K}_1 \underset{\gamma}{\times} \tilde{K}_2)(e_i, e_j)(x_k y_l)(x_m y_n) = \begin{cases} \tilde{F}_1(e_i)(x_k) \wedge \tilde{K}_2(e_j)(y_l y_n) & if\ x_k = x_m, y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{K}_1(e_i)(x_k x_m) & if\ y_l = y_n, x_k x_m \in E_1 \\ \tilde{F}_1(e_i)(x_k) \wedge \tilde{F}_1(e_i)(x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & if\ x_k \neq x_m, \\ & y_l y_n \in E_2 \\ \tilde{F}_2(e_j)(y_l) \wedge \tilde{F}_2(e_j)(y_n) \wedge \tilde{K}_1(e_i)(x_k x_m) & if\ y_l \neq y_n, \\ & x_k x_m \in E_1 \\ \tilde{K}_1(e_i)(x_k x_m) \wedge \tilde{K}_2(e_j)(y_l y_n) & if\ x_k x_m \in E_1, y_l y_n \in E_2 \end{cases}$$

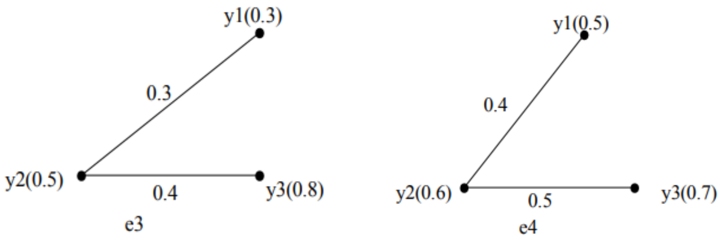
Example 3.4: Consider two fuzzy soft graphs $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively such that $V_1 = \{x_1, x_2\}$, $E_1 = \{x_1 x_2\}$, $V_2 = \{y_1, y_2, y_3\}$, $E_2 = \{y_1 y_2, y_2 y_3\}$, $A_1 = \{e_i\}$ where i=1,2 and $A_2 = \{e_j\}$ where j=3,4. Let $(\tilde{F}_1, A_1), (\tilde{F}_2, A_2), (\tilde{K}_1, A_1)$ and $(\tilde{K}_2, A_2)$ be represented by the following Table 1.

$\tilde{F}_1$	x_1	x_2	$\tilde{F}_2$	y_1	y_2	y_3
e_1	0.4	0.6	e_3	0.3	0.5	0.8
e_2	0.7	0.5	e_4	0.5	0.6	0.7
$\tilde{K}_1$	$x_1 x_2$		$\tilde{K}_2$	$y_1 y_2$		$y_2 y_3$
e_1	0.1		e_3	0.3		0.4
e_2	0.3		e_4	0.4		0.5

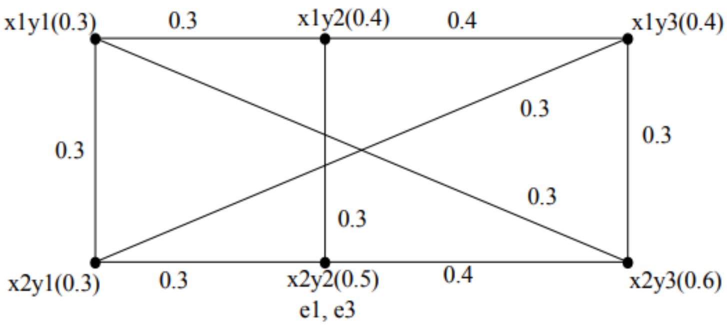
Table 1 : Tabular representation of two fuzzy soft graphs



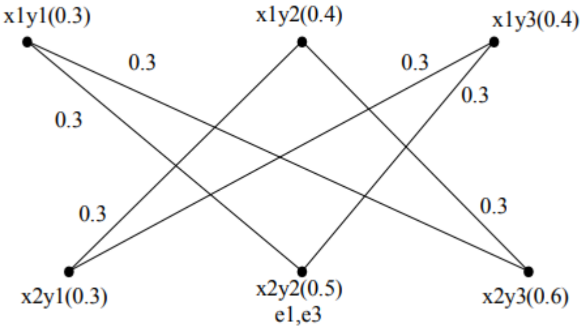
$Fig.1.\ \tilde{G}_1$



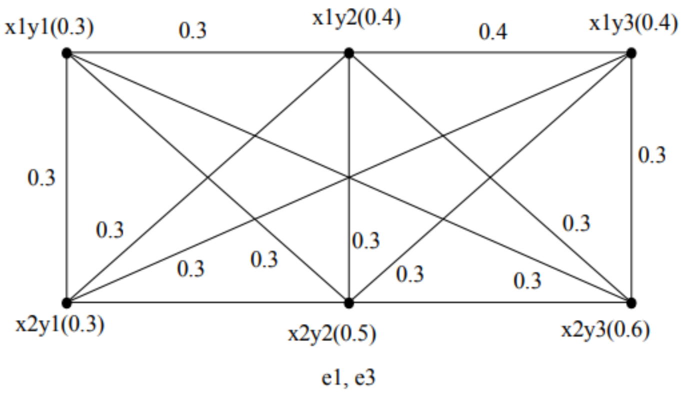
$Fig.2.\ \tilde{G}_2$



$Fig.3.\ \tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$



$Fig.4.\ \tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$



$Fig.5.\ \tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$

4. Degree of a vertex in Alpha product (α – *product*) of two fuzzy soft graphs and its regular properties

Theorem 4.1: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = [1 + d_{G_1^{*c}}(x_1)]|e_i| d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^{*c}}(y_1)]|e_j| d_{\tilde{G}_1}(x_1)$

Proof: $\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \notin E_1 \\ y_1y_2 \in E_2}} \tilde{K}_2(e_j)(y_1y_2)$$

$$= |e_i| d_{\tilde{G}_2}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1)$$

$$= [1 + d_{G_1^{*c}}(x_1)] |e_i| d_{\tilde{G}_2}(y_1) + [1 + d_{G_2^{*c}}(y_1)] |e_j| d_{\tilde{G}_1}(x_1)$$

This is true for any vertex (x_1, y_1) in $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ with $\tilde{F}_1 \geq \tilde{K}_2$ and $\tilde{F}_2 \geq \tilde{K}_1$.

Theorem 4.2 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)$ is a constant function with $\tilde{F}_1(e_i)(x_k) = c \;\; \forall x_k \in V_1$ then

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = c|e_i||e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)]$$

Proof: Given $\tilde{F}_1(e_i) \leq \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$. $\tilde{F}_1(e_i)(x_k) = c \;\; \forall x_k \in V_1$, for any $(x_k, y_l) \in V_1 \times V_2$

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1y_1)(x_2y_2) \in E} (\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1y_1)(x_2y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \notin E_1 \\ y_1y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1x_2 \in E_1}} \tilde{K}_1(e_i)(x_1x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \notin E_2}} \tilde{K}_1(e_i)(x_1x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \notin E_1 \\ y_1y_2 \in E_2}} \tilde{F}_1(e_i)(x_1)$$

$$= |e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) +$$

$$|e_j| d_{G_1^{*c}}(x_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) d_{G_2^*}(y_1)$$

$$= |e_j| d_{G_2^*}(y_1)\, c|e_i| + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_j| d_{G_1^{*c}}(x_1) c|e_i| d_{G_2^*}(y_1)$$

$$= c|e_i||e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)]$$

Theorem 4.3 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)$ is a constant function with $\tilde{F}_2(e_i)(y_l) = m \ \ \forall y_l \in V_2$ then

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = m|e_i||e_j| d_{G_1^*}(x_1)[1 + d_{G_2^{*c}}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x_1)]$$

Proof: *Given* $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$ then $\tilde{F}_1(e_i) \ge \tilde{K}_2(e_j)$

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{F}_2(e_j)(y_1) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2)$$

$$= |e_i| d_{\tilde{G}_2}(y_1) + |e_i| d_{G_1^*}(x_1) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_1) + |e_i| d_{G_2^{*c}}(y_1) \sum_{e_j \in A_2} \tilde{F}_2(e_j)(y_1) d_{G_1^*}(x_1) +$$

$$|e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1)$$

$$= m|e_i||e_j| d_{G_1^*}(x_1)[1 + d_{G_2^{*c}}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x_1)]$$

Theorem 4.4 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on complete graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \ge \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$ then $\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) + |e_j| d_{\tilde{G}_1}(x_1)$

Proof: $\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\in E_1\\ y_1y_2\notin E_2}}\tilde{F}_2(e_j)(y_1)\wedge\tilde{F}_2(e_j)(y_2)\wedge\tilde{K}_1(e_i)(x_1x_2)+$$

$$\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\notin E_1\\ y_1y_2\in E_2}}\tilde{F}_1(e_i)(x_1)\wedge\tilde{F}_1(e_i)(x_2)\wedge\tilde{K}_2(e_j)(y_1y_2)$$

$$=\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1=x_2\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{y_1=y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\in E_1\\ y_1y_2\notin E_2}}\tilde{K}_1(e_i)(x_1x_2)+$$

$$\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\notin E_1\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)$$

$$=[1+d_{G_1^{*c}}(x_1)]|e_i|d_{\tilde{G}_2}(y_1)+[1+d_{G_2^{*c}}(y_1)]|e_j|d_{\tilde{G}_1}(x_1)$$

$=|e_i|\,d_{\tilde{G}_2}(y_1)+|e_j|d_{\tilde{G}_1}(x_1)$ (since G_1^* and G_2^* are complete graphs)

Theorem 4.5 : Let $\tilde{G}_1:(\tilde{F}_1,\tilde{K}_1,A_1)$ and $\tilde{G}_2:(\tilde{F}_2,\tilde{K}_2,A_2)$ be two fuzzy soft graphs on regular graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ respectively. If $\tilde{F}_1(e_i)\geq\tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j)\geq\tilde{K}_1(e_i)$ then $\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2$ is a regular fuzzy soft graph if and only if $\tilde{G}_1$ and $\tilde{G}_2$ are regular fuzzy soft graphs.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular fuzzy soft graphs of degree k_1 and k_2 respectively. For any vertex $(x_1y_1)\in V_1\times V_2$,

$\deg_{\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2}(x_1,y_1)=[1+d_{G_1^{*c}}(x_1)]|e_i|d_{\tilde{G}_2}(y_1)+[1+d_{G_2^{*c}}(y_1)]|e_j|d_{\tilde{G}_1}(x_1)$ (From Theorem 4.1)

$\deg_{\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2}(x_1,y_1)=[1+|E_1^{\,c}|]|e_i|k_2+[1+|E_2^{\,c}|]|e_j|k_1$

This is true $\forall(x_1y_1)\in V_1\times V_2$

Hence, $\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2$ is a regular fuzzy soft graph.

Conversely, Let $\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2$ be a regular fuzzy soft graph. For any two vertices (x_1y_1) and (x_2y_2) in $V_1\times V_2$,

$\deg_{\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2}(x_1,y_1)=\deg_{\tilde{G}_1\underset{\alpha}{\times}\tilde{G}_2}(x_2,y_2)$

$[1+d_{G_1^{*c}}(x_1)]|e_i|d_{\tilde{G}_2}(y_1)+[1+d_{G_2^{*c}}(y_1)]|e_j|d_{\tilde{G}_1}(x_1)$

$=[1+d_{G_1^{*c}}(x_2)]|e_i|d_{\tilde{G}_2}(y_2)+[1+d_{G_2^{*c}}(y_2)]|e_j|d_{\tilde{G}_1}(x_2)$ (From Theorem 4.1)

Fix $x\in V_1$, consider (x_1y_1) and (x_2y_2) in $V_1\times V_2$,

$$[1+d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_1)+[1+d_{G_2^{*c}}(y_1)]|e_j|d_{\tilde{G}_1}(x)$$
$$=[1+d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_2)+[1+d_{G_2^{*c}}(y_2)]|e_j|d_{\tilde{G}_1}(x)$$
$$\Rightarrow [1+d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_1)=[1+d_{G_1^{*c}}(x)]|e_i|d_{\tilde{G}_2}(y_2)$$
$$\Rightarrow d_{\tilde{G}_2}(y_1)=d_{\tilde{G}_2}(y_2)$$

This is true for all vertices of V_2 .$\therefore \tilde{G}_2$ is a regular fuzzy soft graph.

Fix $y \in V_2$, consider $(x_1 y_1)$ and $(x_2 y_2)$ in $V_1 \times V_2$,

$$[1+d_{G_1^{*c}}(x_1)]|e_i|d_{\tilde{G}_2}(y)+[1+d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_1)$$
$$=[1+d_{G_1^{*c}}(x_2)]|e_i|d_{\tilde{G}_2}(y)+[1+d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_2)$$
$$\Rightarrow [1+d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_1)=[1+d_{G_2^{*c}}(y)]|e_j|d_{\tilde{G}_1}(x_2)$$
$$\Rightarrow d_{\tilde{G}_1}(x_1)=d_{\tilde{G}_1}(x_2)$$

This is true for all vertices of V_1. $\therefore \tilde{G}_1$ is a regular fuzzy soft graph.

Theorem 4.6 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on complete graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ is regular if and only if $\tilde{G}_1$ and $\tilde{G}_2$ are regular fuzzy soft graphs.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular fuzzy soft graphs of degree k_1 and k_2 respectively. Let G_1^* and G_2^* are complete graphs.

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = |e_i|d_{\tilde{G}_2}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) \quad \text{(From Theorem 4.4)}$$

This is true $\forall (x_1 y_1) \in V_1 \times V_2$

Hence, $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ is a regular fuzzy soft graph.

Conversely, Let $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ be a regular fuzzy soft graph. For any two vertices $(x_1 y_1)$ and $(x_2 y_2)$ in $V_1 \times V_2$,

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_2, y_2)$$
$$|e_i|d_{\tilde{G}_2}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) = |e_i|d_{\tilde{G}_2}(y_2) + |e_j|d_{\tilde{G}_1}(x_2)$$

Fix $x \in V_1$, consider $(x_1 y_1)$ and $(x_2 y_2)$ in $V_1 \times V_2$,

$$|e_i|d_{\tilde{G}_2}(y_1) + |e_j|d_{\tilde{G}_1}(x) = |e_i|d_{\tilde{G}_2}(y_2) + |e_j|d_{\tilde{G}_1}(x)$$
$$\Rightarrow d_{\tilde{G}_2}(y_1)=d_{\tilde{G}_2}(y_2)$$

This is true for all vertices of V_2 . $\therefore \tilde{G}_2$ is a regular fuzzy soft graph.

Fix $y \in V_2$, consider (x_1y_1) and (x_2y_2) in $V_1 \times V_2$,

$|e_i| d_{\tilde{G}_2}(y) + |e_j| d_{\tilde{G}_1}(x_1) = |e_i| d_{\tilde{G}_2}(y) + |e_j| d_{\tilde{G}_1}(x_2)$

$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$

This is true for all vertices of V_1. $\therefore \tilde{G}_1$ is a regular fuzzy soft graph.

Theorem 4.7 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs on complete graphs G_1^* and G_2^* respectively then $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ becomes a Cartesian product of fuzzy soft graphs.

Proof: By the definition of alpha product of fuzzy soft graphs, for any two vertices (x_1y_1) and (x_2y_2) in $V_1 \times V_2$,

$$(\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1y_1)(x_2y_2) = \begin{cases} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1y_2) \ \ if \ x_1 = x_2, y_1y_2 \in E_2 \\ \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1x_2) \ \ if \ y_1 = y_2, x_1x_2 \in E_1 \\ \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1x_2) \ if \ x_1x_2 \in E_1, \\ \qquad y_1y_2 \notin E_2 \\ \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1y_2) \ if \ x_1x_2 \notin E_1, \\ \qquad y_1y_2 \in E_2 \end{cases}$$

Since G_1^* and G_2^* are complete graphs,

$$(\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1y_1)(x_2y_2) = \begin{cases} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1y_2) \ \ if \ x_1 = x_2, y_1y_2 \in E_2 \\ \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1x_2) \ \ if \ y_1 = y_2, x_1x_2 \in E_1 \end{cases}$$

$$\Rightarrow (\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1y_1)(x_2y_2) = (\tilde{K}_1 \times \tilde{K}_2)(e_i, e_j)(x_1y_1)(x_2y_2)$$

is a Cartesian product of $\tilde{G}_1$ and $\tilde{G}_2$.

Theorem 4.8 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs such that $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$. Let $\tilde{F}_1(e_i)$ be a constant and G_1^* is a complete graph then $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ is regular fuzzy soft graph if and only if $\tilde{G}_1$ is a regular fuzzy soft graph and G_2^* is a regular graph.

Proof: Let $\tilde{F}_1(e_i)(x_k) = c \ \ \forall x_k \in V_1$

Given $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$. Let G_1^* be a complete graph.

From Theorem 4.2,

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = c|e_i||e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)]$$

$$= c|e_i||e_j| d_{G_2^*}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1)$$

Let $\tilde{G}_1$ be a regular fuzzy soft graph with degree m and G_2^* is a regular graph of degree n.

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = c|e_i||e_j|n + |e_j|m + |e_j|d_{G_2^{*c}}(y_1)m$$

$$= |e_i||e_j|cn + |e_j|m\,[1 + |E_2{}^c|]$$

Hence, $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ is regular fuzzy soft graph.

Conversely, assume that $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ is regular fuzzy soft graph.

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_2, y_2)$$

$$c|e_i||e_j|d_{G_2^*}(y_1) + |e_j|d_{\tilde{G}_1}(x_1) + |e_j|d_{G_2^{*c}}(y_1)d_{\tilde{G}_1}(x_1) + |e_j|d_{G_1^{*c}}(x_1)c|e_i|d_{G_2^*}(y_1) =$$

$$c|e_i||e_j|d_{G_2^*}(y_2) + |e_j|d_{\tilde{G}_1}(x_2) + |e_j|d_{G_2^{*c}}(y_2)d_{\tilde{G}_1}(x_2) + |e_j|d_{G_1^{*c}}(x_2)c|e_i|d_{G_2^*}(y_2)$$

Fix $x \in V_1$, consider (xy_1) and (xy_2) in $V_1 \times V_2$,

$$c|e_i||e_j|d_{G_2^*}(y_1) + |e_j|d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_1)] = c|e_i||e_j|d_{G_2^*}(y_2) + |e_j|d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_2)]$$

This is true when degree of all vertices in G_2^* as well as in its complement are equal. Hence, G_2^* is regular.

Fix $y \in V_2$,

$$c|e_i||e_j|d_{G_2^*}(y) + |e_j|d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y)] = c|e_i||e_j|d_{G_2^*}(y) + |e_j|d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y)]$$

$$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$$

Hence, $\tilde{G}_1$ is a regular fuzzy soft graph.

Theorem 4.9 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs such that $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$. If $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$, $\tilde{G}_1$ are regular fuzzy soft graphs, G_1^* and G_2^* are regular graphs then $\sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k)$ is same for all k=1,2,3.....

Proof: Let $\tilde{G}_1$ be a regular fuzzy soft graph of degree m, G_1^* and G_2^* are regular graphs of degree s and n respectively.

Given $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$. Using the definition of $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$,

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\alpha}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \notin E_2}} \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \notin E_1 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2)$$

$$= \left|e_j\right| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + \left|e_j\right| d_{\tilde{G}_1}(x_1) + \left|e_j\right| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + \left|e_j\right| d_{G_1^{*c}}(x_1)\, d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_k)$$

$$= \left|e_j\right| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1)[1 + d_{G_1^{*c}}(x_1)] + \left|e_j\right| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)]$$

Since $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ is a regular fuzzy soft graph,

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_2, y_2)$$

$$\left|e_j\right| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1)[1 + d_{G_1^{*c}}(x_1)] + \left|e_j\right| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)] =$$

$$\left|e_j\right| d_{G_2^*}(y_2) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2)[1 + d_{G_1^{*c}}(x_2)] + \left|e_j\right| d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y_2)]$$

$$\left|e_j\right| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1)[1 + d_{G_1^{*c}}(x_1)] + \left|e_j\right| m[1 + d_{G_2^{*c}}(y_1)] =$$

$$\left|e_j\right| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2)[1 + d_{G_1^{*c}}(x_2)] + \left|e_j\right| m[1 + d_{G_2^{*c}}(y_2)]$$

$\Rightarrow \left|e_j\right| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1)[1 + d_{G_1^{*c}}(x_1)] = \left|e_j\right| n \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2)[1 + d_{G_1^{*c}}(x_2)]$ (Since G_1^{*c} is regular)

$$\Rightarrow \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) = \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2)$$

Theorem 4.10 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs such that $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$. If $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$, $\tilde{G}_2$ are regular fuzzy soft graphs, G_1^* and G_2^* are regular graphs then $\sum_{e_j \in A_{21}} \tilde{F}_1(e_j)(y_l)$ is same for all $l = 1,2,3...$

Proof: Proof is similar to the proof of theorem 4.9.

Theorem 4.11 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs with $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j) = c$ then $\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2$ is a regular FSG if and only if G_1^* and G_2^* is a regular graph and $\tilde{G}_2$ is a regular FSG.

Proof: *Given* $\tilde{F}_2(e_j) = c$, Since $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$, $\tilde{F}_1(e_i) \ge \tilde{K}_2(e_j)$.

Let $\tilde{G}_2$ be a regular FSG of degree m, G_1^* and G_2^* are regular graphs of degree n and p respectively.

From Theorem 4.3,

$\deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_1, y_1) = c|e_i||e_j|d_{G_1^*}(x_1)[1 + d_{G_2^{*c}}(y_1)] + |e_i|d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x_1)]$

$= nc|e_i||e_j|[1 + d_{G_2^{*c}}(y_1)] + m|e_i|[1 + d_{G_1^{*c}}(x_1)]$

$\Rightarrow \tilde{G}_1 \times_\alpha \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \times_\alpha \tilde{G}_2$ be a regular FSG.

$\deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_2, y_2)$

$c|e_i||e_j|d_{G_1^*}(x_1)[1 + d_{G_2^{*c}}(y_1)] + |e_i|d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x_1)] =$

$c|e_i||e_j|d_{G_1^*}(x_2)[1 + d_{G_2^{*c}}(y_2)] + |e_i|d_{\tilde{G}_2}(y_2)[1 + d_{G_1^{*c}}(x_2)]$

Fix $x \in V_1$,

$c|e_i||e_j|d_{G_1^*}(x)[1 + d_{G_2^{*c}}(y_1)] + |e_i|d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x)] =$

$c|e_i||e_j|d_{G_1^*}(x)[1 + d_{G_2^{*c}}(y_2)] + |e_i|d_{\tilde{G}_2}(y_2)[1 + d_{G_1^{*c}}(x)]$

$\Rightarrow d_{\tilde{G}_2}(y_1) = d_{\tilde{G}_2}(y_2)$ and $d_{G_2^{*c}}(y_1) = d_{G_2^{*c}}(y_2)$

$\Rightarrow \tilde{G}_2$ is a regular FSG and G_2^* is regular.

Fix $y \in V_2$,

$c|e_i||e_j|d_{G_1^*}(x_1)[1 + d_{G_2^{*c}}(y)] + |e_i|d_{\tilde{G}_2}(y)[1 + d_{G_1^{*c}}(x_1)] =$

$c|e_i||e_j|d_{G_1^*}(x_2)[1 + d_{G_2^{*c}}(y)] + |e_i|d_{\tilde{G}_2}(y)[1 + d_{G_1^{*c}}(x_2)]$

$\Rightarrow d_{G_1^*}(x_1) = d_{G_1^*}(x_2)$ and $d_{G_1^{*c}}(x_1) = d_{G_1^{*c}}(x_2)$

This is true when G_1^* is regular.

Theorem 4.12 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs where G_1^* is a complete graph. If $\tilde{F}_1(e_i)$ and $\tilde{K}_1(e_i)$ are constant and $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$ then $\tilde{G}_1 \times_\alpha \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is regular and G_2^* is a regular graph.

Proof: Let $\tilde{F}_1(e_i) = \tilde{K}_1(e_i) = c$ and $\tilde{G}_1 \times_\alpha \tilde{G}_2$ be regular.

Given $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$.

$\deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_2, y_2)$

Using the result of theorem 4.9,

$|e_j|d_{G_2^*}(y_1)\sum_{e_i \in A_1}\tilde{F}_1(e_i)(x_1)[1 + d_{G_1^{*c}}(x_1)] + |e_j|d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)] =$

$|e_j| d_{G_2^*}(y_2) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2)[1 + d_{G_1^{*c}}(x_2)] + |e_j| d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y_2)]$

$|e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)] =$

$|e_j| d_{G_2^*}(y_2) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_2) + |e_j| d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y_2)]$ (Since G_1^* is a complete graph)

Given $\tilde{F}_1(e_i) = c$

$c|e_i| d_{G_2^*}(y_1) + d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)] = c|e_i| d_{G_2^*}(y_2) + d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y_2)]$

Fix $x \in V_1$,

$c|e_i| d_{G_2^*}(y_1) + d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_1)] = c|e_i| d_{G_2^*}(y_2) + d_{\tilde{G}_1}(x)[1 + d_{G_2^{*c}}(y_2)]$

$\Rightarrow d_{G_2^*}(y_1) = d_{G_2^*}(y_2)$ and $d_{G_2^{*c}}(y_1) = d_{G_2^{*c}}(y_2)$

This is true only when G_2^* is a regular graph.

Similarly, fix $y \in V_2$,

$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$

$\Rightarrow \tilde{G}_1$ is regular FSG.

Conversely, Let $\tilde{G}_1$ and G_2^* be regular with degree m and n respectively.

$\deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_1, y_1) = |e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1)[1 + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)]$

$= |e_j| n |e_i| c + |e_j| m(1 + p)$

Therefore, $\tilde{G}_1 \times_\alpha \tilde{G}_2$ is a regular FSG.

5. Degree of a vertex in Beta product (β – *product*) of two fuzzy soft graphs and its regular properties

Theorem 5.1: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively.

i) If $\tilde{K}_1(e_i) \le \tilde{K}_2(e_j)$ then $\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$

ii) If $\tilde{K}_2(e_j) \le \tilde{K}_1(e_i)$ then $\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1)$

Proof: $\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \times_\beta \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$

$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \ne x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) +$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \in E_2}} \tilde{K}_1(e_i)(x_1x_2) \wedge \tilde{K}_2(e_j)(y_1y_2)$$

i) For any vertex $(x_1y_1) \in V_1 \times V_2$

Since G_1^* and G_2^* are complete graphs, we have

$$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \in E_2}} \tilde{K}_1(e_i)(x_1x_2) \wedge \tilde{K}_2(e_j)(y_1y_2)$$

Given $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$

$$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \in E_2}} \tilde{K}_1(e_i)(x_1x_2)$$

$$= |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$$

ii) $$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \in E_2}} \tilde{K}_1(e_i)(x_1x_2) \wedge \tilde{K}_2(e_j)(y_1y_2)$$

Since $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$

$$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \in E_2}} \tilde{K}_2(e_j)(y_1y_2)$$

$$= |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1)$$

Theorem 5.2: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then

i) If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1)[d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_1}(y_1)$$

ii) If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1)[d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1)$$

Proof: $$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1y_1)(x_2y_2) \in E} (\tilde{K}_1 \times_\beta \tilde{K}_2)(e_i, e_j)(x_1y_1)(x_2y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1y_2) +$$

$$\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{y_1\neq y_2\\ x_1x_2\in E_1}}\tilde{F}_2(e_j)(y_1)\wedge\tilde{F}_2(e_j)(y_2)\wedge\tilde{K}_1(e_i)(x_1x_2)+$$

$$\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_1(e_i)(x_1x_2)\wedge\tilde{K}_2(e_j)(y_1y_2)$$

i) Given $\tilde{F}_1(e_i)\geq\tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j)\geq\tilde{K}_1(e_i)$. Also $\tilde{K}_1(e_i)\leq\tilde{K}_2(e_j)$

$$\deg_{\tilde{G}_1\underset{\beta}{\times}\tilde{G}_2}(x_1,y_1)=\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{x_1\neq x_2\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{y_1\neq y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)$$

$$+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_1(e_i)(x_1x_2)$$

$$=|e_i|d_{G_1^{*c}}(x_1)d_{\tilde{G}_2}(y_1)+|e_j|[d_{G_2^{*c}}(y_1)d_{\tilde{G}_1}(x_1)+|e_j|d_{\tilde{G}_1}(x_1)d_{G_2^*}(y_1)$$

$$=|e_j|d_{\tilde{G}_1}(x_1)[d_{G_2^{*c}}(y_1)+d_{G_2^*}(y_1)]+|e_i|d_{G_1^{*c}}(x_1)d_{\tilde{G}_1}(y_1)$$

ii) Given $\tilde{F}_1(e_i)\geq\tilde{K}_2(e_j)$, $\tilde{F}_2(e_j)\geq\tilde{K}_1(e_i)$ & $\tilde{K}_2(e_j)\leq\tilde{K}_1(e_i)$

$$\deg_{\tilde{G}_1\underset{\beta}{\times}\tilde{G}_2}(x_1,y_1)=\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{x_1\neq x_2\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{y_1\neq y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)$$

$$+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)$$

$$=|e_i|d_{G_1^{*c}}(x_1)d_{\tilde{G}_2}(y_1)+|e_j|[d_{G_2^{*c}}(y_1)d_{\tilde{G}_1}(x_1)+|e_i|d_{\tilde{G}_2}(y_1)d_{G_1^*}(x_1)$$

$$=|e_i|d_{\tilde{G}_2}(y_1)[d_{G_1^{*c}}(x_1)+d_{G_1^*}(x_1)]+|e_j|d_{G_2^{*c}}(y_1)d_{\tilde{G}_1}(x_1)$$

Theorem 5.3: Let $\tilde{G}_1:(\tilde{F}_1,\tilde{K}_1,A_1)$ and $\tilde{G}_2:(\tilde{F}_2,\tilde{K}_2,A_2)$ be two fuzzy soft graphs on $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ respectively. If $\tilde{F}_1(e_i)\leq\tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)$ is a constant function with $\tilde{F}_1(e_i)(x_k)=c$ $\forall x_k\in V_1$ then $\deg_{\tilde{G}_1\underset{\beta}{\times}\tilde{G}_2}(x_1,y_1)=|e_j|d_{\tilde{G}_1}(x_1)[d_{G_2^{*c}}(y_1)+d_{G_2^*}(y_1)]+c|e_i||e_j|d_{G_1^{*c}}(x_1)d_{G_2^*}(y_1)$

Proof: $\deg_{\tilde{G}_1\underset{\beta}{\times}\tilde{G}_2}(x_1,y_1)=\sum_{(e_i,e_j)\in A}\ \sum_{(x_1y_1)(x_2y_2)\in E}(\tilde{K}_1\underset{\beta}{\times}\tilde{K}_2)(e_i,e_j)(x_1y_1)(x_2y_2)$

Using the given conditions,

$$=\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{x_1\neq x_2\\ y_1y_2\in E_2}}\tilde{F}_1(e_i)(x_1)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{y_1\neq y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\ \sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_1(e_i)(x_1x_2)$$

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = |e_i| \, c \, d_{G_1^{*c}}(x_1) d_{G_2^*}(y_1) |e_j| + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^{*c}}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$$

$$= |e_j| d_{\tilde{G}_1}(x_1)[d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c|e_i||e_j| d_{G_1^{*c}}(x_1) d_{G_2^*}(y_1)$$

Theorem 5.4: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. If $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)$ is a constant function with $\tilde{F}_2(e_j)(y_l) = m$ $\forall y_l \in V_2$ then

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1)[d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + |e_i||e_j| m d_{G_2^{*c}}(y_1) d_{G_1^*}(x_1)$$

Proof: $\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\beta}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$

Using the given conditions,

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \ne x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \ne y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2)$$

$$= |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^{*c}}(x_1) + |e_j| m d_{G_2^{*c}}(y_1) d_{G_1^*}(x_1) |e_i| + |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1)$$

$$= |e_i| d_{\tilde{G}_2}(y_1)[d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + |e_i||e_j| m d_{G_2^{*c}}(y_1) d_{G_1^*}(x_1)$$

Theorem 5.5: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively. $\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ and $\tilde{G}_2$ are regular.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular FSGs with degrees m and n respectively. Let G_1^* and G_2^* be complete graphs with degrees p and q respectively.

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\beta}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

Case (i): If $\tilde{K}_1(e_i) \le \tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$$= |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$$

$$= |e_j| m q$$

$\therefore \tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ is a regular FSG.

Case (i): If $\tilde{K}_2(e_j) \le \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1x_2 \in E_1 \\ y_1y_2 \in E_2}} \tilde{K}_2(e_j)(y_1y_2)$$

$$= |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^*}(x_1)$$

$$= |e_i| n\, p$$

$\therefore \tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ be a regular FSG.

$$\deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \underset{\alpha}{\times} \tilde{G}_2}(x_2, y_2)$$

Case (i): If $\tilde{K}_1(e_i) \le \tilde{K}_2(e_j)$ then

$$|e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1) = |e_j| d_{\tilde{G}_1}(x_2) d_{G_2^*}(y_2)$$

Fix $x \in V_1$,

$$d_{\tilde{G}_1}(x) d_{G_2^*}(y_1) = d_{\tilde{G}_1}(x) d_{G_2^*}(y_2)$$

$$\Rightarrow d_{G_2^*}(y_1) = d_{G_2^*}(y_2)$$

$\therefore G_2^*$ is a regular graph.

Fix $y \in V_2$,

$$d_{\tilde{G}_1}(x_1) d_{G_2^*}(y) = d_{\tilde{G}_1}(x_2) d_{G_2^*}(y)$$

$$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$$

$\therefore \tilde{G}_1$ is regular.

Case (ii): Similarly for $\tilde{K}_2(e_j) \le \tilde{K}_1(e_i)$, we get G_1^* and $\tilde{G}_2$ as regular.

Theorem 5.6: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs and G_1^* is a complete graph and G_2^* is a regular graph. If $\tilde{F}_1(e_i) \ge \tilde{K}_2(e_j), \tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) = \tilde{K}_2(e_j)$ then $\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is a regular FSG.

Proof: Let G_2^* be a regular graph of degree p and G_1^* is a complete graph. Let $\tilde{K}_1(e_i) = \tilde{K}_2(e_j) = c$, $\tilde{F}_1(e_i) \ge \tilde{K}_2(e_j), \tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$.

Let us assume that $\tilde{G}_1$ is a regular FSG of degree m.

Using the definition of degree of a vertex in beta product of FSGs and the above conditions, we get

$$\deg_{\tilde{G}_1 \times_\beta \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$ (Since G_1^* is a complete graph)

$= |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$

$= |e_j| d_{\tilde{G}_1}(x_1)[d_{G_2^*}(y_1) + d_{G_2^{*c}}(y_1)]$

$= |e_j| m[p+s]$

$\therefore \tilde{G}_1 \times_\beta \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \times_\beta \tilde{G}_2$ be a regular FSG, G_2^* is a regular graph of degree p and G_1^* is a complete graph.

$\deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \times_\alpha \tilde{G}_2}(x_2, y_2)$

$\Rightarrow |e_j| d_{\tilde{G}_1}(x_1)[d_{G_2^*}(y_1) + d_{G_2^{*c}}(y_1)] = |e_j| d_{\tilde{G}_1}(x_2)[d_{G_2^*}(y_2) + d_{G_2^{*c}}(y_2)]$

$\Rightarrow |e_j| d_{\tilde{G}_1}(x_1)[p+t] = |e_j| d_{\tilde{G}_1}(x_2)[p+t]$ where $d_{G_2^{*c}}(y) = t$

$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$

$\therefore \tilde{G}_1$ is regular.

Theorem 5.7: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs and G_2^* is a complete graph and G_1^* is a regular graph. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j), \tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ and $\tilde{K}_1(e_i) = \tilde{K}_2(e_j)$ then $\tilde{G}_1 \times_\beta \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is a regular FSG.

Proof: Proof is similar to the proof of Theorem 5.6.

Theorem 5.8: If $\tilde{G}_1$ and $\tilde{G}_2$ are two regular FSGs with $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j), \tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, G_1^* and G_2^* are regular but not complete graphs, then beta product of two FSGs $\tilde{G}_1$ and $\tilde{G}_2$ is a regular FSG.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular FSGs with degrees m and n respectively. Let G_1^* and G_2^* be regular graphs of degrees p and q respectively and suppose that G_1^* and G_2^* are not complete graphs. Let $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j), \tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$.

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\beta}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

Case (i) : Assume that G_1^* and G_2^* are isomorphic graphs. Let $\tilde{K}_1(e_i) = \tilde{K}_2(e_j) = c$

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$$= |e_i| d_{G_1^{*c}}(x_1) d_{\tilde{G}_2}(y_1) + |e_j| d_{G_2^{*c}}(y_1) d_{\tilde{G}_1}(x_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$$

$= |e_j| m\,(q + s) + |e_i| s\, n$ (Since G_1^* and G_2^* are regular graphs of degree p and q and are isomorphic, $d_{G_1^{*c}}(x_1) = d_{G_2^{*c}}(y_1) = s$)

$\therefore \tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ is regular.

Case (ii) : Assume that G_1^* and G_2^* are not isomorphic but are regular graphs of degrees p and q .
By the definition,

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2)$$

If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$,

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$$= |e_j| d_{\tilde{G}_1}(x_1) [d_{G_2^*}(y_1) + d_{G_2^{*c}}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) d_{G_1^{*c}}(x_1)$$

$$= |e_j| m[q + t] + |e_i| n\, s$$

$\therefore \tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ is regular.

If $\tilde{K}_2(e_j) \le \tilde{K}_1(e_i)$,

$$\deg_{\tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \ne x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \ne y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2)$$

$$= |e_i| d_{\tilde{G}_2}(y_1)[d_{G_1^*}(x_1) + d_{G_1^{*c}}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^{*c}}(y_1)$$

$$= |e_i| n\,[p + s] + |e_j| mt$$

$\therefore \tilde{G}_1 \underset{\beta}{\times} \tilde{G}_2$ is regular.

6. Degree of a vertex in Gamma product ($\gamma - product$) of two fuzzy soft graphs and its regular properties

Theorem 6.1: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively and $\tilde{F}_1(e_i) \ge \tilde{K}_2(e_j), \tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$

i) If $\tilde{K}_1(e_i) \le \tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1)$$

ii) If $\tilde{K}_2(e_j) \le \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^*}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1)$$

Proof: Given $\tilde{F}_1(e_i) \ge \tilde{K}_2(e_j), \tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$.For any vertex $(x_1 y_1) \in V_1 \times V_2$,

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\gamma}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) \wedge \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \ne x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_j)(x_1) \wedge \tilde{F}_1(e_i)(x_2) \wedge \tilde{K}_2(e_j)(y_1 y_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \ne y_2 \\ x_1 x_2 \in E_1}} \tilde{F}_2(e_j)(y_1) \wedge \tilde{F}_2(e_j)(y_2) \wedge \tilde{K}_1(e_i)(x_1 x_2)$$

$$\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_1(e_i)(x_1x_2)\wedge\tilde{K}_2(e_j)(y_1y_2)$$

i) Given $\tilde{K}_1(e_i)\le\tilde{K}_2(e_j)$,

$$\deg_{\tilde{G}_1\times_\gamma\tilde{G}_2}(x_1,y_1)=\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1=x_2\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{y_1=y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)+$$

$$\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1\neq x_2\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{y_1\neq y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_1(e_i)(x_1x_2)$$

Since G_1^* and G_2^* are complete graphs,

$$=\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1=x_2\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{y_1=y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_1(e_i)(x_1x_2)$$

$$\deg_{\tilde{G}_1\times_\gamma\tilde{G}_2}(x_1,y_1)=|e_j|d_{\tilde{G}_1}(x_1)+|e_i|d_{\tilde{G}_2}(y_1)+|e_j|d_{\tilde{G}_1}(x_1)d_{G_2^*}(y_1)$$

$$=|e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^*}(y_1)]+|e_i|d_{\tilde{G}_2}(y_1)$$

ii) Given $\tilde{K}_2(e_j)\le\tilde{K}_1(e_i)$

$$\deg_{\tilde{G}_1\times_\gamma\tilde{G}_2}(x_1,y_1)=\sum_{(e_i,e_j)\in A}\sum_{(x_1y_1)(x_2y_2)\in E}(\tilde{K}_1\times_\gamma\tilde{K}_2)(e_i,e_j)(x_1y_1)(x_2y_2)$$

$$=\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{y_1=y_2\\ x_1x_2\in E_1}}\tilde{K}_1(e_i)(x_1x_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1=x_2\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)+\sum_{\substack{e_i\in A_1\\ e_j\in A_2}}\sum_{\substack{x_1x_2\in E_1\\ y_1y_2\in E_2}}\tilde{K}_2(e_j)(y_1y_2)$$

$$\deg_{\tilde{G}_1\times_\gamma\tilde{G}_2}(x_1,y_1)=|e_j|d_{\tilde{G}_1}(x_1)+|e_i|d_{\tilde{G}_2}(y_1)+|e_i|d_{\tilde{G}_2}(y_1)d_{G_1^*}(x_1)$$

$$=|e_i|d_{\tilde{G}_2}(y_1)[1+d_{G_1^*}(x_1)]+|e_j|d_{\tilde{G}_1}(x_1)$$

Theorem 6.2: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on crisp graphs G_1^* and G_2^* respectively and $\tilde{F}_1(e_i)\ge\tilde{K}_2(e_j)$, $\tilde{F}_2(e_j)\ge\tilde{K}_1(e_i)$

i) If $\tilde{K}_1(e_i)\le\tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1\times_\gamma\tilde{G}_2}(x_1,y_1)=|e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^{*c}}(y_1)+d_{G_2^*}(y_1)]+|e_i|d_{\tilde{G}_2}(y_1)[1+d_{G_1^{*c}}(x_1)]$$

ii) If $\tilde{K}_2(e_j)\le\tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1\times_\gamma\tilde{G}_2}(x_1,y_1)=|e_i|d_{\tilde{G}_2}(y_1)[1+d_{G_1^{*c}}(x_1)+d_{G_1^*}(x_1)]+|e_j|d_{\tilde{G}_1}(x_1)[1+d_{G_2^{*c}}(y_1)]$$

Proof: Given $\tilde{F}_1(e_i)\ge\tilde{K}_2(e_j)$, $\tilde{F}_2(e_j)\ge\tilde{K}_1(e_i)$

i) If $\tilde{K}_1(e_i)\le\tilde{K}_2(e_j)$ then

Using the definition of degree of $\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\gamma}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x_1)]$$

ii) If $\tilde{K}_2(e_j) \le \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\gamma}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

$$= \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_2(e_j)(y_1 y_2)$$

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1)]$$

Theorem 6.3: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. If $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)$ is a constant function with $\tilde{F}_1(e_i)(x_k) = c \quad \forall x_k \in V_1$, then

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c|e_i||e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x_1)]$$

Proof: Given $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$ then $\tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$, $\tilde{K}_1(e_i) \le \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)(x_1) = c$

By the definition,

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 = x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_i)(x_1) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 = y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) +$$

$$\sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 \neq x_2 \\ y_1 y_2 \in E_2}} \tilde{F}_1(e_j)(x_1) \wedge \tilde{F}_1(e_i)(x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{y_1 \neq y_2 \\ x_1 x_2 \in E_1}} \tilde{K}_1(e_i)(x_1 x_2) + \sum_{\substack{e_i \in A_1 \\ e_j \in A_2}} \sum_{\substack{x_1 x_2 \in E_1 \\ y_1 y_2 \in E_2}} \tilde{K}_1(e_i)(x_1 x_2)$$

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = |e_j| d_{G_2^*}(y_1) \sum_{e_i \in A_1} \tilde{F}_1(e_i)(x_1) + |e_j| d_{\tilde{G}_1}(x_1) + |e_i| c d_{G_1^{*c}}(x_1) d_{G_2^*}(y_1) |e_j| +$$

$$|e_j| d_{\tilde{G}_1}(x_1) d_{G_2^{*c}}(y_1) + |e_j| d_{\tilde{G}_1}(x_1) d_{G_2^*}(y_1)$$

$$= |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c|e_i||e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x_1)]$$

Theorem 6.4: Let $\tilde{G}_1 : (\tilde{F}_1, \tilde{K}_1, A_1)$ and $\tilde{G}_2 : (\tilde{F}_2, \tilde{K}_2, A_2)$ be two fuzzy soft graphs on G_1^* and G_2^* respectively. If $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)$ is a constant function with $\tilde{F}_2(e_j)(y_l) = c$ $\forall y_l \in V_2$, then $\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = |e_i| d_{\tilde{G}_2}(y_1)[1 + d_{G_1^{*c}}(x_1) + d_{G_1^*}(x_1)] + c|e_i||e_j| d_{G_1^*}(x_1)[1 + d_{G_2^{*c}}(y_1)]$

Proof: Proof is analogues to the proof of Theorem 6.3.

Theorem 6.5 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two fuzzy soft graphs with $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$ and $\tilde{F}_1(e_i)(x_k) = c$ $\forall x_k \in V_1$ then $\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ is regular, G_1^* and G_2^* are regular graphs.

Proof: Given $\tilde{F}_1(e_i)(x_k) = c$, $\tilde{F}_1(e_i) \le \tilde{K}_2(e_j)$, $\tilde{F}_2(e_j) \ge \tilde{K}_1(e_i)$ then $\tilde{K}_1(e_i) \le \tilde{K}_2(e_j)$. Let $\tilde{G}_1$ be a regular FSG of degree m, G_1^* and G_2^* are regular graphs of degree p and q respectively.

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = \sum_{(e_i, e_j) \in A} \sum_{(x_1 y_1)(x_2 y_2) \in E} (\tilde{K}_1 \underset{\gamma}{\times} \tilde{K}_2)(e_i, e_j)(x_1 y_1)(x_2 y_2)$$

Using Theorem 6.3,

$$= |e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c|e_i||e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x_1)]$$

$$= |e_j| m\,[1 + |E_2{}^c| + |E_2|] + c|e_i||e_j| p[1 + |E_1{}^c|]$$

$\therefore \tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ be a regular FSG.

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_2, y_2)$$

$$|e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^{*c}}(y_1) + d_{G_2^*}(y_1)] + c|e_i||e_j| d_{G_2^*}(y_1)[1 + d_{G_1^{*c}}(x_1)] =$$

$$= |e_j| d_{\tilde{G}_1}(x_2)[1 + d_{G_2^{*c}}(y_2) + d_{G_2^*}(y_2)] + c|e_i||e_j| d_{G_2^*}(y_2)[1 + d_{G_1^{*c}}(x_2)]$$

Fix $y \in V_2$,

$$\left|e_j\right|d_{\tilde{G}_1}(x_1)[1+d_{G_2^{*c}}(y)+d_{G_2^*}(y)]+c\left|e_i\right|\left|e_j\right|d_{G_2^*}(y)[1+d_{G_1^{*c}}(x_1)]=$$

$$=\left|e_j\right|d_{\tilde{G}_1}(x_2)[1+d_{G_2^{*c}}(y)+d_{G_2^*}(y)]+c\left|e_i\right|\left|e_j\right|d_{G_2^*}(y)[1+d_{G_1^{*c}}(x_2)]$$

$$\Rightarrow d_{\tilde{G}_1}(x_1)=d_{\tilde{G}_1}(x_2) \text{ and } d_{G_1^{*c}}(x_1)=d_{G_1^{*c}}(x_2)$$

i.e. $\tilde{G}_1$ and G_1^* are regular.

Fix $x \in V_1$,

$$\left|e_j\right|d_{\tilde{G}_1}(x)[1+d_{G_2^{*c}}(y_1)+d_{G_2^*}(y_1)]+c\left|e_i\right|\left|e_j\right|d_{G_2^*}(y_1)[1+d_{G_1^{*c}}(x)]=$$

$$=\left|e_j\right|d_{\tilde{G}_1}(x)[1+d_{G_2^{*c}}(y_2)+d_{G_2^*}(y_2)]+c\left|e_i\right|\left|e_j\right|d_{G_2^*}(y_2)[1+d_{G_1^{*c}}(x)]$$

This holds good when $d_{G_2^*}(y_1)=d_{G_2^*}(y_2)$ and $d_{G_2^{*c}}(y_1)=d_{G_2^{*c}}(y_2)$

$\Rightarrow G_2^*$ is regular.

Theorem 6.6: Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs on complete graphs G_1^* and G_2^* respectively. If $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$ then $\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_1$ and $\tilde{G}_1$ are regular FSGs.

Proof: Let $\tilde{G}_1$ and $\tilde{G}_2$ be regular FSGs of degree m and n respectively. Given $\tilde{F}_1(e_i) \geq \tilde{K}_2(e_j)$ and $\tilde{F}_2(e_j) \geq \tilde{K}_1(e_i)$, G_1^* and G_2^* are complete graphs of degree p and q respectively.

From Theorem 6.1,

i) If $\tilde{K}_1(e_i) \leq \tilde{K}_2(e_j)$ then

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1,y_1)=\left|e_j\right|d_{\tilde{G}_1}(x_1)[1+d_{G_2^*}(y_1)]+\left|e_i\right|d_{\tilde{G}_2}(y_1)$$

$$=\left|e_j\right|m[1+q]+\left|e_i\right|n$$

$\therefore \tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ is a regular FSG.

ii) If $\tilde{K}_2(e_j) \leq \tilde{K}_1(e_i)$ then

$$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1,y_1)=\left|e_i\right|d_{\tilde{G}_2}(y_1)[1+d_{G_1^*}(x_1)]+\left|e_j\right|d_{\tilde{G}_1}(x_1)$$

$$=\left|e_i\right|n[1+p]+\left|e_j\right|m$$

$\therefore \tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ is a regular FSG.

Conversely, Let $\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ be a regular FSG.

$\deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_1, y_1) = \deg_{\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2}(x_2, y_2)$

For $\tilde{K}_1(e_i) \le \tilde{K}_2(e_j)$,

$|e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) = |e_j| d_{\tilde{G}_1}(x_2)[1 + d_{G_2^*}(y_2)] + |e_i| d_{\tilde{G}_2}(y_2)$

Fix $x \in V_1$,

$|e_j| d_{\tilde{G}_1}(x)[1 + d_{G_2^*}(y_1)] + |e_i| d_{\tilde{G}_2}(y_1) = |e_j| d_{\tilde{G}_1}(x)[1 + d_{G_2^*}(y_2)] + |e_i| d_{\tilde{G}_2}(y_2)$

Since G_1^* and G_2^* are complete graphs, we get $d_{\tilde{G}_2}(y_1) = d_{\tilde{G}_2}(y_2)$

$\therefore \tilde{G}_2$ is a regular FSG.

Fix $y \in V_2$,

$|e_j| d_{\tilde{G}_1}(x_1)[1 + d_{G_2^*}(y)] + |e_i| d_{\tilde{G}_2}(y) = |e_j| d_{\tilde{G}_1}(x_2)[1 + d_{G_2^*}(y)] + |e_i| d_{\tilde{G}_2}(y)$

$\Rightarrow d_{\tilde{G}_1}(x_1) = d_{\tilde{G}_1}(x_2)$

$\therefore \tilde{G}_1$ is a regular FSG.

ii) For $\tilde{K}_2(e_j) \le \tilde{K}_1(e_i)$, similar process is followed and we get $\tilde{G}_1$ and $\tilde{G}_2$ as regular FSGs.

Theorem 6.7 : Let $\tilde{G}_1$ and $\tilde{G}_2$ be two FSGs with $\tilde{F}_2(e_j) \le \tilde{K}_1(e_i)$ and $\tilde{F}_2(e_j)(y_l) = c \ \ \forall y_l \in V_2$ then $\tilde{G}_1 \underset{\gamma}{\times} \tilde{G}_2$ is a regular FSG if and only if $\tilde{G}_2$ is regular, G_1^* and G_2^* are regular graphs.

Proof: Proof is similar to the proof of Theorem 6.5.

7 Conclusions

In this paper, different products of fuzzy soft graphs are defined and demonstrated with some examples. The degree of a vertex in these product FSGs under some conditions and the regular properties associated with it are analyzed. When fuzzy soft graphs are very large these formulas will play a major role which helps in studying some of its properties.

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