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Some common fixed point theorems in complex valued fuzzy metric spaces

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Abstract

In this paper, we aim to prove some common fixed point theorems for pairs of any mappings, for pairs of occasionally weakly compatible mappings satisfying some conditions in complex valued fuzzy metric spaces.

Keywords: Complex fuzzy set, complex valued continuous t-norm, complex valued fuzzy metric spaces, occasionally weakly compatible mappings, common fixed point.

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1 Introduction

The concept of fuzzy sets was initiated by L. A. Zadeh [Zadeh, 1965] in 1965. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Since then it has become a vigorous area of research in engineering, medical sciennce, social science, graph theory, metric space theory, complex analysis etc. Deng [Deng, 1982], Erceg [Ercez, 1979], Kaleva and Seikkala [Kaleva and Seikkala, 1984], Kramosil and Michalek [Kramosil and Michalek, 1975] have introduced the concepts of fuzzy metric spaces in different ways. George and Veermani [George and Veeramani, 1994] modified the notion of fuzzy metric spaces with the help of continuous t-norms. The concepts of compatible, weakly compatible, occasionally weakly compatible mappings in fuzzy metric space are present in the paper by C. T. Aage and J. N. Salunke [Aage and Salunke, 2009]. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions in fuzzy metric spaces.

The concept of fuzzy complex numbers and fuzzy complex analysis were first introduced by Buckley (see [Buckley, 1987], [Buckley, 1989], [Buckley, 1991], [Buckley, 1992]). Motivated by the work of Buckley some authers continued research in fuzzy complex numbers. In this series Ramot *et al.* [D. Ramot and Kandel, 2002] extended fuzzy sets to complex fuzzy sets as a generalization. According to Ramot *et al.* [D. Ramot and Kandel, 2002], the complex fuzzy set is characterized by a membership function, whose range is not limited to $[0, 1]$ but extended to the unit cirlce in the complex plane.

Azam *et al.* [A. Azam and Khan, 2011] introduced the notion of complex valued metric space which is a generalization of classical metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces. Additionally it offers numerous research activities in mathematical analysis.

Later, D. Singh *et al.* [D. Singh and Kumam, 2016] defined the notion of complex valued fuzzy metric spaces with the help of complex valued continuous t -norm and also defined the notion of convergent sequence, cauchy sequence in complex valued fuzzy metric spaces.

This paper presents some common fixed point theorems for pairs of any mappings and pairs of occasionally weakly compatible mappings satisfying some conditions in the complex valued fuzzy metric spaces. We also provide some examples which support the main results here.

2 Definitions and Notations

Here and in the following, let $\mathbb{R}, \mathbb{R}_0^+, \mathbb{C}$ and $\mathbb{N}$ be the sets of real numbers, non negative real numbers, complex numbers and positive integers, respectively.

Definition 2.1. *[J. Choi and Islam, 2017] Define a partial order relation $\precsim$ on $\mathbb{C}$ as follows: For $z_1,\ z_2 \in \mathbb{C}$,*

$$z_1 \precsim z_2 \textit{ if and only if } \Re(z_1) \leq \Re(z_2) \textit{ and } \Im(z_1) \leq \Im(z_2).$$

Thus $z_1 \precsim z_2$ if any one of the following statements holds:
(o_1) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$
(o_2) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$
(o_3) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$
(o_4) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$,
where $\Re(z)$ and $\Im(z)$ indicates respectively the real and imaginary parts of the complex number z.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$, i.e., any one of (o_2), (o_3) and (o_4) is satisfied and we write $z_1 \prec z_2$ if only (o_4) is satisfied. Considering (o_1)-(o_4), the following properties for the partial order $\precsim$ on $\mathbb{C}$ hold true:
(i) $0 \precsim z_1 \precsim z_2 \Longrightarrow |z_1| \leq |z_2|$
(ii) $z_1 \precsim z_2$ and $z_2 \precsim z_3 \Longrightarrow z_1 \precsim z_3$
(iii) $z_1 \precsim z_2$ and $\lambda > 0$ $(\lambda \in \mathbb{R}) \Longrightarrow \lambda\, z_1 \precsim \lambda\, z_2$.

Note: $z_1 \precsim z_2$ and $z_2 \succsim z_1$ have the same meaning.

Definition 2.2. *[J. Choi and Islam, 2017] The* $\max$ *function for the partial order $\precsim$ on $\mathbb{C}$ is defined as follows:*
(i) $\max\{u, v\} = v \Leftrightarrow u \precsim v$
(ii) $u \precsim \max\{u, v\} \Rightarrow u \precsim v$
(iii) $u \precsim \max\{v, w\} \Rightarrow u \precsim v$ *or* $u \precsim w$.

For any $0 \precsim u$, $0 \precsim v$, we can easily prove that $|\max\{u, v\}| = \max\{|u|, |v|\}$, where $|.|$ denotes the usual modulus of complex number.

Definition 2.3. *[J. Choi and Islam, 2017] The* $\min$ *function for the partial order $\precsim$ on $\mathbb{C}$ is defined as follows:*
(i) $\min\{u, v\} = u \Leftrightarrow u \precsim v$
(ii) $\min\{u, v\} \precsim v \Rightarrow u \precsim v$
(iii) $\min\{u, w\} \precsim v \Rightarrow u \precsim v$ *or* $w \precsim v$.

For any $0 \precsim u$, $0 \precsim v$, we can easily prove that $|\min\{u, v\}| = \min\{|u|, |v|\}$.

Definition 2.4. *[Zadeh, 1965] A fuzzy set on a non empty set X is just a function* $\mu : X \to [0, 1]$.

Definition 2.5. *[D. Ramot and Kandel, 2002] A complex fuzzy set S, defined on a universe discourse U, is characterized by a membership function $\mu_s(x)$ that assigns every element $x \in U$, a complex valued grade of membership in S. The values $\mu_s(x)$ lie within the unit circle in the complex plane and are thus of the form $\mu_s(x) = r_s(x)e^{iw_s(x)}, (i = \sqrt{-1})$, where $r_s(x)$ and $w_s(x)$ both real valued, with $r_s(x) \in [0,1]$. The complex fuzzy set S may be represented as the set of ordered pairs given by $S = \{(x, \mu_s(x)) : x \in U\}$.*

Definition 2.6. *[D. Singh and Kumam, 2016] A binary operation $* : r_s e^{i\theta} \times r_s e^{i\theta} \to r_s e^{i\theta}$, where $r_s \in [0,1]$ and a fix $\theta \in [0, \frac{\pi}{2}]$, is called complex valued continuous t-norm if it satisfies the followings:*

$(t1)$ $*$ *is associative and commutative,*
$(t2)$ $*$ *is continuous,*
$(t3)$ $a * e^{i\theta} = a, \forall a \in r_s e^{i\theta}$,
$(t4)$ $a * b \precsim c * d$ *whenever* $a \precsim c$ *and* $b \precsim d$, $\forall\, a, b, c, d \in r_s e^{i\theta}$.

Example 2.1. *[D. Singh and Kumam, 2016] The followings are examples for complex valued continuous t-norm: (i) $a * b = \min(a, b)$, $\forall a, b \in r_s e^{i\theta}$ and a fix $\theta \in [0, \frac{\pi}{2}]$ (ii) $a * b = \max(a + b - e^{i\theta}, 0)$, $\forall a, b \in r_s e^{i\theta}$ and a fix $\theta \in [0, \frac{\pi}{2}]$.*

Definition 2.7. *[D. Singh and Kumam, 2016, Demir, 2021] The triplet $(X, M, *)$ is said to be complex valued fuzzy metric space if X is an arbitrary non empty set, $*$ is a complex valued continuous t-norm and $M : X \times X \times (0, \infty) \to r_s e^{i\theta}$ is a complex valued fuzzy set, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$, satisfying the following conditions:*

(cf1) $0 \prec M(x, y, t)$,
(cf2) $M(x, y, t) = e^{i\theta}, \forall t \in (0, \infty)$ *if and only if* $x = y$,

(cf3) $M(x, y, t) = M(y, x, t)$,
(cf4) $M(x, y, t + s) \succsim M(x, z, t) * M(z, y, s)$,
(cf5) $M(x, y, t) : (0, \infty) \to r_s e^{i\theta}$ *is continuous,*

for all $x, y, z \in X$; $t, s \in (0, \infty)$; $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. The pair $(M, *)$ is called a complex valued fuzzy metric and $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t. It is noted that if we take $\theta = 0$, then complex valued fuzzy metric simply goes to real valued fuzzy metric.

Note: It is clear that $r_s e^{i\theta} \precsim e^{i\theta}$ and consequently, $M(x, y, t) \precsim e^{i\theta}$ for all $x, y \in X$, $t \in (0, \infty)$, $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$.

Example 2.2. *[D. Singh and Kumam, 2016] Let (X, d) be a real valued metric space. Let $a * b = min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$.*

For each $t > 0$, $x, y \in X$, we define

$$M(x,y,t) = e^{i\theta}\frac{kt^n}{kt^n + md(x,y)},$$

where $k, m, n \in \mathbb{N}$. Then $(X, M, *)$ is a complex valued fuzzy metric space. By choosing $k = m = n = 1$, we get

$$M(x,y,t) = e^{i\theta}\frac{t}{t + d(x,y)}.$$

This complex valued fuzzy metric space induced by a metric d is referred to as a standard complex valued fuzzy metric space.

Example 2.3. *[D. Singh and Kumam, 2016] Let $X = \mathbb{R}$. Let $a * b = min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$, $x, y \in X$, we define*

$$M(x,y,t) = e^{i\theta}e^{-\frac{|x-y|}{t}}.$$

Then $(X, M, *)$ is a complex valued fuzzy metric space.

Definition 2.8. *[D. Singh and Kumam, 2016] Let $(X, M, *)$ be a complex valued fuzzy metric space. We define an open ball $B(x, r, t)$ with centre $x \in X$ and radius $r \in \mathbb{C}$ with $0 \prec r \prec e^{i\theta}, t > 0$ as: $B(x, r, t) = \{y \in X : M(x, y, t) \succ e^{i\theta} - r\}$, where $\theta \in [0, \frac{\pi}{2}]$. A point $x \in X$ is said to be interior point of a set $A \subset X$, whenever there exists $r \in \mathbb{C}$ with $0 \prec r \prec e^{i\theta}$ such that $B(x, r, t) = \{y \in X : M(x, y, t) \succ e^{i\theta} - r\} \subset A$, where $\theta \in [0, \frac{\pi}{2}]$. A subset A of X is called open if every element of A is an interior point of A. If we define $\tau = \{A \subset X : x \in A$ if and only if there exists $t > 0$ and $r \in \mathbb{C}$, $0 \prec r \prec e^{i\theta}, \theta \in [0, \frac{\pi}{2}]$ such that $B(x, r, t) \subset A$ $\}$. Then one can easily check that τ is a topology on X.*

Definition 2.9. *[D. Singh and Kumam, 2016] Let $(X, M, *)$ be complex valued fuzzy metric space and τ be the topology induced by complex valued fuzzy metric. Let $\{x_n\}$ be any sequence in X. The sequence $\{x_n\}$ is said to converges to $x \in X$ if and only if for any $t > 0$, $M(x_n, x, t) \to e^{i\theta}$ as $n \to \infty$ or $|M(x_n, x, t)| \to 1$ as $n \to \infty$. A sequence $\{x_n\}$ in a complex valued fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if for any $t > 0$, $M(x_m, x_n, t) \to e^{i\theta}$ as $m, n \to \infty$ or $|M(x_m, x_n, t)| \to 1$ as $m, n \to \infty$. A complex valued fuzzy metric space in which every cauchy sequece is convergent is called complex valued complete fuzzy metric space.*

For example (see [D. Singh and Kumam, 2016]), let $X = \mathbb{R}$ and $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define $M(x, y, t) = \frac{te^{i\theta}}{t+|x-y|}$. Then $(X, M, *)$ is complex valued complete fuzzy metric space.

Definition 2.10. *Let $(X, M, *)$ be a complex valued fuzzy metric space and S, T : $X \to X$ be two mappings. A point $x \in X$ is said to be a coincidence point of S and T if and only if $Sx = Tx$. We shall call $w = Sx = Tx$ a point of coincidence of S and T. Moreover, if $Sx = Tx = x$, then the point $x \in X$ is called common fixed point of S and T.*

By study of the paper by C. T. Aage and J. N. Salunke (see [Aage and Salunke, 2009]), we find the definitions of compatible, weakly compatible, occasionally weakly compatible mappings in fuzzy metric space. In the similar way, we can define the same definitions in complex valued fuzzy metric space as follows:

Definition 2.11. *Let $(X, M, *)$ be a complex valued fuzzy metric space and S, T : $X \to X$ be two self mappings. The self maps S and T on X are said to be commuting if $STx = TSx$, for all $x \in X$. The self maps S and T are said to be compatible if*

$$\lim_{n\to\infty} M(STx_n, TSx_n, t) = e^{i\theta}, t > 0 \textit{ or } \lim_{n\to\infty} |M(STx_n, TSx_n, t)| = 1, t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$, for some $x \in X$.

Definition 2.12. *Let $(X, M, *)$ be a complex valued fuzzy metric space and S, T : $X \to X$ be two mappings. The self-maps S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, that is, they commute at their coincidence points.*

Definition 2.13. *Let $(X, M, *)$ be a complex valued fuzzy metric space and S, T : $X \to X$ be two mappings. The self-maps S and T are said to be occasionally weakly compatible if and only if there is a coincidence point x (say) in X of S and T at which S and T commute, i.e., $STx = TSx$.*

It is needed to mention that every weakly compatible map is occasionally weakly compatible but converse is not always true. The supporting example is given below:

Example 2.4. *Let $\mathbb{R}$ be the set of real numbers with standard complex valued fuzzy metric space. Define $S, T : \mathbb{R} \to \mathbb{R}$ by $Sx = x^2 + x$ and $Tx = 2x$ for all $x \in \mathbb{R}$. Then $Sx = Tx$ for $x = 0, 1$ but $ST0 = TS0$ and $ST1 \neq TS1$. Therefore S and T are occasionally weakly compatible self maps but not weakly compatible.*

3 Lemmas

Here we set lemmas which will be required in the sequel.

Lemma 3.1. *[Jungck and Rhoades, 2006] Let X be any set and S, T be occasionally weakly compatible self maps of X. If S and T have a unique point of coincidence $w = Sx = Tx$, then w is the unique common fixed point of S and T.*

D. Singh *et al.* [D. Singh and Kumam, 2016] state and prove the following lemma for complex valued complete fuzzy metric space, but it is also true without the restriction of completeness. So we state the lemma without the restriction of completeness.

Lemma 3.2. *Let $(X, M, *)$ be a complex valued fuzzy metric space such that*

$$\lim_{t\to\infty} M(x, y, t) = e^{i\theta}, \text{for all } x, y \in X.$$

If $M(x, y, kt) \succsim M(x, y, t)$, for some $0 < k < 1$, for all $x, y \in X$, $t \in (0, \infty)$, then $x = y$.

Lemma 3.3. *[D. Singh and Kumam, 2016] Let $\{x_n\}$ be a sequence in a complex valued fuzzy metric space $(X, M, *)$ with*

$$\lim_{t\to\infty} M(x, y, t) = e^{i\theta}, \text{for all } x, y \in X.$$

If there exists $k \in (0, 1)$ such that $M(x_{n+1}, x_{n+2}, kt) \succsim M(x_n, x_{n+1}, t)$, for all $t > 0$ and $n = 0, 1, 2, 3, ...$ Then $\{x_n\}$ is a Cauchy sequence in X.

We can easily prove the following lemma.

Lemma 3.4. *Let $(X, M, *)$ be a complex valued fuzzy metric space. Let $\{x_n\}$ be a sequence in X converging to $x \in X$. Then any subsequence of $\{x_n\}$ converges to the same point $x \in X$.*

4 Main Results

Here we present some fixed point theorems on complex valued fuzzy metric spaces.

Theorem 4.1. *Let $(X, M, *)$ be a complex valued complete fuzzy metric space with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X, t > 0$ and let S and T be self-mappings on X. If there exists $k \in (0, 1)$ such that*

$$M(Sx, Ty, kt) \succsim M(x, y, t), \tag{1}$$

for all $x, y \in X$ and for all $t > 0$, then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and we define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}; n = 0, 1, 2, 3, ...$$

Now, for $k \in (0, 1)$ and for all $t > 0$, we have by the condition (1)

$$\begin{aligned} M(x_{2n+1}, x_{2n+2}, kt) &= M(Sx_{2n}\ , Tx_{2n+1}, kt) \\ &\succsim M(x_{2n}\ , x_{2n+1}, t). \end{aligned}$$

and

$$\begin{aligned} M(x_{2n}, x_{2n+1}, kt) &= M(Sx_{2n-1}\ , Tx_{2n}, kt) \\ &\succsim M(x_{2n-1}\ , x_{2n}, t). \end{aligned}$$

In general, we have $M(x_{n+1}, x_{n+2}, kt) \succsim M(x_n, x_{n+1}, t)$, for all $t > 0$ and $k \in (0, 1); n = 0, 1, 2, 3, ...$ By Lemma 3.3, $\{x_n\}$ is a cauchy sequence in X. Since X is complete, then there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Obviously $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ in X, by Lemma 3.4, they are also converge to the same point $u \in X$, i.e., $x_{2n} \to u$ and $x_{2n+1} \to u$ as $n \to \infty$. Now, we claim that u is the common fixed point of S and T.
For this, using condition (1), we have

$$\begin{aligned} M(Su, u, kt) &= M(Su, u, \frac{kt}{2} + \frac{kt}{2}) \\ &\succsim M(Su, x_{2n+2}, \frac{kt}{2}) * M(x_{2n+2}, u, \frac{kt}{2}) \\ &= M(Su, Tx_{2n+1}, \frac{kt}{2}) * M(x_{2n+2}, u, \frac{kt}{2}) \\ &\succsim M(u, x_{2n+1}, \frac{t}{2}) * M(x_{2n+2}, u, \frac{kt}{2}). \end{aligned}$$

Taking limit as $n \to \infty$, we get, $M(Su, u, kt) \succsim e^{i\theta} * e^{i\theta} = e^{i\theta}$. So $Su = u$.
Again,

$$\begin{aligned} M(u, Tu, kt) &= M(u, Tu, \frac{kt}{2} + \frac{kt}{2}) \\ &\succsim M(u, x_{2n+1}, \frac{kt}{2}) * M(x_{2n+1}, Tu, \frac{kt}{2}) \\ &= M(u, x_{2n+1}, \frac{kt}{2}) * M(Sx_{2n}, Tu, \frac{kt}{2}) \\ &\succsim M(u, x_{2n+1}, \frac{kt}{2}) * M(x_{2n}, u, \frac{t}{2}). \end{aligned}$$

Taking limit as $n \to \infty$, we get $M(u, Tu, kt) \succsim e^{i\theta} * e^{i\theta} = e^{i\theta}$. Therefore $Tu = u$.

Thus $Su = Tu = u$ and therefore u is a common fixed point of S and T.
For uniqueness, let z be another fixed point of S and T.
Now, using condition (1),

$$\begin{aligned} M(u, z, kt) &= M(Su, Tz, kt). \\ &\succsim M(u, z, t). \end{aligned}$$

By Lemma 3.2, $u = z$. This completes the theorem. □

If we consider $S = T$ in Theorem 4.1, we get the following corollary.

Corollary 4.1. *Let $(X, M, *)$ be a complex valued complete fuzzy metric space with $\lim\limits_{t\to\infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$, $t > 0$ and let S be self-mapping on X. If there exists $k \in (0, 1)$ such that*

$$M(Sx, Sy, kt) \succsim M(x, y, t),$$

for all $x, y \in X$ and for all $t > 0$, then S has a unique fixed point in X.

A supporting example to Corollary 4.1 is given below.

Example 4.1. *Let $X = \mathbb{R}$ and $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define*

$$M(x, y, t) = \frac{te^{i\theta}}{t + |x - y|}.$$

Then certainly $(X, M, *)$ is complex valued complete fuzzy metric space with

$$\lim_{t\to\infty} M(x, y, t) = e^{i\theta}.$$

We define the self map S on X by

$$Sx = \frac{x + 1}{2}, \text{for all } x \in X.$$

Now, for any $t > 0$ and for $k = \frac{1}{2}$,

$$\begin{aligned} M(Sx, Sy, \frac{t}{2}) &= \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2} + |Sx - Sy|}. \\ &= \frac{te^{i\theta}}{t + 2\left|\frac{x+1}{2} - \frac{y+1}{2}\right|} = \frac{te^{i\theta}}{t + |x - y|} = M(x, y, t). \end{aligned}$$

Thus all the conditions of Corollary 4.1 are satisfied and $x = 1$ is the unique fixed point of S.

Theorem 4.2. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$ and let A, B, S and T be self-mappings on X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$\begin{aligned} M(Ax, By, kt) &\succsim \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\}, \end{aligned} \tag{2}$$

for all $x, y \in X$ and for all $t > 0$, then A, B, S and T have a unique common fixed point in X.

Proof. Since the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$.
Now, by the given condition (2), we get

$$\begin{aligned} M(Ax, By, kt) &\succsim \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\} \\ &= \min\{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), \\ &\quad M(Ax, By, t), M(By, Ax, t)\} \\ &= \min\{M(Ax, By, t), e^{i\theta}, e^{i\theta}, M(Ax, By, t), M(By, Ax, t)\} \\ &= M(Ax, By, t). \end{aligned}$$

In view of Lemma 3.2, we have $Ax = By$ and therefore

$$Ax = Sx = By = Ty. \tag{3}$$

Suppose that the pair $\{A, S\}$ have an another coincidence point $z \in X$, i.e., $Az = Sz$.
Now,

$$\begin{aligned} M(Az, By, kt) &\succsim \min\{M(Sz, Ty, t), M(Sz, Az, t), M(By, Ty, t), \\ &\quad M(Az, Ty, t), M(By, Sz, t)\} \\ &= \min\{M(Az, By, t), M(Az, Az, t), M(By, By, t), \\ &\quad M(Az, By, t), M(By, Az, t)\} \\ &= \min\{M(Az, By, t), e^{i\theta}, e^{i\theta}, M(Az, By, t), M(By, Az, t)\} \\ &= M(Az, By, t). \end{aligned}$$

Again, in view of Lemma 3.2, we have $Az = By$. Therefore

$$Az = Sz = By = Ty. \tag{4}$$

From (3) and (4), $Ax = Az$ and therefore the pair $\{A, S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus by Lemma 3.1, w is the unique common fixed point of $\{A, S\}$. Similarly, we can show that the pair $\{B, T\}$ have also a unique common fixed point. Suppose this is $u \in X$.
Now,

$$\begin{aligned} M(w,u,kt) &= M(Aw,Bu,kt) \\ &\succsim \min\{M(Sw,Tu,t), M(Sw,Aw,t), M(Bu,Tu,t), \\ &\quad M(Aw,Tu,t), M(Bu,Sw,t)\} \\ &= \min\{M(w,u,t), M(w,w,t), M(u,u,t), \\ &\quad M(w,u,t), M(u,w,t)\} \\ &= \min\{M(w,u,t), e^{i\theta}, e^{i\theta}, M(w,u,t), M(u,w,t)\} \\ &= M(w,u,t). \end{aligned}$$

Therefore using Lemma 3.2, we have $w = u$ and consequently, w is common fixed point of A, B, S and T.
To show, this common fixed point is unique, let v is an another common fixed point of A, B, S and T.
Now,

$$\begin{aligned} M(w,v,kt) &= M(Aw,Bv,kt) \\ &\succsim \min\{M(Sw,Tv,t), M(Sw,Aw,t), M(Bv,Tv,t), \\ &\quad M(Aw,Tv,t), M(Bv,Sw,t)\} \\ &= \min\{M(w,v,t), M(w,w,t), M(v,v,t), \\ &\quad M(w,v,t), M(v,w,t)\} \\ &= \min\{M(w,v,t), e^{i\theta}, e^{i\theta}, M(w,v,t), M(v,w,t)\} \\ &= M(w,v,t). \end{aligned}$$

By Lemma 3.2, we have $w = v$. Hence A, B, S and T have a unique common fixed point. □

In the following an supporting example to Theorem 4.2 is given.

Example 4.2. *Let $X = \mathbb{R}$. Consider the metric $d(x,y) = \mid x \mid + \mid y \mid$, for all $x \neq y$ and $d(x,y) = 0$, for $x = y$ on X. Let $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define*

$$M(x,y,t) = \frac{te^{i\theta}}{t + d(x,y)}.$$

Then certainly $(X, M, *)$ *is complex valued fuzzy metric space with*

$$\lim_{t\to\infty} M(x, y, t) = e^{i\theta}.$$

Now, we define the self maps A, B, S *and* T *on* X *by*

$$Ax = \frac{x}{3}, Bx = \frac{x}{4}, Sx = x \text{ and } Tx = \frac{x}{2}, \text{for all } x \in X.$$

Set $k = \frac{1}{2}$*. Now, for* $x \neq y$*,*

$$\begin{aligned} M(Ax, By, \frac{t}{2}) &= \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2}+ \mid Ax \mid + \mid By \mid} \\ &= \frac{te^{i\theta}}{t + 2\frac{|x|}{3} + 2\frac{|y|}{4}} \\ &\succsim \frac{te^{i\theta}}{t + 2\frac{|x|}{2} + 2\frac{|y|}{4}} \\ &= \frac{te^{i\theta}}{t + |x| + \frac{|y|}{2}} \\ &= \frac{te^{i\theta}}{t + |Sx| + |Ty|} = M(Sx, Ty, t). \end{aligned}$$

For the case $x = y$*,*

$$M(Ax, By, \frac{t}{2}) = \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2} + 0} = e^{i\theta} = M(Sx, Ty, t).$$

Therefore, for any $x, y \in X$*,*

$$\begin{aligned} M(Ax, By, \frac{t}{2}) &\succsim M(Sx, Ty, t) \\ &= \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\}. \end{aligned}$$

Therefore, the maps A, B, S *and* T *satisfies the condition (2) of Theorem* 4.2 *for* $k = \frac{1}{2}$*. Also the pairs* $\{A, S\}$ *and* $\{B, T\}$ *are obviously occasionally weakly compatible. Thus all the conditions of Theorem* 4.2 *are satisfied and* $x = 0$ *is the unique common fixed point of* A, B, S *and* T *in* X*.*

Setting $A = B$ and $S = T$ in Theorem 4.2, we get the following corollary.

Corollary 4.2. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim\limits_{t\to\infty} M(x,y,t) = e^{i\theta}$, for all $x, y \in X$ and let A, S be self-mappings on X. Let the pair $\{A, S\}$ be occasionally weakly compatible. If there exists $k \in (0,1)$ such that*

$$\begin{aligned} M(Ax, Ay, kt) &\succsim \min\{M(Ax, Sy, t), M(Ay, Sx, t), M(Sx, Sy, t), \\ &\quad M(Sx, Ax, t), M(Sy, Ay, t)\}, \end{aligned}$$

for all $x, y \in X$ and for all $t > 0$, then A and S have a unique common fixed point in X.

Theorem 4.3. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim\limits_{t\to\infty} M(x,y,t) = e^{i\theta}$, for all $x, y \in X$ and let A, B, S and T be self-mappings on X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0,1)$ such that*

$$\begin{aligned} M(Ax, By, kt) &\succsim g(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\}), \end{aligned}$$

for all $x, y \in X$ and for all $t > 0$, where $g : r_s e^{i\theta} \to r_s e^{i\theta}$ with $g(x) \succsim x$ for all $x \in r_s e^{i\theta}$, where $r_s \in (0,1)$ and $\theta \in [0, \frac{\pi}{2}]$. Then A, B, S and T have a unique common fixed point in X.

Proof. The proof follows from Theorem 4.2. □

Theorem 4.4. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim\limits_{t\to\infty} M(x,y,t) = e^{i\theta}$, for all $x, y \in X$, and let A, B, S and T be self-mappings on X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0,1)$ such that*

$$M(Ax, By, kt) \succsim M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t), \tag{5}$$

for all $x, y \in X$ and for all $t > 0$. Then A, B, S and T have a unique common fixed point in X.

Proof. The pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$.
Now, the condition (5) gives

$$\begin{aligned} M(Ax, By, kt) &\succsim M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ &\quad * M(Ax, Ty, t) \\ &= M(Ax, By, t) * M(Ax, Ax, t) * M(By, By, t) \\ &\quad * M(Ax, By, t) \\ &= M(Ax, By, t) * e^{i\theta} * e^{i\theta} * M(Ax, By, t) \\ &= M(Ax, By, t). \end{aligned}$$

In view of Lemma 3.2, we have $Ax = By$ and therefore

$$Ax = Sx = By = Ty. \tag{6}$$

Suppose that the pair $\{A, S\}$ have an another coincidence point z (say)$\in X$, i.e., $Az = Sz$.
Now,

$$\begin{aligned} M(Az, By, kt) &\succsim M(Sz, Ty, t) * M(Az, Sz, t) * M(By, Ty, t) \\ &\quad * M(Az, Ty, t) \\ &= M(Az, By, t) * M(Az, Az, t) * M(By, By, t) \\ &\quad * M(Az, By, t) \\ &= M(Az, By, t) * e^{i\theta} * e^{i\theta} * M(Az, By, t) \\ &= M(Az, By, t). \end{aligned}$$

Using Lemma 3.2, $Az = By$ and consequently

$$Az = Sz = By = Ty. \tag{7}$$

From (6) and (7) we get, $Ax = Az$ and therefore the pair $\{A, S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus by Lemma 3.1, w is the unique common fixed point of $\{A, S\}$. Similarly, we can show that there is a unique common fixed point u (say)$\in X$ of $\{B, T\}$.
Now,

$$\begin{aligned} M(w, u, kt) &= M(Aw, Bu, kt) \\ &\succsim M(Sw, Tu, t) * M(Aw, Sw, t) * M(Bu, Tu, t) \\ &\quad * M(Aw, Tu, t) \\ &= M(Aw, Bu, t) * M(Aw, Aw, t) * M(Bu, Bu, t) \\ &\quad * M(Aw, Bu, t) \\ &= M(Aw, Bu, t) * e^{i\theta} * e^{i\theta} * M(Aw, Bu, t) \\ &= M(Aw, Bu, t) \\ &= M(w, u, t). \end{aligned}$$

By Lemma 3.2, we have $w = u$ and consequently w is common fixed point of A, B, S and T. For uniqueness, let v is an another common fixed point of A, B, S and T.

Therefore

$$\begin{aligned} M(w,v,kt) &= M(Aw,Bv,kt) \\ &\succsim M(Sw,Tv,t) * M(Aw,Sw,t) * M(Bv,Tv,t) \\ &\quad * M(Aw,Tv,t) \\ &= M(w,v,t) * M(w,w,t) * M(v,v,t) * \\ &\quad M(w,v,t) \\ &= M(w,v,t) * e^{i\theta} * e^{i\theta} * M(w,v,t). \\ &= M(w,v,t) \end{aligned}$$

and in view of Lemma 3.2, we have $w = v$. Hence A, B, S and T have a unique common fixed point. □

A supporting example to Corollary 4.4 is given below.

Example 4.3. *Let $X = \mathbb{R}$. Define the metric $(x, y) =\mid x \mid + \mid y \mid$, for all $x \neq y$ and $d(x, y) = 0$, for $x = y$ on X. Let $a * b = \min\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in [0, \frac{\pi}{2}]$. For each $t > 0$ and $x, y \in X$, we define*

$$M(x,y,t) = e^{i\theta} e^{-\frac{d(x,y)}{t}}.$$

Then certainly $(X, M, *)$ is complex valued fuzzy metric space with

$$\lim_{t\to\infty} M(x,y,t) = e^{i\theta}.$$

Now, we define the self maps A, B, S and T on X by

$$Ax = \frac{x}{5}, Bx = \frac{x}{15}, Sx = x \text{ and } Tx = \frac{x}{4}, \text{for all } x \in X.$$

Let $k = \frac{1}{3}$. Now for $x \neq y$,

$$\begin{aligned} M(Ax,By,\frac{t}{3}) &= \frac{\frac{t}{3}e^{i\theta}}{\frac{t}{3} + \mid Ax \mid + \mid By \mid} \\ &= \frac{te^{i\theta}}{t + 3\frac{|x|}{5} + 3\frac{|y|}{15}} \\ &\succsim \frac{te^{i\theta}}{t + |x| + \frac{|y|}{4}} \\ &= \frac{te^{i\theta}}{t + |Sx| + |Ty|} = M(Sx,Ty,t). \end{aligned}$$

For $x = y$,

$$M(Ax, By, \frac{t}{3}) = \frac{\frac{t}{3}e^{i\theta}}{\frac{t}{3} + 0} = e^{i\theta} = M(Sx, Ty, t).$$

Thus for any $x, y \in X$,

$$\begin{aligned} M(Ax, By, \frac{t}{3}) &\succsim M(Sx, Ty, t) \\ &= \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(By, Sx, t)\}. \end{aligned}$$

Therefore the maps A, B, S and T satisfies the condition (5) of Theorem 4.4 for $k = \frac{1}{3}$. Also the pairs $\{A, S\}$ and $\{B, T\}$ are obviously occasionally weakly compatible. Thus all the conditions of Theorem 4.3 are satisfied and $x = 0$ is the unique common fixed point of A, B, S and T in X.

Setting $A = B$ and $S = T$ in Theorem 4.4, we get the following corollary.

Corollary 4.3. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$, and let A, S be self-mappings on X. Let the pair $\{A, S\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$M(Ax, Ay, kt) \succsim M(Sx, Sy, t) * M(Ax, Sx, t) * M(Ay, Sy, t) * M(Ax, Sy, t),$$

for all $x, y \in X$ and for all $t > 0$, then A and S have a unique common fixed point in X.

Theorem 4.5. *Let $(X, M, *)$ be a complex valued fuzzy metric space with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$, for all $x, y \in X$. Let the pair $\{A, S\}$ be occasionally weakly compatible self maps on X. If there exists $k \in (0, 1)$ such that*

$$\begin{aligned} M(Sx, Sy, kt) &\succsim pM(Ax, Ay, t) + \\ &\quad q \min\{M(Ax, Ay, t), M(Sx, Ax, t), M(Sy, Ay, t)\}, \end{aligned} \tag{8}$$

for all $x, y \in X$ and for all $t > 0$, where $p, q > 0$ are reals with $p + q \geq 1$. Then A and S have a unique common fixed point in X.

Proof. The pair $\{A, S\}$ are occasionally weakly compatible, then there exists $x \in X$ such that $Ax = Sx$. Suppose that there exists another $y \in X$ for which $Ay = Sy$.

From the condition (8),

$$\begin{aligned} M(Sx,Sy,kt) &\succsim pM(Ax,Ay,t)+ \\ &\quad q\min\{M(Ax,Ay,t),M(Sx,Ax,t),M(Sy,Ay,t)\} \\ &= pM(Sx,Sy,t)+ \\ &\quad q\min\{M(Sx,Sy,t),M(Sx,Sx,t),M(Sy,Sy,t)\} \\ &= pM(Sx,Sy,t)+q\min\{M(Sx,Sy,t),e^{i\theta},e^{i\theta}\} \\ &= pM(Sx,Sy,t)+qM(Sx,Sy,t) \\ &= (p+q)M(Sx,Sy,t). \end{aligned}$$

Since $p+q \geq 1$, $M(Sx,Sy,kt) \succsim M(Sx,Sy,t)$.
In view of Lemma 3.2, we have $Sx = Sy$ and consequently $Ax = Ay$. Therefore the pair $\{A,S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus by Lemma 3.1, A and S have a unique common fixed point in X. □

The following corollary is the direct consequence of Theorem 4.5.

Corollary 4.4. *Let $(X,M,*)$ be a complex valued fuzzy metric space with $\lim_{t\to\infty} M(x,y,t) = e^{i\theta}$, for all $x,y \in X$. Let A be self map on X. If there exists $k \in (0,1)$ such that*

$$M(Ax,Ay,kt) \succsim \alpha M(Ax,Ay,t),$$

for all $x,y \in X$ and for all $t > 0$, where $\alpha \geq 1$ real. Then A has a unique fixed point in X.

5 Conclusions

In the Theorem 4.1 we have shown a sufficient condition for existence of a unique common fixed point of pair of any mappings in complex valued fuzzy metric spaces. Similar results are found in the remaining theorems 4.2, 4.3, 4.4 and 4.5 but for pairs of occasionally weakly compatible mappings.

References

F. Brain A. Azam and M. Khan. Common fixed point theorems in complex valued metric spaces. *Numer. Funct. Anal. Optim.*, 32(3):243–253, 2011.

C. T. Aage and J. N. Salunke. Common fixed point theorems in fuzzymetric spaces. *International Journal of Pure and Applied Mathematics*, 56(2):155–164, 2009.

J. J. Buckley. Fuzzy complex numbers. *Proceedings of ISFK,Guangzhou, China*, pages 597–700, 1987.

J. J. Buckley. Fuzzy complex numbers. *Fuzzy sets and systems*, 33(3):333–345, 1989.

J. J. Buckley. Fuzzy complex analysis i: Differentiation. *Fuzzy sets and systems*, 41:269–284, 1991.

J. J. Buckley. Fuzzy complex analysis ii: Integration. *Fuzzy sets and systems*, 49: 171–179, 1992.

M. Friedman D. Ramot, R. Milo and A. Kandel. Complex fuzzy sets. *IEEE Transactions of Fuzzy systems*, 10(2):171–186, 2002.

M. Imdad D. Singh, V. Joshi and P. Kumam. A novel framework of complex fuzzy metric spaces and fixed point theorems. *Journal of Intelligent and Fuzzy Systems*, 30(6):3227–3238, 2016.

I. Demir. Fixed point theorems in complex valued fuzzy $b$application to integral equations. *Miskolc Mathematical Notes*, 22(1):153–171, 2021.

Z.K. Deng. Fuzzy pseudo metric spaces. *J. Math. Anal. Appl.*, 86:74–95, 1982.

M.A. Ercez. A metric space in fuzzy set theory. *J. Math. Anal. Appl.*, 69:205–230, 1979.

A. George and P. Veeramani. On some results in fuzzy metric spaces. *Fuzzy Sets and Systems*, 64:395–399, 1994.

T. Biswas J. Choi, S.K. Datta and M.N. Islam. Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. *Honam Mathematical J.*, 39(1):115–126, 2017.

G. Jungck and B.E. Rhoades. Fixed point theorems for occasionally weakly compatible mappings. *Fixed Point Theory*, 7(2):287–296, 2006.

O. Kaleva and S. Seikkala. On fuzzy metric spaces. *Fuzzy Sets and Systems*, 12: 215–229, 1984.

I. Kramosil and J. Michalek. Fuzzy metric and statistical metric spaces. *Kybernetika*, 11:336–344, 1975.

L.A. Zadeh. Fuzzy sets. *Information and control*, 8:338–353, 1965.