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Atanassov's intuitionistic fuzzy index of hypergroupoids

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Abstract

In this work we introduce the concept of Atanassov's intuitionistic fuzzy index of a hypergroupoid based on the notion of intuitionistic fuzzy grade of a hypergroupoid. We calculate it for some particular hypergroups, making evident some of its special properties.

Key words: (Atanassov's intuitionistic) fuzzy set, (Atanassov's intuitionistic) fuzzy grade, Hypergroup.

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1 Introduction

In 1986 Atanassov [5, 8] defined the notion of intuitionistic fuzzy set as a generalization of the elder one of fuzzy set introduced by Zadeh [30]. Since then this new tool of investigation of various uncertain problems bears his name: Atanassov's intuitionistic fuzzy set. More exactly, for any element x in a finite nonempty set X one assigns two values in the interval $[0, 1]$: the membership degree $\mu(x)$ and the non-membership degree $\lambda(x)$ such that $0 \leq \mu(x) + \lambda(x) \leq 1$. For any $x \in X$, the value $\pi(x) = 1 - \mu(x) - \lambda(x)$ is called the Atanassov's intuitionistic fuzzy index or the hesitation degree of x to X. This index is an important characteristic of the intuitionistic fuzzy sets since it provides valuable information on each element x in X (see [6, 7]). Recently, Bustince et al. [9] have given a construction method to obtain a generalized Atanassov's intuitionistic fuzzy index. For other applications of this index see [10, 11].

The concept of fuzzy grade appeared in hypergroup theory in 2003 and seven years later it was extended to the intuitionistic fuzzy case. Based on the two connections between hypergroupoids and fuzzy sets introduced by Corsini [13, 14], one may associate with any hypergroupoid H a sequence of join spaces and fuzzy sets, whose length is called the fuzzy grade of H. Corsini and Cristea determined the fuzzy grade of all i.p.s. hypergroups of order less than 8 (see [15, 16]). The same problem was treated by Cristea [21], Angheluţă and Cristea [1] for the complete hypergroups, by Corsini et al. for the hypergraphs and hypergroupoids obtained from multivalued functions [18, 19, 20].

In 2010 Cristea and Davvaz [22] introduced and studied the Atanassov's intuitionistic fuzzy grade of a hypergroupoid as the length of the sequence of join spaces and intuitionistic fuzzy sets associated with a hypergroupoid. These sequences have been determined for all i.p.s. hypergroups of order less than 8 and for the complete hypergroups of order less than 7 by Davvaz et al. [23, 24, 25].

Any hypergroupoid H may be endowed with an intuitionistic fuzzy set $\bar{A} = (\bar{\mu}, \bar{\lambda})$ in the sense of Cristea-Davvaz [22]. Based on this construction, we define here the notion of Atanassov's intuitionistic fuzzy index of a hypergroupoid H.

Throughout the paper we use, for simplicity, the term of intuitionistic fuzzy set instead of Atanassov's intuitionistic fuzzy set.

The paper is structured as follows. After some background information regarding hypergroups theory, we recall in Section 2 the construction of the sequence of join spaces and intuitionistic fuzzy sets associated with a hypergroupoid H, presenting some technical results for the membership functions $\widetilde{\mu}, \bar{\mu}, \bar{\lambda}$. In Section 3 we introduce the notion of intuitionistic fuzzy index of a hypergroupoid, giving its formula for some particular hypergroups. We conclude with final remarks and some open problems.

2 Atanassov's intuitionistic fuzzy grade of hypergroupoids

First we recall some definitions from [12, 17], needed in what follows.

Let H be a nonempty set and let $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*.

This hyperoperation can be extended to a binary operation on $\mathcal{P}^*(H)$. If

A and B are nonempty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \ \text{ and } \ A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if, for all x, y, z in H, we have the associative law: $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if, for all $x \in H$, we have the reproducibility axiom: $x \circ H = H \circ x = H$. A hypergroup $(H, \circ)$ is called *total hypergroup* if, for any $x, y \in H$, $x \circ y = H$.

For each pair of elements $a, b \in H$, we denote: $a/b = \{x \in H \mid a \in x \circ b\}$ and $b \backslash a = \{y \in H \mid a \in b \circ y\}$.

A commutative hypergroup $(H, \circ)$ is called a *join space* if the following condition holds:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset.$$

A commutative hypergroup $(H, \circ)$ is *canonical* if and only if it is a join space with a scalar identity.

The notion of join space has been introduced and studied for the first time by Prenowitz. Later on, together with Jantosciak, he reconstructed, from an algebraic point of view, several branches of geometry: the projective, the descriptive and the spherical geometry (see [28]).

Several connections between hypergroups and (intuitionistic) fuzzy sets have been investigated till now (see for example [2, 3, 4, 26]). Here we focus our study on that initiated by Corsini [14] in 2003, when he defined a sequence of join spaces associated with a hypergroupoid endowed with a fuzzy set. Based on this idea, Cristea and Davvaz [22] extended later on this connection to the intuitionistic fuzzy grade. We recall now briefly these constructions.

For any hypergroup $(H, \circ)$, Corsini [14] defined a fuzzy subset $\widetilde{\mu}$ of H in the following way: for any $u \in H$, one considers:

$$\widetilde{\mu}(u) = \frac{\displaystyle\sum_{(x,y) \in Q(u)} \frac{1}{|x \circ y|}}{q(u)}, \tag{1}$$

where $Q(u) = \{(a, b) \in H^2 \mid u \in a \circ b\}$, $\quad q(u) = |Q(u)|$. If $Q(u) = \emptyset$, set $\widetilde{\mu}(u) = 0$.

On the other hand, with any hypergroupoid H endowed with a fuzzy set α, we can associate a join space $(H, \circ_\alpha)$ as follows (see [13]): for any $(x, y) \in H^2$,

$$x \circ_\alpha y = \{z \in H \mid \alpha(x) \wedge \alpha(y) \leq \alpha(z) \leq \alpha(x) \vee \alpha(y)\}. \tag{2}$$

Then, from $({}^1H, \circ_1)$ we obtain, in the same way as in (1), a membership function $\widetilde{\mu}_1$ and then the join space ${}^2H$ and so on. A sequence of fuzzy sets and of join spaces $({}^rH, \widetilde{\mu}_r)$ is determined.

We denote $\tilde{\mu}_0 = \tilde{\mu}$, ${}^0H = H$. If two consecutive hypergroups of the obtained sequence are isomorphic, then the sequence stops.

The length of this sequence has been called by Corsini and Cristea [15, 16] the (strong) fuzzy grade of the hypergroupoid H.

Let now $(H, \circ)$ be a finite hypergroupoid of cardinality $n, n \in \mathbb{N}^*$. Cristea and Davvaz [22] defined on H an Atanassov's intuitionistic fuzzy set $(\overline{\mu}, \overline{\lambda})$ in the following way: for any $u \in H$, we set:

$$\begin{aligned} \overline{\mu}(u) = & \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{n^2}, \\ \overline{\lambda}(u) = & \frac{\displaystyle\sum_{(x,y)\in \overline{Q}(u)} \frac{1}{|x \circ y|}}{n^2}, \end{aligned} \tag{3}$$

where $Q(u) = \{(x, y) \mid (x, y) \in H^2, u \in x \circ y\}, \overline{Q}(u) = \{(x, y) \mid (x, y) \in H^2, u \notin x \circ y\}$. If $Q(u) = \emptyset$, then we put $\overline{\mu}(u) = 0$ and similarly, if $\overline{Q}(u) = \emptyset$, then we put $\overline{\lambda}(u) = 0$.

It follows immediately the following relation.

Corolary 2.1. *For all $u \in H$, $\tilde{\mu}(u) \geq \overline{\mu}(u)$.*

Proof. Let $|H| = n$. Then $q(u) \leq n^2$, for all $u \in H$. Thus, by definitions (1) and (3), we have $\tilde{\mu}(u) \geq \overline{\mu}(u)$. □

Moreover, for all $u \in H$, we define: $A(u) = \sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}, \overline{A}(u) = \sum_{(x,y)\in \overline{Q}(u)} \frac{1}{|x \circ y|}$.

Remark 2.1. There exist hypergroups H such that there exists $u \in H$ with

$$(i) |Q(u)| = |\overline{Q}(u)| \quad \text{and} \quad \overline{\mu}(u) \neq \overline{\lambda}(u).$$

$$(ii) |Q(u)| \neq |\overline{Q}(u)| \quad \text{and} \quad \overline{\mu}(u) = \overline{\lambda}(u).$$

An example for the case (i) is the hypergroup H_1 and the case (ii) is illustrated by the hypergroup H_2 given below; both hypergroups are commutative and are represented by the following tables:

H_1	1	2	3	4
1	1,4	1,2	1,3	1,4
2		2	2,3	2,4
3			3	3,4
4				1,2,3,4

where $|Q(1)| = |\overline{Q}(1)| = 8$, $A(1) = \frac{15}{4}$, $\overline{A}(1) = \frac{20}{4}$ and by consequence $\overline{\lambda}(1) = \frac{15}{64} \neq \frac{20}{64} = \overline{\mu}(1)$.

H_2	1	2	3
1	1	1,2	1,3
2		2	2,3
3			3

where $|Q(1)| = 5 \neq 4 = |\overline{Q}(1)|$ and by consequence $\overline{\lambda}(1) = \overline{\mu}(1) = \frac{1}{3}$.

3 Atanassov's intuitionistic index of hypergroupoids

Definition 3.1. [5] Let (μ, λ) be an intuitionistic fuzzy set on the nonempty set H. We call the *hesitation degree* or the *intuitionistic index* of the element x in the set H the following expression

$$\pi(x) = 1 - \mu(x) - \lambda(x).$$

Evidently $0 \leq \pi(x) \leq 1$, for all $x \in H$.

Remark 3.1. Let $(\overline{\mu}, \overline{\lambda})$ be the intuitionistic fuzzy set associated with a hypergroupoid $(H, \circ)$ like in Cristea-Davvaz [22]. Since $\overline{\mu}(u)+\overline{\lambda}(u) = \frac{\sum_{(x,y)\in H^2} \frac{1}{|x\circ y|}}{n^2}$, for all $u \in H$, according with (ω'), it follows that $\overline{\pi}(u) = constant$, for all $u \in H$. Therefore we introduce the following definition.

Definition 3.2. Let $(H, \circ)$ be a hypergroupoid and let $\overline{\mu}$, $\overline{\lambda}$ be the membership functions defined in (w'). For all $u \in H$, we define

$$\overline{\pi}(H) = \overline{\pi}(u) = 1 - \overline{\mu}(u) - \overline{\lambda}(u),$$

and we call it the *intuitionistic index* of the hypergroupoid H.

In the following we determine the intuitionistic index for some particular hypergroupoids.

Proposition 3.1. *Let $(H, \circ)$ be a total hypergroup of cardinality n. Then $\overline{\pi}(u) = \dfrac{n-1}{n}$, for any $u \in H$.*

Proof. Let $(H, \circ)$ be a total hypergroup and $|H| = n$. Then $x \circ y = H$, for all $x, y \in H$. Thus $A(u) = n$, $\overline{A}(u) = 0$, for all $u \in H$. Since $\overline{Q}(u) = \emptyset$, it follows that $\overline{\mu}(u) = \frac{1}{n}$, $\overline{\lambda}(u) = 0$, which imply that $\overline{\pi}(u) = \frac{n-1}{n}$. □

Proposition 3.2. *Let $H = \{x_1, x_2, x_3, ..., x_{n-1}, x_n\}$, be the hypergroupoid defined as it follows: $x_i \circ x_j = \{x_i, x_j\}$, $1 \leq i, j \leq n$. Then $\overline{\mu}(x) = \frac{1}{n}, \overline{\pi}(x) = \frac{n-1}{2n}$, for any $x \in H$.*

Proof. The table of the commutative hyperoperation $\circ$ is the following one:

H	x_1	x_2	x_3	$\dots$	x_{n-1}	x_n
x_1	x_1	x_1, x_2	x_1, x_3	$\dots$	x_1, x_{n-1}	x_1, x_n
x_2		x_2	x_2, x_3	$\dots$	x_2, x_{n-1}	x_2, x_n
x_3			x_3	$\dots$	x_3, x_{n-1}	x_3, x_n
$\dots$				$\dots$	$\dots$	$\dots$
x_{n-1}					x_{n-1}	x_{n-1}, x_n
x_n						x_n

Let $x_i \in H$. Then $|Q(x_i)| = \{(x_i, x_i), (x_i, x_s), (x_p, x_i),\ 1 \leq s, p \leq n,\ s, p \neq i\}$. It follows that $|Q(x_i)| = 2n-1$. Thus $|\overline{Q}(x_i)| = n^2-(2n-1) = (n-1)^2$. So, we have $\overline{\mu}(x_i) = \frac{1}{n^2}[1+\frac{1}{2}(2n-2)] = \frac{1}{n}$. Similarly, we obtain $\overline{\lambda}(x_i) = \overline{\pi}(x_i) = \frac{n-1}{2n}$. □

Corolary 3.1. *Let $(H, \circ)$ be the hypergroupoid defined in Proposition 3.2. Then, for any $x \in H$, we obtain $\overline{\mu}(x) = \frac{1}{n} > \overline{\lambda}(x) = \overline{\pi}(x) = \frac{n-1}{2n}$, for any $n < 3$, and $\overline{\mu}(x) = \frac{1}{n} < \overline{\lambda}(x) = \overline{\pi}(x) = \frac{n-1}{2n}$ for any $n \geq 3$.*

With any hypergroupoid H one may associate a sequence of join spaces and fuzzy sets denoted by $((^iH, \circ_i), \overline{\mu}_i(u))_{i \geq 1}$.

Then, for any $i \geq 1$, we can divide ^iH in the classes $\{^iC_j\}_{j=1}^r$, where $x, y \in {^iC_j} \Longleftrightarrow \mu_{i-1}(x) = \mu_{i-1}(y)$. Moreover, we define the following ordering relation: $j < k$ if, for elements $x \in {^iC_j}$ and $y \in {^iC_k}$, we have $\mu_{i-1}(x) < \mu_{i-1}(y)$. We need the following notations: for all j, s, set

$$k_j = |^iC_j|, \quad {_sC} = \bigcup_{1 \leq j \leq s} {^iC_j}, \quad {^sC} = \bigcup_{s \leq j \leq r} {^iC_j}, \quad {_sk} = |_sC|, \quad {^sk} = |^sC|.$$

Therefore with any ordered chain $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ we may associate an ordered r-tuple $(k_1, k_2, ..., k_r)$, where $k_l = |{}^iC_l|$, for all l, $1 \le l \le r$. Using these notations we determine the general formula for calculating the values of the membership functions $\overline{\mu}_i, \overline{\lambda}_i$.

Theorem 3.1. *[22] For any $u \in {}^iC_s$, $i \ge 1$, $s \in \{1, 2, ..., r\}$, we find that*

$$\overline{\mu}_i(z) = \frac{k_s + 2 \displaystyle\sum_{\substack{l \le s \le m \\ l \ne m}} \frac{k_l k_m}{\displaystyle\sum_{l \le t \le m} k_t}}{n^2}$$

and

$$\overline{\lambda}_i(z) = \frac{\displaystyle\sum_{l \ne s} k_l + 2 \sum_{s < l < m \le r} \frac{k_l k_m}{\displaystyle\sum_{l \le t \le m} k_t} + 2 \sum_{1 \le l < m < s} \frac{k_l k_m}{\displaystyle\sum_{l \le t \le m} k_t}}{n^2}.$$

We immediately obtain also the formula for calculating the intuitionistic fuzzy index.

Theorem 3.2. *For any $u \in {}^iC_s$, $i \ge 1$, $s \in \{1, 2, ..., r\}$,*

$$\overline{\pi}_i(u) = 1 - \left[\frac{\sum_{l=1}^{r} k_l + 2 \sum_{1 \le l < m \le r} \frac{k_l k_m}{\sum_{l \le t \le m} k_t}}{n^2} \right].$$

One of the natural problems regarding the intuitionistic fuzzy grade of a hypergroupoid is that of constructing a hypergroupoid H with $i.f.g.(H) = p$, for any arbitrary natural number $p \ge 2$. A such example is given in the next result.

Lemma 3.1. *[22] Let $H = \{x_1, x_2, x_3, ..., x_{n-1}, x_n\}$, where n is an even number, be the hypergroupoid defined as follows:*

i) $x_i \circ x_i = x_i$, $1 \le i \le n$,

ii) $x_i \circ x_j = x_j \circ x_i = \{x_i, x_{i+1}, ..., x_j\}$, $1 \le i < j \le n$.

Then, for any $s \in \{1, 2, ..., \frac{n}{2}\}$, we have
$\overline{\mu}(x_s) = \overline{\mu}(x_{n-s+1})$, $\overline{\lambda}(x_s) = \overline{\lambda}(x_{n-s+1})$ *and* $\overline{\mu}(x_s) \le \overline{\mu}(x_{s+1})$, $\overline{\lambda}(x_s) \ge \overline{\lambda}(x_{s+1})$.

Moreover, we obtain

$$\overline{\mu}(x_1) = \frac{1}{n^2}\left(1 + 2\sum_{m=2}^{n}\frac{1}{m}\right), \quad \overline{\lambda}(x_1) = \frac{1}{n^2}\left(1 - n + 2n\sum_{m=2}^{n}\frac{1}{m}\right)$$

$$\overline{\mu}(x_{s+1}) = \overline{\mu}(x_s) + \frac{2}{n^2}\sum_{m=s+1}^{n-s}\frac{1}{m}, \quad \overline{\lambda}(x_s) = \overline{\lambda}(x_{s+1}) + \frac{2}{n^2}\sum_{m=s+1}^{n-s}\frac{1}{m}, \ \forall s, 2 \leq s \leq \frac{n}{2}.$$

Notation 3.1. In the following we denote the hypergroup of Lemma 3.1 by $\mathcal{H}_n$, $n \in \mathbb{N}^*$. For $\mathcal{H}_n$, we denote the membership functions $\overline{\mu}$ and $\overline{\lambda}$ introduced in (ω') by $\overline{\mu}^n$ and $\overline{\lambda}^n$.

Example 3.1. Let us consider again the hypergroup $\mathcal{H}_n$ that has the following table

H	x_1	x_2	x_3	...	x_{n-1}	x_n
x_1	x_1	x_1, x_2	$x_1 \to x_3$	...	$x_1 \to x_{n-1}$	$x_1 \to x_n$
x_2		x_2	x_2, x_3	...	$x_2 \to x_{n-1}$	$x_2 \to x_n$
x_3			x_3	...	$x_3 \to x_{n-1}$	$x_3 \to x_n$
...				...	...	...
x_{n-1}					x_{n-1}	x_{n-1}, x_n
x_n						x_n

where we use the notation $x_i \to x_j = \{x_i, x_{i+1}, ..., x_j\}$, with $i < j$. Then we obtain

	$\overline{\mu}^n(x_1)$	$\overline{\mu}^n(x_2)$	$\overline{\mu}^n(x_3)$	$\overline{\mu}^n(x_4)$	$\overline{\lambda}^n(x_1)$	$\overline{\lambda}^n(x_2)$	$\overline{\lambda}^n(x_3)$	$\overline{\lambda}^n(x_4)$	$\overline{\pi}^n(x_k)$
$n=2$	0.500				0.250				0.250
$n=3$	0.296	0.407			0.333	0.222			0.370
$n=4$	0.197	0.302			0.354	0.250			0.447
$n=5$	0.142	0.229	0.256		0.353	0.266	0.240		0.504
$n=6$	0.108	0.179	0.212		0.342	0.273	0.240		0.547
$n=7$	0.085	0.144	0.176	0.186	0.332	0.257	0.241	0.231	0.581
$n=8$	0.075	0.125	0.145	0.159	0.226	0.176	0.156	0.141	0.698

where $k \in \{1, ..., n\}$. The missing values in the previous table are equal to the other written values as in the formulas in Lemma 3.1.

Lemma 3.2. *For the hypergroup $\mathcal{H}_n$, with n an even number, for any $s \in \{1, 2, ..., \frac{n}{2}\}$ and $n \geq 6$, we obtain*

$$\overline{\mu}^n(x_s) < \overline{\lambda}^n(x_s).$$

Corolary 3.2. *Let us consider the hypergroup $\mathcal{H}_n$. Then, for any $s \in \{1, 2, ..., n\}$, we obtain*

$$\overline{\pi}(\mathcal{H}_n) = \overline{\pi}^n(x_s) = 1 - \frac{1}{n^2}\left[2 - n + 2(n+1)\sum_{m=2}^{n}\frac{1}{m}\right].$$

Proof. Since

$$\begin{aligned}\overline{\mu}^n(x_1) + \overline{\lambda}^n(x_1) &= \tfrac{1}{n^2}[(1 + 2\textstyle\sum_{m=2}^{n}\frac{1}{m}) + (1 - n + 2n\sum_{m=2}^{n}\frac{1}{m})] \\ &= \tfrac{1}{n^2}[2 - n + 2(n+1)\textstyle\sum_{m=2}^{n}\frac{1}{m}],\end{aligned}$$

the required result is proved. □

Theorem 3.3. *Let us consider the hypergroups $(\mathcal{H}_n, \circ)$ and $(\mathcal{H}_{n+2}, \circ)$, with n an even natural number. Then*

$$\overline{\pi}(\mathcal{H}_n) < \overline{\pi}(\mathcal{H}_{n+2}).$$

Proof. By Corollary 3.2, we will prove that

$$\frac{1}{n^2}[2 - n + 2(n+1)\sum_{m=2}^{n}\frac{1}{m}] > \frac{1}{(n+2)^2}[2 - (n+2) + 2(n+3)\sum_{m=2}^{n+2}\frac{1}{m}].$$

Denoting $\sum_{m=2}^{n}\frac{1}{m}$ by A, we will show that:

$$\frac{1}{n^2}[2 - n + 2(n+1)A] > \frac{1}{(n+2)^2}[-n + 2(n+3)(A + \frac{2n+3}{(n+1)(n+2)})],$$

that is equivalent with

$$(2-n)(n+1)(n+2)^3 + 2(n+1)^2(n+2)^3A >$$

$$> -n^3(n+1)(n+2) + 2n^2(n+1)(n+2)(n+3)A + 2n^2(2n+3)(n+3).$$

Therefore we have

$$4(n+1)(n+2)(n^2+4n+2)A > 2(3n^4 + 10n^3 + n^2 - 16n - 8)$$

if and only if

$$2A > \frac{3n^4 + 10n^3 + n^2 - 16n - 8}{n^4 + 7n^3 + 16n^2 + 14n + 4}.$$

We prove the last relation by induction on n.
For $n = 3$ the relation becomes $2(\frac{1}{2} + \frac{1}{3}) > \frac{466}{460}$, that is $\frac{5}{3} > \frac{466}{460}$, that is true.
We suppose that

$$P(n) : 2\sum_{m=2}^{n} \frac{1}{m} > \frac{3n^4 + 10n^3 + n^2 - 16n - 8}{n^4 + 7n^3 + 16n^2 + 14n + 4} \text{ is true}$$

and we prove that

$$P(n) : 2\sum_{m=2}^{n+1} \frac{1}{m} > \frac{3(n+1)^4 + 10(n+1)^3 + (n+1)^2 - 16(n+1) - 8}{(n+1)^4 + 7(n+1)^3 + 16(n+1)^2 + 14(n+1) + 4} \text{ is fulfilled.}$$

Since

$$2\sum_{m=2}^{n+1} \frac{1}{m} = 2\sum_{m=2}^{n} \frac{1}{m} + \frac{2}{n+1} > \frac{3n^4 + 10n^3 + n^2 - 16n - 8}{n^4 + 7n^3 + 16n^2 + 14n + 4} + \frac{2}{n+1},$$

it remains to prove that

$$\frac{3n^5 + 15n^4 + 25n^3 + 17n^2 + 4n}{n^5 + 8n^4 + 23n^3 + 30n^2 + 18n + 4} > \frac{3n^4 + 22n^3 + 49n^2 + 28n - 10}{n^4 + 11n^3 + 43n^2 + 71n + 42}.$$

After simple computations that we omit here, we prove that the last relation is true, for any natural number n.

Now the proof is complete. □

Generalizing this theorem we obtain the following result.

Corolary 3.3. *For the hypergroups $\mathcal{H}_n$ and $\mathcal{H}_{n'}$, with n, n' two even natural numbers such that $n < n'$, we have the following relation:*

$$\overline{\pi}(\mathcal{H}_n) < \overline{\pi}(\mathcal{H}_{n'}).$$

Proof. Let $n' = n + 2k$, $k \in \mathbb{N}^*$. We will prove the relation by induction on k. For $k = 1$, by Theorem 3.3 we have $\overline{\pi}(\mathcal{H}_n) < \overline{\pi}(\mathcal{H}_{n+2})$. Assume the corollary is true for $k = m$, i.e., $\overline{\pi}(\mathcal{H}_n) < \overline{\pi}^(\mathcal{H}_{n+2m})$. Again by Theorem 3.3 we have $\overline{\pi}(\mathcal{H}_{n+2m}) < \overline{\pi}(\mathcal{H}_{n+2(m+1)})$. Thus, the thesis of the corollary is true for $k = m + 1$. This completes the induction and the proof. □

We conclude this section with a result regarding the membership function $\overline{\mu}^n$.

Theorem 3.4. *Let us consider the hypergroups $\mathcal{H}_n$ and $\mathcal{H}_{n+2}$, with n an even natural number. Then, for any $s \in \{1, 2, ..., n/2\}$, we have the relation*

$$\overline{\mu}^n(x_s) > \overline{\mu}^{n+2}(x_s).$$

Proof. For any $s \in \{1, 2, ..., n/2\}$, we will prove that

$$\frac{1}{n^2}[2s-1+2\sum_{m=1}^{n-2s+1}\frac{s}{s+m}+2\sum_{m=1}^{s-1}\frac{m}{n-m+1}] >$$

$$> \frac{1}{(n+2)^2}[2s-1+2\sum_{m=1}^{n-2s+1}\frac{s}{s+m}+2s(\frac{1}{n-s+2}+\frac{1}{n-s+3})+2\sum_{m=1}^{s-1}\frac{m}{n-m+3}].$$

Denote $A = 2s-1+2\sum_{m=1}^{n-2s+1}\frac{s}{s+m}$. Since

$$\frac{1}{n^2}\left(2\sum_{m=1}^{s-1}\frac{m}{n-m+1}\right) > \frac{1}{(n+2)^2}\left(2\sum_{m=1}^{s-1}\frac{m}{n-m+3}\right),$$

to prove that $\overline{\mu}^n(x_s) > \overline{\mu}^{n+2}(x_s)$ it is enough to show that

$$(n+2)^2 A > n^2\left[A + \frac{2s(2n-2s+5)}{(n-s+2)(n-s+3)}\right],$$

that is true if and only if

$$A > \frac{sn^2(2n-2s+5)}{(2n+2)(n-s+2)(n-s+3)}.$$

Since

$$\sum_{m=1}^{n-2s+1}\frac{s}{s+m} > \frac{s(n-2s+1)}{n-s+1},$$

it follows that

$$A > \frac{(4s-1)n-6s^2+5s-1}{n-s+1}.$$

It remains to prove that

$$\frac{(4s-1)n-6s^2+5s-1}{n-s+1} > \frac{sn^2(2n-2s+5)}{(2n+2)(n^2+(5-2s)n+s^2-5s+6)} =$$

$$= \frac{2sn^3+n^2(-2s^2+5s)}{2n^3+n^2(-4s+12)+n(2s^2-14s+22)+2s^2-10s+12},$$

and this is true if and only if

$$[2n^3+n^2(-4s+12)+n(2s^2-14s+22)+2s^2-10s+12][(4s-1)n-6s^2+5s-1] >$$

$$> (n-s+1)[2sn^3 + n^2(-2s^2+5s)],$$

which is equivalent with

$$E(n) = n^4(6s-29 + n^3(-24s^2+55s-14) + n^2(30s^3-143s^2+161s-34)+$$
$$+n(-12s^4+102s^3-246s^2+182s-34) - 12s^4+70s^3-124s^2+70s-12 > 0,$$

whenever $2 \leq 2s \leq n$. Then we find

$$E^{(4)}(n) = 144s - 48 > 0;$$

it follows that $E'''(n)$ is a strictly increasing function, so

$$E'''(n) \geq E'''(2s) = 144s^2 + 234s - 84 > 0.$$

It follows that $E''(n)$ is a strictly increasing function, so

$$E''(n) \geq E''(2s) = 60s^3 + 278s^2 + 154s - 68 > 0.$$

Then $E'(n)$ is a strictly increasing function, thus

$$E'(n) \geq E'(2s) = 12s^4 + 126s^3 + 230s^2 + 46s - 34 > 0.$$

Thus, we obtain that $E(n)$ is a strictly increasing function, so

$$E(n) \geq E(2s) = 28s^4 + 110s^3 + 104s^2 + 2s - 12 > 0, \text{ for any } s \geq 1,$$

and the required result is proved. □

A generalized corollary follows.

Corolary 3.4. Let us consider the hypergroups $\mathcal{H}_n$ and $\mathcal{H}_{n'}$, with n, n' even numbers such that $n < n'$. Then

$$\overline{\mu}^n(x_s) > \overline{\mu}^{n'}(x_s), \ \forall s \in \{1, 2, ..., n/2\}.$$

Proof. The proof is similar to the proof of Corollary 3.3. □

4 Conclusions

Given a hypergroupoid H, one may calculate two numerical functions associated with it: the fuzzy grade and the intuitionistic fuzzy grade. In this note we define another one, called the Atanassov's intuitionistic fuzzy index of a hypergroupoid. This function depends on the values of the first membership functions in the sequence of join spaces and intuitionistic fuzzy sets associated with H as in [22]. We have determined it for some particular hypergroups. We will investigate further properties and connections with the intuitionistic fuzzy grade in a future work.

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