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Tikhonov type regularization for unbounded operators

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Abstract

In this paper, we introduce a Tikhonov type regularization method for an ill-posed operator equation $Tx = y$, where T is a closed densely defined unbounded operator on a Hilbert space H.
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1 Introduction

Most of the problems arise in the field of science and engineering can be modelled as an operator equation

$$Tx = y \tag{1}$$

where $T : X \to Y$ is a bounded linear map from a normed linear space X to a normed linear space Y. In most of the cases (1) is Ill-posed. Certain regularization procedures are known for solving ill-posed operator equation (1). For example Tikhonov regularization, Mollifier method, Ritz method [5, 3]. In this paper we introduce a Tikhonov type regularization method for solving an ill-posed operator equation (1), where T is a closed densely defined operator on a Hilbert space H and we study the order of convergence.

2 Preliminaries

Let $L(H), C(H)$ and $B(H)$ denote the space of all linear, closed linear and bounded linear operators on a Hilbert space H respectively. For $T \in L(H)$, the domain, range of T are denoted by $D(T), N(T)$ respectively. An operator $T \in L(H)$ is said to be densely defined if $\overline{D(T)} = H$.
For example let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, ..., x_n, ...) = (x_1, 2x_2, 3x_3, ..., nx_n, ...)$$

with domain

$$D(T) = \{(x_1, x_2, x_3, ..., x_n, ...) \in H : \Sigma_{j=1}^{\infty} |jx_j|^2 < \infty\}.$$

Then T is closed and unbounded. Since $c_{00} \subseteq D(T)$ and c_{00} is dense in $l^2(\mathbb{N})$, $D(T)$ is dense in $l^2(\mathbb{N})$.

Proposition 2.1. *Let $T \in C(H)$ be a densely defined operator. Then there exist a unique operator $T^* \in C(H)$ such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \ \ \forall x \in D(T), \ \forall y \in D(T^*).$$

Proof. Let $D(T^*) = \{y \in H : \langle Tx, y\rangle$ is continuous for every $x \in D(T)$ $\}$. For $y \in D(T)$, define $f : D(T) \to \mathbb{C}$ by $f(x) = \langle Tx, y\rangle \ \forall x \in D(T)$. Extend f to $f_0 : H \to \mathbb{C}$ by $f_0(x) = \lim_{n\to\infty} \langle Tx_n, y\rangle$ where (x_n) is a sequence in $D(T)$ such that $x_n \to x$.

Next we prove that f_0 is well defined. For, let (x_n) and (y_n) be two sequences in $D(T)$ converges to x. Since T is closed, $T(x_n - y_n) \to 0$. If $\langle Tx_n, y\rangle \to \langle x, y\rangle$, then

$$\begin{aligned} |\langle Ty_n, y\rangle - \langle x, y\rangle| &= |\langle Ty_n - Tx_n + Tx_n - x, y\rangle| \\ &\leq \|T(y_n - x_n)\|\|y\| + |\langle Tx_n - x, y\rangle| \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Hence f_0 is well defined.

Since f_0 is a bounded linear functional on the Hilbert space H, by Riesz representation theorem there exist a unique $y^* \in H$ such that $f_0(x) = \langle x, y^*\rangle$. Thus $\langle Tx, y\rangle = \langle x, y^*\rangle \; \forall x \in D(T)$. Define $T^* : D(T^*) \to H$ by $T^* y = y^*$. Then T^* is well-defined. Also $\langle Tx, y\rangle = \langle x, T^* y\rangle \; \forall x \in D(T), \; \forall y \in D(T^*)$. □

Consider an ill-posed operator equation

$$Tx = y \tag{2}$$

where T is a closed densely defined operator on H.

Definition 2.1. *[7]*

Let $T \in C(H)$ be densely defined. Then there exist a unique densely defined operator $T^\dagger \in C(H)$ with domain $D(T^\dagger) = R(T) \oplus R(T)^\perp$ satisfies the following properties

(i) $TT^\dagger y = P_{\overline{R(T)}}\, y$ for all $y \in D(T^\dagger)$,

(ii) $T^\dagger Tx = Q_{N(T)^\perp} x$ for all $x \in D(T)$.

(iii) $N(T^\dagger) = R(T)^\perp$.

where P and Q are the orthogonal projection on to $\overline{R(T)}$ and $N(T^\perp)$ respectively. The operator $T^\dagger$ is called the Moore-Penrose inverse of T.

For $y \in D(T^\dagger)$, let $S_y = \{x \in D(T) : \|Tx - y\| \leq \|Tu - y\| \; \forall u \in D(T)\}$. Then $u \in S_y$ is called least square solution of the operator equation (2). Note that $\|T^\dagger y\| \leq \|x\| \; \forall x \in S_y$, is called least square solution of minimal norm and is denoted by $\widehat{x}$ [7].

If $R(T)$ is not closed, then $T^\dagger$ is not continuous. Now we introduce a Tikhonov type regularization procedure for finding an approximate solution for $T^\dagger y$.

3 Tikhonov type regularization

In this section we introduce a Tikhonov type regularization procedure for solving (2).

Lemma 3.1. *Let $T \in C(H)$ be densely defined and $\alpha > 0$. Then $T^*T + \alpha I$ and $TT^* + \alpha I$ are bijective closed densely defined operators on H. Also $(TT^* + \alpha I)^{-1}$ and $(T^*T + \alpha I)^{-1}$ are bounded, self adjoint operators on H.*

Proof. Let $T \in C(H)$ and $\alpha > 0$. By proposition 2.1, we have $T^* \in C(H)$. Hence, $(TT^* + \alpha I)$ and $(T^*T + \alpha I)$ are closed densely defined operators on H. Since $\langle (TT^* + \alpha I)x, x\rangle = \langle T^*x, T^*x\rangle + \alpha\langle x, x\rangle \geq 0, \forall x \in D(T^*)$, we have $(TT^* + \alpha I)$ is a positive operator. Similarly $(T^*T + \alpha I)$ is also a positive operator. Since $T^*T + \alpha I$ is positive,

$$\begin{aligned} \|(T^*T + \alpha I)x\|\|x\| &\geq \langle (T^*T + \alpha I)x, x\rangle \\ &= \langle T^*Tx, x\rangle + \alpha\|x\|^2 \\ &\geq \alpha\|x\|^2 \ \forall x \in H \end{aligned}$$

Thus

$$\|(T^*T + \alpha I)x\| \geq \alpha\|x\| \ \forall x \in H \tag{3}$$

Since $T^*T + \alpha I$ is bounded below, it is one-one and its inverse from the range is continuous. Also $R(T^*T + \alpha I)$ is closed. Since $T^*T + \alpha I$ is also self adjoint, $R(T^*T + \alpha I) = N(T^*T + \alpha I)^{\perp} = H$. Hence $T^*T + \alpha I$ is onto. Therefore $(T^*T + \alpha I)^{-1} \in B(H)$. Similary $(TT^* + \alpha I)^{-1} \in B(H)$. From (3), $\|(T^*T + \alpha I)^{-1}\| \leq \dfrac{1}{\alpha}$. □

Theorem 3.1. *Let $T \in C(H)$ be densely defined. Then $T^*(TT^* + \alpha I)^{-1}$ and $T(T^*T + \alpha I)^{-1}$ are bounded operators on H. Also $\| T^*(TT^* + \alpha I)^{-1} \| \leq \frac{1}{\sqrt{\alpha}}$ and $\| T(T^*T + \alpha I)^{-1} \| \leq \frac{1}{\sqrt{\alpha}}$.*

Proof. We have $(T^*T + \alpha I)^{-1}T^*T = I - \alpha(T^*T + \alpha I)^{-1}$

Since $\langle (T^*T + \alpha I)^{-1}x, x\rangle \geq 0 \ \forall x \in H$,

$$\begin{aligned} \langle (T^*T + \alpha I)^{-1}T^*Tx, x\rangle &= \langle I - \alpha(T^*T + \alpha I)^{-1}x, x\rangle \\ &= \langle x, x\rangle - \alpha\langle (T^*T + \alpha I)^{-1}x, x\rangle \leq \langle x, x\rangle. \end{aligned}$$

Since $(T^*T + \alpha I)^{-1}T^*T$ self adjoint, $\|(T^*T + \alpha I)^{-1}T^*T\| \leq 1$.

Let $x \in H$.

$$\begin{aligned} \|T^*(TT^* + \alpha I)^{-1}x\|^2 &= \langle T^*(TT^* + \alpha I)^{-1}x, T^*(TT^* + \alpha I)^{-1}x\rangle \\ &= \langle TT^*(TT^* + \alpha I)^{-1}x, (TT^* + \alpha I)^{-1}x\rangle \\ &= \langle (TT^* + \alpha I)^{-1}TT^*x, (TT^* + \alpha I)^{-1}x\rangle \\ &\leq \|(TT^* + \alpha I)^{-1}TT^*x\|\|(TT^* + \alpha I)^{-1}x\| \\ &\leq \frac{1}{\alpha}\|x\|^2 \end{aligned}$$

we have $\|T^*(TT^*+\alpha I)^{-1}x\|^2 \leq \dfrac{1}{\alpha}\|x\|^2 \; \forall x \in H$.

Thus $\|T^*(TT^*+\alpha I)^{-1}\| \leq \dfrac{1}{\sqrt{\alpha}}$. Hence $T^*(TT^*+\alpha I)^{-1}$ is bounded. Similarly $T(T^*T+\alpha I)^{-1}$ is bounded.

□

Lemma 3.2. *[7]*
Let $T \in C(H)$ be densely defined. Then
(i) $(TT^*+I)^{-1}T \subseteq T(T^*T+I)^{-1}$
(ii) $(T^*T+I)^{-1}T^* \subseteq T^*(TT^*+I)^{-1}$

Remark 3.1. *From Theorem 3.1 , we have $T^*(TT^*+\alpha I)^{-1}$ and $T(T^*T+\alpha I)^{-1}$ are bounded. Therefore from Lemma 3.2, we have $(TT^*+\alpha I)^{-1}T$ and $(T^*T+\alpha I)^{-1}T^*$ are bounded.*

Lemma 3.3. *Let $T \in C(H)$ be densely defined. For every $x \in D(T) \cap N(T)^{\perp}$ $\|\alpha(T^*T+\alpha I)^{-1}x\| \longrightarrow 0, as \;\; \alpha \to 0.$*

Proof. Let $T_\alpha = \alpha(T^*T+\alpha I)^{-1}$, $\alpha > 0$.
From (3.1) we have $\|(T^*T+\alpha I)^{-1}\| \leq \dfrac{1}{\alpha}$. Hence $\|T_\alpha\| \leq 1$ for every $\alpha > 0$. Let $u \in R(T^*T)$ then there exist $v \in D(T^*T)$ such that $T^*Tv = u$.

$$\begin{aligned}\|T_\alpha u\| &= \|T_\alpha T^*Tv\| \\ &= \alpha\|(T^*T+\alpha I)^{-1}T^*Tv\| \\ &\leq \alpha\|(T^*T+\alpha I)^{-1}T^*T\|\|v\| \\ &\leq \alpha\|v\|\end{aligned}$$

Hence $\|T_\alpha u\| \leq \alpha\|v\| \; \forall u \in R(T^*T)$.
Thus for every $u \in R(T^*T)$, $\|\alpha(T^*T+\alpha I)^{-1}u\| \longrightarrow 0$ as $\alpha \longrightarrow 0$. Since $\overline{R(T^*T)} = N(T)^{\perp}$, $\|\alpha(T^*T+\alpha I)^{-1}x\| \longrightarrow 0$, $\forall x \in D(T) \cap N(T)^{\perp}$.

□

Theorem 3.2. *Let $T \in C(H)$ be densely defined and $R_\alpha = (T^*T+\alpha I)^{-1}T^*$. Then $\{R_\alpha\}_{\alpha>0}$ is a regularization family for (2).*

Proof. Let $y \in D(T^*)$. Then $(T^*T+\alpha I)\widehat{x} = T^*y + \alpha\widehat{x}$.
Hence $\widehat{x} = (T^*T+\alpha I)^{-1}(T^*y+\alpha\widehat{x})$. Thus

$$\begin{aligned}T^\dagger y - R_\alpha y &= \widehat{x} - (T^*T+\alpha I)^{-1}T^*y \\ &= (T^*T+\alpha I)^{-1}(T^*y+\alpha\widehat{x}) - (T^*T+\alpha I)^{-1}T^*y \\ &= (T^*T+\alpha I)^{-1}\alpha\widehat{x}\end{aligned}$$

Hence $\|T^{\dagger}y - R_{\alpha}y\| = \alpha\|(T^*T + \alpha I)^{-1}\widehat{x}\|$.
Since $\widehat{x} \in D(T) \cap N(T)^{\perp}$, by Lemma 3.3, $\|T^{\dagger}y - R_{\alpha}y\| \longrightarrow 0$ as $\alpha \longrightarrow 0$.
Thus $\{R_{\alpha}\}_{\alpha>0}$ is a regularization family for (2).

□

4 Order estimate

In this section we find an error estimate for the regularization family $R_{\alpha} = (T^*T + \alpha I)^{-1}T^*$, where T is a closed densely defined operator. We use the following lemmas.

Lemma 4.1. *[7]*
For $T \in C(H)$ we have the following
(i) If $\mu \in \mathbb{C}$ and $\lambda \in \sigma(T)$ then $\lambda + \mu \in \sigma(T + \mu I)$
(ii) If $\alpha \in \mathbb{C}$ and $\lambda \in \sigma(T)$ then $\alpha\lambda \in \sigma(\alpha T)$
(iii) $\sigma(T^2) = \{\lambda^2 : \lambda \in \sigma(T)\}$

Lemma 4.2. *[7]*
Let $T \in L(H)$ be a positive operator. Then the following results bold.
(i) $T^{\dagger}$ is positive.
(ii) $\sigma(T) = \sigma_a(T)$
(iii) $0 \notin \sigma(I + T)$ that is $(I + T)^{-1} \in B(H)$
(iv) If $0 \notin \sigma(T)$ then $0 \neq \lambda \in \sigma(T)$ if and only if $\frac{1}{\lambda} \in \sigma(T^{-1})$

Theorem 4.1. *Suppose $T \in C(H)$ is densely defined positive operator. Then for every $\alpha > 0$*
$$\sigma\Big((T + \alpha I)^{-2}T\Big) = \Big\{\frac{\lambda}{(\lambda + \alpha)^2} : \lambda \in \sigma(T)\Big\}$$

Proof. Since T is positive, $T + \alpha I$ is bijective.
Also $(T + \alpha I)^{-2}T = (T + \alpha I)^{-1} - \alpha(T + \alpha I)^{-2}$.
From Lemmas 4.1, 4.2 for $\alpha, \lambda > 0$, we have
$\lambda \in \sigma(T)$ if and only if $(\lambda + \alpha)^{-1} \in \sigma\Big((T + \alpha I)^{-1}\Big)$.
Hence
$$\begin{aligned}
\sigma\Big((T + \alpha I)^{-2}T\Big) &= \Big\{\mu - \alpha\mu^2 : \mu \in \sigma\Big((T + \alpha I)^{-1}\Big)\Big\} \\
&= \Big\{\frac{1}{\lambda + \alpha} - \frac{\alpha}{(\lambda + \alpha)^2} : \lambda \in \sigma(T)\Big\} \\
&= \Big\{\frac{\lambda}{(\lambda + \alpha)^2} : \lambda \in \sigma(T)\Big\}.
\end{aligned}$$

□

Corolary 4.1. *Let $T \in C(H)$ be densely defined and $\alpha > 0$.*
Then $\|(T^*T+\alpha I)^{-1}T^*\| = sup\left\{\dfrac{\sqrt{\lambda}}{\lambda+\alpha} : \lambda \in \sigma(T^*T)\right\} \leqslant \dfrac{1}{2\sqrt{\alpha}}$

Proof. We have $R_\alpha = (T^*T+\alpha I)^{-1}T^*$. Hence $R_\alpha^* R_\alpha = T(T^*T+\alpha I)^{-2}T^*$.
From Lemma 2.2 in [2], we have $R_\alpha^* R_\alpha = (TT^*+\alpha I)^{-2}TT^*$.
Since $R_\alpha^* R_\alpha$ is self adjoint and bounded, $\|R_\alpha\|^2 = \|R_\alpha^* R_\alpha\|$

$$= \text{Sup}\left\{|k| : k \in \sigma(R_\alpha^* R_\alpha)\right\}$$
$$= \text{Sup}\left\{\frac{\lambda}{(\lambda+\alpha)^2} : \lambda \in \sigma(TT^*)\right\}$$
$$\|R_\alpha\| = \text{Sup}\left\{\frac{\sqrt{\lambda}}{\lambda+\alpha} : \lambda \in \sigma(TT^*)\right\}.$$

Since $2\sqrt{\alpha\lambda}(\lambda+\alpha)^{-1} \leqslant 1$ for $\lambda, \alpha > 0$, we have $\|R_\alpha\| \leqslant \dfrac{1}{2\sqrt{\alpha}}$. □

Now we find an order estimate for R_α.

Corolary 4.2. *Let $T \in C(H)$ is densely defined and $R_\alpha = (T^*T+\alpha I)^{-1}T^*$. For every $\alpha > 0$ and $\delta > 0$, let $y^\delta \in H$ be such that $\|y - y^\delta\| \leqslant \delta$. Then $\|R_\alpha y - R_\alpha y^\delta\| \leqslant \dfrac{\delta}{2\sqrt{\alpha}}$.*

Proof. For $\|y - y^\delta\| \leqslant \delta$,

$$\begin{aligned} \|R_\alpha y - R_\alpha y^\delta\| &\leqslant \|R_\alpha\|\|y - y^\delta\| \\ &\leqslant \frac{1}{2\sqrt{\alpha}}\|y - y^\delta\| \\ &\leqslant \frac{\delta}{2\sqrt{\alpha}} \end{aligned}$$

□

Theorem 4.2. *Let $T \in C(H)$ is densely defined and $R_\alpha = (T^*T+\alpha I)^{-1}T^*$. Then $\|\widehat{x} - R_\alpha y^\delta\| \leqslant \|\widehat{x} - R_\alpha y\| + \dfrac{\delta}{2\sqrt{\alpha}}$. If $\alpha = \alpha(\delta)$ is chosen such that $\alpha(\delta) \longrightarrow 0$ and $\dfrac{\delta}{\sqrt{\alpha}(\delta)} \longrightarrow 0$ as $\delta \longrightarrow 0$, then $\|\widehat{x} - R^\delta_{\alpha(\delta)}\| \longrightarrow 0$ as $\delta \longrightarrow 0$.*

Proof.
$$\begin{aligned} \|\widehat{x} - R_\alpha y^\delta\| &\le \|\widehat{x} - R_\alpha y\| + \|R_\alpha y - R_\alpha y^\delta\| \\ &\le \|\widehat{x} - R_\alpha y\| + \frac{\delta}{2\sqrt{\alpha}} \end{aligned}$$

by Theorem 3.6, $\|\widehat{x} - R_\alpha y\| \longrightarrow 0$ as $\alpha \longrightarrow 0$.

□

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