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A Result on b-metric space using $\alpha-$compatible mappings

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Abstract

The objective of this paper is to generate a common fixed point theorem in b-metric space using α-compatible and α-continuous mappings. This result generalizes the Theorem proved by J.R. Roshan and others. Further our findings are supported by discussing some valid examples.

Keywords: b-metric space; fixed point; α-fixed point; α-compatible; α-continuous mappings.

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1. Introduction

Fixed point theory plays an important role in mathematics and it is fast growing in the fields of analysis because of its applications in mathematics and allied subjects. Several authors [1, 2, 3, 4] established many results in fixed point theory using various weaker conditions. In the recent past, b-metric space was emerged as one of the generalizations of metric space. During this period Czerwik [5] introduced the concept of $b-$metric space. In recent years, number of well-known fixed point theorems have been established in b-metric space such as [6, 7, 8, 9]. The concept of alpha compatible and alpha continuous mappings were introduced in metric space [10] and some results were established in the recent past under certain weaker conditions. J.R. Roshan, N. Shobkolaei, S.Sedghi and M.Abbas [11] proved a common fixed point theorem using compatible and continuous mappings in b-metric space. In this paper, we use the concept of α-compatible and α-continuous maps and generate a fixed point theorem in b-metric space.

2.2. Preliminaries

Definition 2.1. A function $d: X \times X \to R^{+}$ where X is a nonempty set and $m \geq 1$ is a b-metric space if and only if for each $\phi, \varphi, \gamma \in X$

(i) $d(\phi,\varphi) = 0 \Leftrightarrow \phi = \varphi$

(ii) $d(\phi,\varphi) = d(\varphi,\phi)$

(iii) $d(\phi,\gamma) \leq m[d(\phi, \varphi) + d(\varphi, \gamma)]$.

Definition 2.2. Two self maps M, N of a b-metric space X is said to be compatible if $d(MN\eta_K, NM\eta_K) = 0$ whenever $\{\eta_k\}$ sequence in X such that $M\eta_K = N\eta_K = \gamma$ for some $\gamma \in X$ as $k \to \infty$.

Definition 2.3. A point $\mu \in X$ is said to be an α- fixed point of map $M: X \to X$ if $(\alpha o M)\mu = \mu$.

Remark 2.4. A fixed point is not necessarily α- fixed point and α- fixed point is not necessarily a fixed point. If α=1, the identity map then they coincide.

Example2.5. Let M, $\alpha: R \to R$ be defined by M(u) = $u^2 - 1$ and $\alpha(u) = u^3 + 1$.

Then $(\alpha o M)(0) = \alpha(-1) = 0$ and $(\alpha o M)(1) = \alpha(0) = 1$.

Therefore 0 and 1 are α - fixed points but not fixed points.

Example 2.6. Let M, $\alpha : R \to R$ be defined by M(u) = $\frac{u^2}{3}$ and $\alpha(u) = u^3$.

Here $(\alpha o M)(0) = \alpha(0) = 0$ and M(0) = 0.

Therefore 0 is α - fixed point of M which is also fixed point of M.

Definition 2.7. A pair of self maps M and N of a b-metric space X is called $\alpha-$commuting if $\left((\alpha o M) o (\alpha o N)\right)\mu = \left((\alpha o N) o (\alpha o M)\right)\mu$ for all $\mu \in X$.

The preceding example show the relation between commuting and $\alpha-$commuting mappings.

Example 2.8. Let M, N, $\alpha : R \to R$ be defined by $M(u) = u^4$, $N(u) = \sqrt{u}$ and $\alpha(u) = 3u$ for all $u \in R$.

Then $MN(u) = M(\sqrt{u}) = (\sqrt{u})^4 = u^2$ and $NM(u) = N(u^4) = u^2$.

Therefore $MN(u) = NM(u)$.

Hence M and N are commuting mappings.

`Also for $u \in R$, $(\alpha o M)(u) = \alpha(u^4) = 3u^4$, $(\alpha o N)(u) = \alpha(\sqrt{u}) = 3\sqrt{u}$.

$(\alpha o M) o (\alpha o N)(u) = (\alpha o M)(3\sqrt{u}) = \alpha(3^4 u^2) = 3^5 u^2$

and $(\alpha o N) o (\alpha o M)(u) = (\alpha o N)(3u^4) = \alpha(\sqrt{3u^4}) = 3\sqrt{3}u^2$.

Therefore $(\alpha o N) o (\alpha o M)(u) \neq (\alpha o M) o (\alpha o N)(u)$.

Hence M and N are not $\alpha-$commuting mappings.

Example 2.9. Suppose $M, N, \alpha : R - \{0\} \to R - \{0\}$ given by

$M(u) = u^4$, $N(u) = u^5$ and $\alpha(u) = \frac{1}{u}$ for all $u \in R.$

Then $MN(u) = M(u^5) = u^{20}$ and $NM(u) = N(u^4) = u^{20}$.

Therefore $MN(u) = NM(u)$.

Hence M and N are commuting mappings.

Also for $u \in R - \{0\}$, $(\alpha o M)(u) = \alpha(u^4) = \frac{1}{u^4}$, $(\alpha o N)(u) = \alpha(u^5) = \frac{1}{u^5}$

$(\alpha o M) o (\alpha o N)(u) = (\alpha o M)\left(\frac{1}{u^5}\right) = \alpha\left(T\left(\frac{1}{u^5}\right)\right) = \alpha\left(\frac{1}{u^{20}}\right) = u^{20}$.

$$and\ \ (\alpha oN)o(\alpha oM)(u) = (\alpha oN)\left(\frac{1}{u^4}\right) = \alpha\left(S\left(\frac{1}{u^4}\right)\right) = \alpha\left(\frac{1}{u^{20}}\right) = u^{20}.$$

Therefore $(\alpha oM)o(\alpha oN)(u) = (\alpha oN)o(\alpha oM)(u)$

Hence M and N are $\alpha-$commuting mappings.

Example 2.10. Let M, N, $\alpha : R \to R$ be defined by $M(u) = u^4$, $N(u) = 4u$ and $\alpha(u) = \frac{u}{4}$ for all $u \in R$.

$MN(u) = M(4u) = (4u)^4 = 4^4u^4$, $NM(u) = N(u^4) = 4u^4$.

Therefore $MN(u) \neq NM(u)$.

Hence M and N are not commuting mappings.

Also for $u \in R$, $(\alpha oM)(u) = \alpha(u^4) = \frac{u^4}{4}$, $(\alpha oN)(u) = \alpha(4u) = u$

$$(\alpha oM)o(\alpha oN)(u) = (\alpha oM)(u) = \alpha(u^4) = \frac{u^4}{4}$$

$$(\alpha oN)o(\alpha oM)(u) = (\alpha oN)\left(\frac{u^4}{4}\right) = \alpha(u^4) = \frac{u^4}{4}.$$

Therefore, $(\alpha oM)o(\alpha oN)(u) = (\alpha oN)o(\alpha oM)(u)$.

Hence M and N are $\alpha-$commuting mappings.

Example 2.11. Suppose $M, N, \alpha : R - \{0\} \to R - \{0\}$ given by $M(u) = u^3$, $N(u) = 3u^2$ and $\alpha(u) = \frac{1}{u^2}$ for all $u \in R$.

$MN(u) = M(3u^2) = 3^3u^6$, $NM(u) = N(u^3) = 3u^6$.

Therefore $MN(u) \neq NM(u)$.

Hence M and N are not commuting mappings.

Also for $u \in R - \{0\}$, $(\alpha oM)(u) = \alpha(u^3) = \frac{1}{u^6}$, $(\alpha oN)(u) = \alpha(3u^2) = \frac{1}{3^2u^4}$

$$(\alpha oM)o(\alpha oN)(u) = (\alpha oM)\left(\frac{1}{9u^4}\right) = \alpha\left(M\left(\frac{1}{9u^4}\right)\right) = \alpha\left(\frac{1}{9^3u^{12}}\right) = 9^6u^{24}.$$

$$(\alpha oN)o(\alpha oM)(u) = (\alpha oN)\left(\frac{1}{u^6}\right) = \alpha\left(S\left(\frac{1}{u^6}\right)\right) = \alpha\left(3\left(\frac{1}{u^{12}}\right)\right) = \frac{u^{24}}{9}.$$

Therefore, $(\alpha oM)o(\alpha oN)(u) \neq (\alpha oN)o(\alpha oM)(u)$.

Hence, M and N are not $\alpha-$commuting mappings.

Definition 2.12. A pair of self maps of a b-metric space (X, d) is called weakly $\alpha-$commuting mappings if $(\alpha o M)$ and $(\alpha o N)$ are weakly commuting maps. i.e $d\big(((\alpha o M)o(\alpha o N))(u),((\alpha o N)o(\alpha o M))(u)\big)\le d\big((\alpha o M)(u),(\alpha o N)(u)\big)$ for all $u\in X$.

Definition 2.13. The self maps M,Nofa b-metric space X are called α-compatible maps if $(\alpha o M)$ and $(\alpha o N)$ are compatible if whenever $\{u_n\}$ is a Sequence in X such that $d\big((\alpha o M)(u_n),(\alpha o N)(u_n)\big)\to\mu\in X$, *then*

$d\big(((\alpha o M)o(\alpha o N))(u_n),((\alpha o N)o(\alpha o M))(u_n)\big)\to 0 \quad as \quad n\to\infty.$

Definition 2.14. Two self maps M and N are called weakly $\alpha-$compatible if $(\alpha o M)$ and $(\alpha o N)$ are weakly compatible,i.e $(\alpha o M)$ and $(\alpha o N)$ commute at their coincidence points.

Remark 2.15. It may be observed that $\alpha-$commuting maps are weakly $\alpha-$commuting maps,weakly $\alpha-$commuting maps are $\alpha-$compatible maps and $\alpha-$compatible mappings are weakly $\alpha-$compatible maps.But converse is not true in each case. These facts are presented in the following example.

Example2.16. Let M, N, $\alpha: R-\{0\}\to R-\{0\}$ given by $M(u)=u^5$, $N(u)=u^4$ and $\alpha(u)=\dfrac{1}{u}$ for all $u\in R$.

Here $(\alpha o M)(u)=\alpha(u^5)=\dfrac{1}{u^5}$, $(\alpha o N)(u)=\alpha(u^4)=\dfrac{1}{u^4}$

$$(\alpha o M)o(\alpha o N)(u)=(\alpha o M)\left(\frac{1}{u^4}\right)=\alpha\left(\frac{1}{u^{20}}\right)=u^{20}.$$

$$and\ (\alpha o N)o(\alpha o M)(u)=(\alpha o N)\left(\frac{1}{u^5}\right)=\alpha\left(\frac{1}{u^{20}}\right)=u^{20}.$$

Therefore, $(\alpha o M)o(\alpha o N)(u)=(\alpha o N)o(\alpha o M)(u)$

Hence, N and M are $\alpha-$commuting mappings.

Also for $u\in R-\{0\}$

$$d\big(((\alpha o M)o(\alpha o N))(u),((\alpha o N)o(\alpha o M))(u)\big)=\left|u^{20}-u^{20}\right|^2=0$$

$$d\big((\alpha o M)u,(\alpha o N)u\big)=\left|\frac{1}{u^5}-\frac{1}{u^4}\right|^2=\left|\frac{1-u}{u^5}\right|^2$$

$$d\big(((\alpha oM)o(\alpha oN))(u),((\alpha oN)o(\alpha oM))(u)\big)=0$$

$$\leq\left|\frac{1-u}{u^5}\right|^2=d\big((\alpha oM)u,(\alpha oN)u\big).$$

Therefore M and N are weakly $\alpha-$commuting maps.

Now $(\alpha oM)(u_n)=\frac{1}{u_n^5}$, $(\alpha oN)(u_n)=\frac{1}{u_n^4}$

$$d\big((\alpha oM)(u_n),(\alpha oN)(u_n)\big)=\left|\frac{1}{u_n^5}-\frac{1}{u_n^4}\right|^2\to 0 \ \ as\ u_n\to 1.$$

Hence M and N are $\alpha-$compatible mappings.

Definition 2.16. The self mapping M of a b-metric space (X,d) is said to be $\alpha-$continuous if (αoM) is continuous .In other words for every $\varepsilon\geq 0,$ for all $\delta>0$ such that $d(x,y)\leq\delta\Rightarrow d\big((\alpha oM)u,\ (\alpha oM)v\big)\leq\varepsilon.$

The following theorem was proved in [11].

Theorem 2.17[10]. Let f, g, S and T be four self mappings defined on a complete b-metric space(X, d) with the following conditions:

(C1) $f(X)\subseteq T(X)$ and $g(X)\subseteq S(X)$

(C2)

$$d(fu,gv)\leq\frac{q}{k^4}\max\left\{d(Su,Tv),\ d(fu,SDu),\ d(gv,Tv),\ \frac{1}{2}\big(d(Su,gv)+d(fu,Tv)\big)\right\}$$

holds for every $u,v\ \in X$ with $0<q<1$.

(C3) The self mappings T and S are both continuous

(C4) two pairs $(f,\ S)$and $(g,\ T)$ are compatible.

Then the above four maps will be having a unique fixed point which is common.

Now we prove the generalization of **Theorem (2.17)** in the preceding Theorem under some modified conditions.

To do so, we'll need to recall the following lemmas.

Lemma 2.18[10]. Let (X, d) be a $b-$metric space with $k \geq 1$ and two sequences $\{u_n\}$ and $\{v_n\}$ are b-convergent to u and v respectively. Then we have

$$\frac{1}{k^2} d(u, v) \leq \liminf_{n\to\infty} d(u_n, v_n) \leq \limsup_{n\to\infty} d(u_n, v_n) \leq k^2 d(u, v).$$

Lemma 2.19[9]. Let M and N be $\alpha-$ compatible mappings from a b-metric space (X,d) into itself such that $\lim_{j\to\infty}(\alpha o M)u_n = \lim_{j\to\infty}(\alpha o N)u_n = \mu,\ for\ some\ \mu \in X.$ Then $\lim_{j\to\infty}((\alpha o M) o (\alpha o N))u_n = (\alpha o M)\mu = \mu$, if M is $\alpha-$continuous.

3. Main Result

Theorem 3.1. Let M, N, P and Q be four self maps and α as defined on a b-metric space (X,d) which is complete with the given conditions:

(b1) $(\alpha o M)(X) \subseteq (\alpha o Q)(X)$ and $(\alpha o N)(X) \subseteq (\alpha o P)(X)$

(b2) $$d((\alpha o M)u, (\alpha o N)v) \leq \frac{q}{k^4} \max\left\{ \begin{array}{l} d((\alpha o P)u, (\alpha o Q)v), \\ d((\alpha o M)u, (\alpha o P)u), \\ d((\alpha o N)v, (\alpha o Q)v), \\ \frac{1}{2}\left(\begin{array}{l} d((\alpha o P)u, (\alpha o N)v) \\ + d((\alpha o M)u, (\alpha o Q)v) \end{array} \right) \end{array} \right\},$$

holds for every $u, v \in X$ with $q \in (0,1)$.

(b3) The mappings Q and P are $\alpha-$continuous

(b4) the pair of maps (M, P) and (N, Q) are $\alpha-$compatible.

Then the above four maps will be having a unique fixed point which is common.

Proof:

Using the condition (b1) for the point $u_0 \in X\ \exists\ u_1 \in X$ such that $(\alpha o M)u_0 = (\alpha o Q)u_1$. For this point u_1 we can select a point $u_2 \in X$ such that $(\alpha o N)u_1 = (\alpha o P)u_2$ and so on. Continuing this process it is possible to construct a sequence $\{v_j\}$ such that $v_{2j} = (\alpha o M)u_{2j} = (\alpha o Q)u_{2j+1}$ and $v_{2j+1} = (\alpha o N)u_{2j+1} = (\alpha o P)u_{2j+2}\ \forall\ j \geq 0.$

We now demonstrate that $\{v_j\}$ is a cauchy sequence.

Take $d(v_{2j}, v_{2j+1}) = d((\alpha o M)u_{2j}, (\alpha o N)u_{2j+1})$

$$\leq \frac{q}{k^4}\max\{d((\alpha oP)u_{2j},(\alpha oQ)u_{2j+1}),d((\alpha oM)u_{2j},(\alpha oP)u_{2j}),d((\alpha oN)u_{2j+1},(\alpha oQ)u_{2j+1}),$$
$$\frac{1}{2}(d((\alpha oP)u_{2j},(\alpha oN)u_{2j+1})+d((\alpha oM)u_{2j},(\alpha oQ)u_{2j+1}))\}$$

$$=\frac{q}{k^4}\max\{d(v_{2j-1},v_{2j}),d(v_{2j},v_{2j-1}),d(v_{2j+1},v_{2j}),$$
$$\frac{1}{2}(d(v_{2j-1},v_{2j+1})+d(v_{2j},v_{2j}))\}.$$
$$=\frac{q}{k^4}\max\{d(v_{2j-1},v_{2j}),d(v_{2j},v_{2j+1}),\frac{d(v_{2j-1},v_{2j+1})}{2}\}$$
$$\leq \frac{q}{k^4}\max\{d(v_{2j-1},v_{2j}),d(v_{2j},v_{2j+1}),\frac{k}{2}(d(v_{2j-1},v_{2j})+d(v_{2j},v_{2j+1}))\}.$$

If $d(v_{2j},v_{2j+1})>d(v_{2j-1},v_{2j})$ for some j, then the above inequality gives

$$d(v_{2j},v_{2j+1})\leq \frac{q}{k^3}d(v_{2j},v_{2j+1})$$

a contradiction.

Hence $d(v_{2j},v_{2j+1})\leq d(v_{2j-1},v_{2j})$ for all $j\in N$.

Now the above inequality gives

$d(v_{2j},v_{2j+1})\leq \frac{q}{k^3}d(v_{2j-1},v_{2j})$. -------------------(1)

Similarly $d(v_{2j-1},v_{2j})\leq \frac{q}{k^3}d(v_{2j-2},v_{2j-1})$. ------------(2)

From (1) and (2) we have

$d(v_j,v_{j-1})\leq \lambda d(v_{j-1},v_{j-2})$, where $\lambda=\frac{q}{k^3}<1$ and $j\geq 2$.

Hence for all $j\geq 2$, we obtain

$d(v_j,v_{j-1})\leq \lambda d(v_{j-1},v_{j-2})\leq ....\leq \lambda^{j-1}d(v_1,v_0)$.----------(3)

So for all $j>l$, we have

$$d(y_j,y_l)\leq kd(y_l,y_{l+1})+k^2d(y_{l+1},y_{l+2})+.........+k^{j-l-1}d(y_{j-1},y_j).$$

Now from (3), we have

$$d(v_j,v_l)\leq (k\lambda^l+k^2\lambda^{l+1}+.............+k^{j-l-1}\lambda^{j-1})d(v_1,v_0)$$
$$\leq k\lambda^l(1+k\lambda+k^2\lambda^2+.............)d(v_1,v_0)$$
$$\leq \frac{k\lambda^l}{1-k\lambda}d(v_1,v_0).$$

Taking limits as $l,j\to\infty$, we have $d(v_j,v_l)\to 0$ as $k\lambda$ is less than one.

Therefore $\{v_j\}$ is a cauchy sequence in X and by completeness of X, it converges to some point μ in X such that

$$\lim_{j\to\infty}(\alpha oM)u_{2j} = \lim_{j\to\infty}(\alpha oQ)u_{2j+1} = \lim_{j\to\infty}(\alpha oN)u_{2j+1} = \lim_{j\to\infty}(\alpha oP)u_{2j+2} = \mu.$$

Since P is $\alpha-$continuous, therefore $\lim_{j\to\infty}(\alpha oP)o(\alpha oP)u_{2j+2} = (\alpha oP)\mu$ and $\lim_{j\to\infty}(\alpha oP)o(\alpha oM)u_{2j} = (\alpha oP)\mu.$

By (b4) we have (M,P) is $\alpha-$compatible,

$$\lim_{j\to\infty} d\big((\alpha oM)o(\alpha oP)u_{2j}, (\alpha oP)o(\alpha oM)u_{2j}\big) = 0$$

so by Lemma (2.19), we have $\lim_{j\to\infty}(\alpha oM)o(\alpha oP)u_{2j} = (\alpha oP)\mu$.

Now putting $u = (\alpha oP)u_{2j}$ and $v = u_{2j+1}$ in (b2), we get

take sup limit as $j \to \infty$ on both the sides and by Lemma (2.18),we get

$$d\big((\alpha oM)o(\alpha oP)u_{2j}, (\alpha oN)u_{2j+1}\big) \le \frac{q}{k^4}\max\left\{\begin{array}{l} d\big((\alpha oP)o(\alpha oP)u_{2j}, (\alpha oQ)u_{2j+1}\big), \\ d\big((\alpha oM)o(\alpha oP)u_{2j}, (\alpha oP)o(\alpha oP)u_{2j}\big), \\ d\big((\alpha oN)u_{2j+1}, (\alpha oQ)u_{2j+1}\big), \\ \frac{1}{2}\left(\begin{array}{l} d\big((\alpha oP)o(\alpha oP)u_{2j}, (\alpha oN)u_{2j+1}\big) \\ + d\big((\alpha oM)o(\alpha oP)u_{2j}, (\alpha oQ)u_{2j+1}\big) \end{array}\right) \end{array}\right\},$$

$$\frac{d\big((\alpha oP)\mu, \mu\big)}{k^2} \le \lim_{j\to\infty} d\big((\alpha oM)o(\alpha oP)u_{2j}, (\alpha oN)u_{2j+1}\big)$$

$$\le \frac{q}{k^4}\max\left\{\begin{array}{l} \limsup_{j\to\infty} d\big((\alpha oP)o(\alpha oP)u_{2j}, (\alpha oQ)u_{2j+1}\big), \\ \limsup_{j\to\infty} d\big((\alpha oM)o(\alpha oP)u_{2j}, (\alpha oP)o(\alpha oP)u_{2j}\big), \\ \limsup_{j\to\infty} d\big((\alpha oN)u_{2j+1}, (\alpha oQ)u_{2j+1}\big), \\ \frac{1}{2}\left(\begin{array}{l} \limsup_{j\to\infty} d\big((\alpha oP)o(\alpha oP)u_{2j}, (\alpha oN)u_{2j+1}\big) \\ + \limsup_{j\to\infty} d\big((\alpha oM)o(\alpha oP)u_{2j}, (\alpha oQ)u_{2j+1}\big) \end{array}\right) \end{array}\right\},$$

$$\le \frac{q}{k^4}\max\left\{\begin{array}{l} k^2 d\big((\alpha oP)\mu, \mu\big), d\big((\alpha oP)\mu, (\alpha oP)\mu\big), d(\mu,\mu), \\ \frac{1}{2}\Big(d\big((\alpha oP)\mu, \mu\big) + d\big((\alpha oP)\mu, \mu\big)\Big) \end{array}\right\}$$

$$\le \frac{q}{k^4}\max\left\{\begin{array}{l} k^2 d\big((\alpha oP)\mu, \mu\big), 0, 0, \\ d\big((\alpha oP)\mu, \mu\big) \end{array}\right\}$$

$$\le \frac{q}{k^4}\max\left\{\begin{array}{l} k^2 d\big((\alpha oP)\mu, \mu\big), 0, 0, \\ d\big((\alpha oP)\mu, \mu\big) \end{array}\right\}$$

$$= \frac{q}{k^4} k^2 d\big((\alpha oP)\mu, \mu\big)$$

$= \frac{q}{k^2} d((\alpha o P)\mu, \mu).$

Therefore $d((\alpha o P)\mu, \mu) \le q d((\alpha o P)\mu, \mu)$.

As $0 < q < 1,$ so $(\alpha o P)\mu = \mu$.

Since Q is $\alpha-$continuous, therefore $\lim_{j\to\infty}(\alpha o T)o(\alpha o T)u_{2j+2} = (\alpha o T)\mu$ and $\lim_{j\to\infty}(\alpha o Q)o(\alpha o N)u_{2j} = (\alpha o Q)\mu.$

Since the pair (N,Q) is $\alpha-$compatible, we have

$\lim_{j\to\infty} d((\alpha o N)o(\alpha o Q)u_{2j}, (\alpha o Q)o(\alpha o N)u_{2j}) = 0.$

So by Lemma (2.18) we have $\lim_{j\to\infty}(\alpha o N)o(\alpha o Q)u_{2j} = (\alpha o Q)\mu$.

Now putting $u = u_{2j}$ and $v = (\alpha o Q)u_{2j+1}$ in (b2), we get

$$d((\alpha o M)u_{2j}, (\alpha o N)o(\alpha o Q)u_{2j+1}) \le \frac{q}{k^4}\max\left\{\begin{array}{l} d((\alpha o P)u_{2j}, (\alpha o Q)o(\alpha o Q)u_{2j+1}), \\ d((\alpha o M)u_{2j}, (\alpha o P)u), \\ d((\alpha o N)o(\alpha o Q)u_{2j+1}, (\alpha o Q)o(\alpha o Q)u_{2j+1}), \\ \frac{1}{2}\left(\begin{array}{l} d((\alpha o P)u_{2j}, (\alpha o N)o(\alpha o Q)u_{2j+1}) \\ + d((\alpha o M)u_{2j}, (\alpha o Q)o(\alpha o Q)u_{2j+1}) \end{array}\right) \end{array}\right\}$$

take sup limit as $j \to \infty$ on both the sides and by Lemma (2.18),we get

$$d(\mu, (\alpha o Q)\mu) \le d((\alpha o M)u_{2j}, (\alpha o N)o(\alpha o Q)u_{2j+1}) \le \frac{q}{k^4}\max\left\{\begin{array}{l} k^2 d(\mu, (\alpha o Q)\mu), 0, 0, \\ \frac{k^2}{2}\left(\begin{array}{l} d(\mu, (\alpha o Q)\mu) \\ + d(\mu, (\alpha o Q)\mu) \end{array}\right) \end{array}\right\}$$

$$d(\mu, (\alpha o Q)\mu) \le \frac{q}{k^2} d(\mu, (\alpha o Q)\mu)$$

which implies that $\mu = (\alpha o Q)\mu.$

Therefore $(\alpha o P)\mu = (\alpha o Q)\mu = \mu$.---------(4)

Again putting $u = \mu$ and $v = u_{2j+1}$ in (b2)

$$d((\alpha o M)\mu, (\alpha o N)u_{2j+1}) \le \frac{q}{k^4}\max\left\{\begin{array}{l} d((\alpha o P)\mu, (\alpha o Q)u_{2j+1}), \\ d((\alpha o M)\mu, (\alpha o P)\mu), \\ d((\alpha o N)u_{2j+1}, (\alpha o Q)u_{2j+1}), \\ \frac{1}{2}\left(\begin{array}{l} d((\alpha o P)\mu, (\alpha o N)u_{2j+1}) \\ + d((\alpha o M)\mu, (\alpha o Q)u_{2j+1}) \end{array}\right) \end{array}\right\},$$

take sup limit as $j \to \infty$ on both the sides and by Lemma (2.18) we get

$$d\left(\left(\alpha o M\right)\mu,\mu\right)\le\frac{q}{k^4}\max\left\{\begin{array}{l}k^2 d\left(\left(\alpha o P\right)\mu,\mu\right),k^2 d\left(\left(\alpha o M\right)\mu,\left(\alpha o P\right)\mu\right),\\ k^2 d\left(\mu,\mu\right),\\ \frac{k^2}{2}\left(\begin{array}{l}d\left(\left(\alpha o P\right)\mu,\mu\right)\\ +d\left(\left(\alpha o M\right)\mu,\mu\right)\end{array}\right)\end{array}\right\},$$

$$=\frac{q}{k^2}d\left(\left(\alpha o M\right)\mu,\mu\right).$$

This implies that $d\left(\left(\alpha o M\right)\mu,\mu\right)=0$.

That gives $\left(\alpha o M\right)\mu=\mu$ as $0<q<1$.

Again putting $u=\mu$ and $v=\mu$ in (b2), we get

$$d\left(\left(\alpha o M\right)\mu,\left(\alpha o N\right)\mu\right)\le\frac{q}{k^4}\max\left\{\begin{array}{l}d\left(\left(\alpha o P\right)\mu,\left(\alpha o Q\right)\mu\right),\\ d\left(\left(\alpha o M\right)\mu,\left(\alpha o P\right)\mu\right),\\ d\left(\left(\alpha o N\right)\mu,\left(\alpha o Q\right)\mu\right),\\ \frac{1}{2}\left(\begin{array}{l}d\left(\left(\alpha o P\right)\mu,\left(\alpha o N\right)\mu\right)\\ +d\left(\left(\alpha o M\right)\mu,\left(\alpha o Q\right)\mu\right)\end{array}\right)\end{array}\right\},$$

$$d\left(\mu,\left(\alpha o N\right)\mu\right)\le\frac{q}{k^4}\max\left\{\begin{array}{l}d\left(\mu,\mu\right),d\left(\mu,\mu\right),d\left(\left(\alpha o N\right)\mu,\mu\right),\\ \frac{1}{2}\left(d\left(\mu,\left(\alpha o N\right)\mu\right)+d\left(\mu,\mu\right)\right)\end{array}\right\},$$

$$d\left(\mu,\left(\alpha o N\right)\mu\right)\le\frac{q}{k^4}\max\left\{\begin{array}{l}0,0,d\left(\left(\alpha o N\right)\mu,\mu\right),\\ \frac{1}{2}\left(d\left(\mu,\left(\alpha o N\right)\mu\right)+0\right)\end{array}\right\},$$

$$d\left(\mu,\left(\alpha o N\right)\mu\right)\le\frac{q}{k^4}\max\left\{\begin{array}{l}0,0,d\left(\left(\alpha o N\right)\mu,\mu\right),\\ \frac{1}{2}\left(d\left(\mu,\left(\alpha o N\right)\mu\right)+0\right)\end{array}\right\},$$

$$=\frac{q}{k^2}d\left(\left(\alpha o N\right)\mu,\mu\right)$$

$$\le q d\left(\left(\alpha o N\right)\mu,\mu\right)$$

which implies that $d\left(\mu,\left(\alpha o N\right)\mu\right)=0$

$\mu=\left(\alpha o N\right)\mu$.

Therefore $\left(\alpha o M\right)\mu=\left(\alpha o N\right)\mu=\mu$.----------(5)

Hence from (4) and (5) we obtain

$(\alpha o P)\mu=(\alpha o Q)\mu=(\alpha o M)\mu=(\alpha o N)\mu=\mu$.

Therefore μ is a common $\alpha-$fixed point of M, N, P and Q.

Uniqueness:
Assume that η $(\eta \neq \mu)$ is another common fixed point of the four mappings M, N, P, and Q**.**
Put $u = \eta$ and $v = \mu$ in (b2)

$$d((\alpha o M)\eta,(\alpha o N)\mu) \leq \frac{q}{k^4}\max\left\{\begin{array}{l} d((\alpha o P)\eta,(\alpha o Q)\mu), \\ d((\alpha o M)\eta,(\alpha o P)\eta), \\ d((\alpha o N)\mu,(\alpha o Q)\mu), \\ \frac{1}{2}\left(\begin{array}{l} d((\alpha o P)\eta,(\alpha o N)\mu) \\ + d((\alpha o M)\eta,(\alpha o Q)\mu)\end{array}\right)\end{array}\right\}$$

$$\leq \frac{q}{k^4}\max\left\{\begin{array}{l} k^2 d(\eta,\mu), k^2 d(\eta,\eta), k^2 d(\mu,\mu), \\ \frac{k^2}{2}\left(d(\eta,\mu)+d(\eta,\mu)\right)\end{array}\right\}$$

$$= \frac{q}{k^4}\max\left\{k^2 d(\eta,\mu), 0, 0, k^2 d(\eta,\mu)\right\}$$

$$\frac{d(\eta,\mu)}{k^2} = \frac{q}{k^4}k^2 d(\eta,\mu)$$

$$d(\eta,\mu) \leq q d(\eta,\mu).$$

As $0 < q < 1$, so $\eta = \mu$.

Hence the four maps M, N, P and Q will be having a unique common fixed point.

Now we give an illustration to support our result.

Example 3.2: Suppose $X = [0, 1]$ is a b-metric space $d(u,v) = |u-v|^2$ where $u, v \in X$.

Define the four self maps M, N, P , Q and α as follows

$$\alpha(u) = u \quad ; \quad M(u) = N(u) = \begin{cases} \frac{u+2}{6}, & 0 \leq u \leq \frac{1}{2} \\ \frac{1}{2}, & u = \frac{1}{2} \\ \frac{1}{6}, & \frac{1}{2} < u \leq 1 \end{cases} \quad ;$$

$$Q(u) = P(u) = \begin{cases} \dfrac{4\alpha+3}{6}, & 0 \le u \le \dfrac{1}{2} \\ 1-u, & \dfrac{1}{2} < u \le 1 \end{cases} \quad ;$$

$$M(X) = N(X) = (0.16, 0.33] \cup \left\{\frac{1}{2}\right\}, \; P(X) = Q(X) = \left[0, \frac{1}{2}\right],$$

clearly the condition (b1) is satisfied.

Take a sequence as $u_j = \frac{1}{2} - \frac{1}{j}$ for $j \ge 0$.

Now $\lim_{j\to\infty} Mu_j = \lim_{j\to\infty} M\left(\frac{1}{2}-\frac{1}{j}\right) = \lim_{j\to\infty} \dfrac{\left(\frac{1}{2}-\frac{1}{j}+2\right)}{6} = \frac{5}{6} - \frac{1}{6n} = \frac{5}{6}$

and $\lim_{j\to\infty} Pu_j = \lim_{j\to\infty} Q\left(\frac{1}{2}-\frac{1}{j}\right) = \lim_{j\to\infty} \dfrac{\left(4\left(\frac{1}{2}-\frac{1}{j}\right)+3\right)}{6} = \frac{5}{6}$

that is $\exists$ a sequence $\{u_j\}$ in X such that $\lim_{j\to\infty} Mu_j = \lim_{j\to\infty} Pu_j = \frac{5}{6}$.

Similarly $\lim_{j\to\infty} Nu_j = \lim_{j\to\infty} Qu_j = \frac{5}{6}$. Also

$$MPu_j = MP\left(\frac{1}{2}-\frac{1}{j}\right) = M\left(\frac{4\left(\frac{1}{2}-\frac{1}{j}\right)+3}{6}\right) = M\left(\frac{5}{6}-\frac{2}{3j}\right) = \frac{1}{2} \textit{ as } k \to \infty \quad \textit{and}$$

$$PMu_j = PM\left(\frac{1}{2}-\frac{1}{j}\right) = P\left(\frac{\left(\frac{1}{2}-\frac{1}{j}\right)+2}{6}\right) = P\left(\frac{5}{6}-\frac{1}{6n}\right) = 1 - \frac{5}{6} + \frac{1}{6n} = \frac{1}{6} \textit{ as } j \to \infty.$$

$$So\ that\ \lim_{j\to\infty} d\left(MPu_j, PMu_j\right) = d\left(\frac{1}{2},\frac{5}{6}\right) = \left|\frac{1}{2}-\frac{5}{6}\right|^2 = \frac{4}{36} \neq 0.$$

$$Similarly\ \lim_{j\to\infty} d\left(NQu_j, QNu_j\right) \neq 0.$$

Showing that the pairs (M,P) and (N,Q) are not compatible mappings.

$$Again\ (\alpha oM)(u_j) = (\alpha oM)\left(\frac{1}{2}-\frac{1}{j}\right) = \alpha\left(\frac{\left(\frac{1}{2}-\frac{1}{j}\right)+2}{6}\right) = \alpha\left(\frac{5}{6}-\frac{1}{6j}\right) = \frac{5}{6}\ as\ j\to\infty$$

$$and\ also\ (\alpha oP)(u_j) = (\alpha oP)\left(\frac{1}{2}-\frac{1}{j}\right) = \alpha\left(\frac{4\left(\frac{1}{2}-\frac{1}{j}\right)+3}{6}\right) = \alpha\left(\frac{5}{6}-\frac{2}{3j}\right) = \frac{5}{6}\ as\ j\to\infty.$$

$$(\alpha oM)o(\alpha oP)u_j = (\alpha oM)\left(\frac{5}{6}-\frac{2}{3j}\right) = \alpha\left[M\left(\frac{5}{6}-\frac{2}{3j}\right)\right] = \alpha\left(\frac{1}{6}\right) = \frac{1}{6}\ as\ j\to\infty\ and$$

$$(\alpha oP)o(\alpha oM)u_j = (\alpha oQ)\left(\frac{5}{6}-\frac{1}{6j}\right) = \alpha\left[1-\frac{5}{6}+\frac{1}{6n}\right] = \alpha\left(\frac{1}{6}+\frac{1}{6n}\right) = \frac{1}{6}\ as\ j\to\infty.$$

$$So\ that\ \lim_{j\to\infty} d\left((\alpha oM)o(\alpha oP)u_j, (\alpha oP)o(\alpha oM)u_j\right) = d\left(\frac{1}{6},\frac{1}{6}\right) = \left|\frac{1}{6}-\frac{1}{6}\right|^2 = 0.$$

$$Similarly\ \lim_{j\to\infty} d\left((\alpha oQ)o(\alpha oN)u_j, (\alpha oN)o(\alpha oQ)u_j\right) = 0.$$

Showing that the pairs $(\alpha oM, \alpha oP)$ and $(\alpha oN, \alpha oQ)$ are α-compatible mappings.

Now we fulfill the requirement that the mappings M, N, P and Q satisfy the condition (b2).

We have $(\alpha o M)(u)=0$, $(\alpha o N)(v)=\dfrac{v}{16}$

$$d\big((\alpha o M)u,(\alpha o N)v\big)=d\left(0,\frac{v}{16}\right)=\left|\frac{v}{16}\right|^2=\frac{v^2}{256}.$$

Also $(\alpha o P)(u)=u$ and $(\alpha o Q)(v)=\dfrac{v}{4}$,

$$d\big((\alpha o P)u,(\alpha o Q)v\big)=d\left(u,\frac{v}{4}\right)=\left|u-\frac{v}{4}\right|^2,$$

$$d\big((\alpha o M)u,(\alpha o P)u\big)=d(0,u)=u^2,$$

$$d\big((\alpha o g)v,(\alpha o T)v\big)=d\left(\frac{v}{16},\frac{v}{4}\right)=\left|\frac{v}{16}-\frac{v}{4}\right|^2=\frac{9v^2}{256},$$

$$d\big((\alpha o S)u,(\alpha o g)v\big)=d\left(u,\frac{v}{16}\right)=\left|u-\frac{v}{16}\right|^2,$$

$$d\big((\alpha o M)u,(\alpha o Q)v\big)=d\left(0,\frac{v}{4}\right)=\frac{v^2}{16}.$$

Substituting all these in the inequality (b2), we obtain

$$\frac{v^2}{256}\le\frac{q}{k^4}\max\left\{k^2\left(u-\frac{v}{4}\right)^2,k\ u^2,k^2\ \frac{9v^2}{256},\frac{k^2}{2}\left(\left(u-\frac{v}{16}\right)^2+\frac{v^2}{16}\right)\right\}$$

If we choose $u=0.5$, $v=0.9$ and k = 2 we obtain

$$0.00316\le\frac{q}{16}\max\{0.3024,1,k^2 0.112,\ 0.4948\}$$

$$0.00316\le\frac{q}{16}(1)$$

$$0.00316\le\frac{q}{16}(1)\quad\Rightarrow q=0.05\in(0,1).$$

Hence the condition (b2) is satisfied.

4. Conclusion

This work is focused to generate the existence of common fixed point theorem proved by J.R.Roshan and others mentioned in Theorem (2.17) by employing some weaker conditions $\alpha-$compatible and $\alpha-$continuous mappings instead of compatible and continuous mappings. At the end of the theorem our result is justified with a suitable example.

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