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Estimation of fuzzy metric spaces from metric spaces

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Abstract

In this paper we estimate the new way for analyzing the fuzzy metric spaces from metric spaces using fuzzy fixed point theorem and vice versa. We derive some definitions and theorems for analyzing the metric spaces with new structure of fuzzy metric space using fixed point theorems. Also, we have given new examples for fuzzy metric spaces using fixed point theorem.

Keywords: metric space, fuzzy logic, fixed point theorem.
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1. Introduction

In earlier of 1906, Maurice Freechet provided the concept of metric spaces. In 1922, Banach developed a reliable result called Banach contraction principle on the basis of fixed point theory [15]. Later in 1965, Zadeh [10] defined the way of fuzzy set in metric spaces. This fuzzy concept used in various field of engineering, science and technology. This fuzzy concept is used to analyze the complex state as to easy by simple condition and conversion. Then several mathematicians gave their various concept towards the concept of fuzzy metric space Erceg [4] [5], Diamond and Kolden [7], George & Veeramani [6], Gregori and Romaguera [11]. The analysis of fuzzy metric spaces was introduced in various way and its topologies developed by many researchers. On the line of this, we frame a new structure to analyze the fuzzy metric spaces from metric spaces using fuzzy fixed point theorem [13] [14]. In this paper, we discuss and give various definition and theorem to estimate the fuzzy metric space from metric space and converse also using fixed point theorem[1][9][12][16].

2. Prefatory

In this part, we look back on some basic concepts and results in both metric and fuzzy based metric spaces.

Definition 2.1: [2] A metric space is given by a set X and a distance function $\bar{\mathrm{d}}$: $\mathrm{X} \times \mathrm{X} \rightarrow \mathrm{R}$ defined on X such that $a, b, c \in X$

(i) $\bar{\mathrm{d}}(a,b) \geq 0, \bar{\mathrm{d}}(a,b) = 0 \leftrightarrow a = b$

(ii) $\bar{\mathrm{d}}(a,b) = \bar{\mathrm{d}}(b,a)$

(iii) $\bar{\mathrm{d}}(a,c) \leq \bar{\mathrm{d}}(a,b) + \bar{\mathrm{d}}\,(b,c)$

Definition 2.2: [10] A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.3: [8] A binary operation $* : [0,1]^2 \rightarrow [0,1]$ is called a continuous triangular norm (called t- norm) if it satisfies the following conditions:

(i) * is associative and commutative,

(ii) * is continuous,

(iii) $p * 1 = p$ for all $p, q, r \in [0,1]$,

(iv) $p * q \leq r * t$ whenever $p \leq r$ and $q \leq p$ for all $p, q, r, t \in [0,1]$

Examples of t-norm are $p * q = pq$,$p * q = \min\{p, q\}$ and $p * q = \max\{p, q\}$.

Definition 2.4: [3, 6] The 3-tuple $(X, F_{\bar{d}},*)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t-norm and $F_{\bar{d}}$ is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, for all $a, b, c \in X$, each t and $u > 0$

(i) $F_{\bar{d}}(a, b, t) > 0$
(ii) $F_{\bar{d}}(a, b, t) = 0$ if and only if $a = b$,
(iii) $F_{\bar{d}}(a, b, t) * F_{\bar{d}}(b, a, t)$,
(iv) $F_{\bar{d}}(a, b, t) * F_{\bar{d}}(b, c, u) \le F_{\bar{d}}(a, c, t + u)$,
(v) $F_{\bar{d}}(a, b, \circ)\colon (0, \infty) \to [0,1]$ is continuous.

Then is $F_{\bar{d}}$ called a fuzzy metric on X. Then $F_{\bar{d}}(a, b, t)$ denotes the degree of nearness between a and b with respect to t.

Lemma 2.5: $(X, F_{\bar{d}},*)$ is non-decreasing for all $a, b \in X$.

3. Main Results

The main aim of this paper is to estimate the fuzzy metric spaces from any ordinary metric spaces and converse also and justify the Banach fixed point.

Theorem 3.1: Let $\bar{d}$ and $F_{\bar{d}}$ are metric and fuzzy metric respectively, so the following diagram

$$\begin{array}{ccc} X \times X \times \acute{\mathcal{R}}^{+} & \overset{F_{\bar{d}}}{\Rightarrow} & I \\ dpr \downarrow & & \uparrow \beta \\ X \times X & \overset{\bar{d}}{\Rightarrow} & \acute{\mathcal{R}}^{+} \end{array}$$

Figure (1): commutative diagram

is commutative. Where, $\bar{d}_{xy}\colon (a, b, t) \to (ta, tb), \bar{d}(ta, tb) \to ty$ for some metric $\bar{d}(a, b) = y > 0$ and $\beta\colon (ty) \to 1 - \sin(ty) =\colon \hat{t}\epsilon I$. Further more $F_{\bar{d}} = \beta \circ \overline{d} \circ \overline{d}_{xy}$.

Proof: According to the below equation it is very easy to check that β is continuous.

$\sin^{-1}(ty)$ in $\acute{\mathcal{R}}^{+}$ is continuous $\Longrightarrow \beta$ is continuous.

Now we justify that $\overline{d} \circ \overline{d}_{xy} = \beta^{-1} \circ F_{\bar{d}}$. For $(\mathrm{a}, \mathrm{b}, \mathrm{c}) \in \mathrm{X} \times \mathrm{X} \times \acute{\mathcal{R}}^{+}$, we have,

$\overline{d} \circ \overline{d}_{xy}(a, b, t) = \bar{d}(ta, tab) = ty \coloneqq q > 0$.

On the other hand,

$\beta^{-1} \circ F_{\bar{d}}(a, b, t) = \beta^{-1}(\hat{t}) = \beta^{-1}(1 - \sin ty) = \sin^{-1}\{1 - (1 - \sin ty)\} = ty$

Therefore, the above diagram (Figure 1) is commutative.

Lemma 3.2:
Let $t_1, t_2 \in \acute{\mathcal{R}}^+$, if $t_1 \leq t_2$, then $\beta(t_1) \geq \beta(t_2)$ and $\beta(t_1 + t_2) \geq \max\{\beta(t_1), \beta(t_2)\}$.
Proof:
If $t_1 \leq t_2$, this $\Rightarrow \sin t_1 \leq \sin t_2 \Rightarrow -\sin t_1 \geq -\sin t_2 \Rightarrow 1 - \sin t_1 \geq 1 - \sin t_2$

Hence, $\beta(t_1) \geq \beta(t_2)$. Now, the second declaration is clear.

Theorem 3.3: Let $(X, \overline{d})$ be the metric space and $(X, F_{\bar{d}}, *)$ is a fuzzy metric space with $p * q = \max\{p, q\}$ for all $p, q \in I$. Then for all $a, b, c \in X$,$t, q, y \in \acute{\mathcal{R}}^+$, we have $(X, \beta \circ \overline{d} \circ \overline{d}_{xy}, *)$ is a fuzzy metric space.
Proof:
We verify the fuzzy metric space conditions (i), (ii), (iii) from the above definition (2.4),
For (i) $\beta \circ \overline{d} \circ \overline{d}_{xy}(a, b, t) = \beta \circ \overline{d}(ta, tb) = \beta\left\{\overline{d}(ta, tb)\right\} = \hat{t} > 0$.
For (ii) $\beta \circ \overline{d} \circ \overline{d}_{xy}(a, a, t) = \beta \circ \overline{d}(ta, ta) = \beta\left\{\overline{d}(ta, ta)\right\} = \beta(0) = 1$
For (iii) $\beta \circ \overline{d} \circ \overline{d}_{xy}(a, b, t) = \beta \circ \overline{d}(ta, tb) = \beta\left\{\overline{d}(ta, tb)\right\} = \beta(ty) = \hat{t}$.
Another side,

$$\beta \circ \overline{d} \circ \overline{d}_{xy}(b, a, t) = \beta \circ \overline{d}(tb, ta) = \beta\left\{\overline{d}(tb, ta)\right\} = \beta(ty) = \hat{t} > 0$$

Therefore, symmetric condition satisfied.
Next, we check the condition (iv)

$$\beta \circ \overline{d} \circ \overline{d}_{xy}(a, c, t+u) = \beta \circ \overline{d}\,\{(t+u)a, (t+u)c)$$
$$=\beta\,\{\bar{d}(t+u)a, (t+u)c)$$
$$=\beta\left\{\overline{d}(t+u)r)\right\}$$
$$=\beta\,(tr + ur)$$

Using the above lemma, $\beta(tr + ur) \geq \max\{\beta(tr), \beta(ur)\}$

$$=\beta(tr) * \beta(ur)$$
$$=\beta\{\bar{d}(ta, tb)\} * \beta\{\bar{d}(ua, uc)\}$$
$$=\beta \circ \overline{d} \circ \overline{d}_{xy}(a, b, t) * \beta \circ \overline{d} \circ \overline{d}_{xy}(b, a, u)$$

For (v) is trivially true. Therefore, $(X, \beta \circ \overline{d} \circ \overline{d}_{xy}, *)$ is a fuzzy metric space.
Comments:
On the other hand we reach the metric space from the fuzzy metric space basis on the above commutative diagram (Figure 1). So, if $(X, \beta \circ \overline{d} \circ \overline{d}_{xy}, *)$ is a fuzzy metric space, then the associative metric is $(X, \beta^{-1} \circ F_{\bar{d}} \circ \overline{d}_{xy}{}^{-1})$.

Definition 3.4: Let $(\mathrm{X}, \bar{d})$ be a metric space on X, and $\{a_n\}$ be a sequence in X the n is $\{a_n\}$ called converge Sequence to some fixed $a \in X\ if\ \varepsilon > 0, N \epsilon \mathbb{N}$, $\bar{d}(a_n, a) < \epsilon\ for\ all\ n > N$
We represent $a_n \rightarrow$ a if $\{a_n\}$ converge to a ; and $\{a_n\}$ is called a Cauchy

sequence. $\bar{d}(a_n, a_m) < \epsilon \; for \; all \; n, m > N$

Definition 3.5: Let $(X, \bar{d})$and$(X, F_{\bar{d}},*)$ are metric and fuzzy metric space on X, respectively. And $\{a_n\}$ is a sequence in X then the following is equivalent.

(i) $\{a_n\}$ is convergent in the metric space $(X, \bar{d})$

(ii) $\bar{d}(a_n, a) < \epsilon \; for \; all \; n > N$

(iii) $\{a_n\}$ is convergent in the fuzzy metric space $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$

(iv) For any $0 < \varepsilon < 1$ and $t > 0$ there exists $n > N$ such that $\beta \circ \bar{d} \circ \bar{d}_{xy}(a_n, a, t) > 1 - \varepsilon$

Definition 3.6: A metric space $(X, \bar{d})$ is complete if every Cauchy sequence in X is convergent.

Definition 3.7: A fuzzy metric space $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$ is complete if and only if $(X, \bar{d})$ is complete.

In the below mentioned theorem we prove that if any self map has fixed point theorems in the metric space, then we induced fuzzy metric space for the same point and vice versa. We point to Mihet & Shen et al. [5] for fixed point theorems in the fuzzy metric spaces.

Theorem 3.8: Let $(X, \bar{d})$ be a complete metric space on X, suppose the mapping $S: X \to X$ satisfy the contractive condition, thus $\bar{d}(Sa, Sb) < k\, \bar{d}(a, b)$, for all $a, b \in X, k \in [0,1)$ is a constant. If T has a unique fixed point in X with respect to the metric $(X, \bar{d})$, then T has a unique fixed point with respect to the induced fuzzy metric $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$.

Proof: Suppose that T has a unique fixed point in X with respect to the metric space $(X, \bar{d})$. So, we have $\bar{d}(Sa, a) = 0$ for some x.

Therefore
$$F_{\bar{d}}(Sa, a, t) = \beta \circ \bar{d} \circ \bar{d}_{xy}(Sa, a, t)$$
$$= \beta \circ \{s(Sa), s(Sb)\}$$
$$= \beta(ty) = \beta(0) = 1$$

This represents that $Sa = a$ with respect to the fuzzy metric space $(X, \beta \circ \bar{d} \circ \bar{d}_{xy}, *)$

If there is another fixed point $b \in X$ then,
$$F_{\bar{d}}(a, b, t) = \beta \circ \bar{d} \circ \bar{d}_{xy}(Sa, Sb, t)$$
$$= \beta \circ \{s(Sa), s(Sb)\}$$
$$= \beta(ty) = \beta(0) = 1$$

and therefore, $a = b$.

Theorem 3.9: Let $\bar{d}$ and $F_{\bar{d}}$ are metric and fuzzy metric respectively and commutative.

Where, $\bar{d}_{xy}: (a, b, t) \to (ta, tb), \bar{d}(ta, tb) \to ty$ for some metric $\bar{d}(a, b) = y > 0$ and $\beta: (ty) \to 1 - \cos(ty) =: \hat{t} \epsilon I$. Further more $F_{\bar{d}} = \beta \circ \bar{d} \circ \bar{d}_{xy}$.

Proof for this theorem is similar to the proof of theorem 3.1.

Example 3.10: Let $\bar{d}$ and $F_{\bar{d}}$ are metric and fuzzy metric respectively.
$\bar{d}_{xy}: (a, b, t) \to (ta, tb), \bar{d}(ta, tb) \to ty$ for some metric $\bar{d}(a, b) = y > 0$ and $\beta: (ty) \to 1 - sin^2(ty) =: \hat{t}\epsilon I$. Furthermore $F_{\bar{d}} = \beta \circ \overline{d} \circ \overline{d}_{xy}$. And we prove it is commutative.

Solution: According to the below equation it is very easy to check that β is continuous.

$$\sin^2(ty) \text{ in } \acute{\mathcal{R}}^+ \text{ is continuous} \Longrightarrow \beta \text{ is continuous.}$$

Now we justify that $\overline{d} \circ \overline{d}_{xy} = \beta^{-1} \circ F_{\bar{d}}$. For $(\mathrm{a}, \mathrm{b}, \mathrm{c}) \in \mathrm{X} \times \mathrm{X} \times \acute{\mathcal{R}}^+$, we have,

$$\overline{d} \circ \overline{d}_{xy}(a, b, t) = \bar{d}(ta, tab) = ty \coloneqq q > 0.$$

On the other hand,

$$\beta^{-1} \circ F_{\bar{d}}(a, b, t) = \beta^{-1}(\hat{t})$$

$$= \beta^{-1}(1 - sin^2(ty)) = \sin^{-1}\{\sqrt{1 - \left(1 - sin^2(ty)\right)}\} = ty$$

Therefore, it is commutative.

4. Conclusion

In this article we design a new structure to estimate the fuzzy metric space with the help of metric space using fuzzy fixed point theorems and converse also. We discussed the various definitions and provided the proof for the same definitions with new structure using fuzzy fixed point theorems. We have also given some examples which satisfies the condition of our new structure basis on fixed point theorem. This paper makes away to analysis the applications of fuzzy metric space in various field of engineering.

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