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On decomposition of multistars into multistars

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Abstract

The multistar $S^{w_1,\dots,w_n}$ is the multigraph whose underlying graph is an n-star and the multiplicities of its n edges are $w_1, ..., w_n$. Let G and H be two multigraphs. An H-decomposition of G is a set D of H-subgraphs of G, such that the sum of $\omega(e)$ over all graphs in D which include an edge e, equals the multiplicity of e in G, for all edges e in G. In this paper, we fully characterize $S^{1,2,3}$, $K_{1,m}$ and S^{m^l} decomposable multistars, where m^l is m repeated l times.

Keywords: decomposition; multigraph; multistar
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1 Introduction

If G and H are two simple graphs with out isolated vertices, then G is H-decomposable or H divides G if there exists a partition of the edge set of G into disjoint isomorphic copies of H.

The above definition can be extended to mutigraphs also. Let G and H be two multigraphs. Then the corresponding H-decomposition problem is to decide for a fixed H and an input G, whether such a partition exists. We can formally define the concepts about multigraphs and the multigraph decomposition problems as follows.

Definition 1.1. *A $multigraph$ (V, E, w) consists of a simple underlying graph (V, E) and a multiplicity function $w : E \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.*

The multigraph on an underlying graph G with constant multiplicity λ is denoted by $\lambda.G$ and this is different from λG, denoting λ disjoint copies of G. When referring to a simple graph G as a multigraph we mean $1.G$. An isomophism between multigraphs is an isomophism between their underlying simple graphs which peserves edge multiplicity.

Definition 1.2. *A subgraph H of a multigraph G is a multigraph H whose underlying graph is a subgraph of that of G and its multiplicity function is dominated by the multiplicity function of G, i.e. the multiplicity of an edge in H does not exceed its multiplicity in G.*

Definition 1.3. *An H-subgraph of G is a subgraph of a multigraph G, isomorphic to a multigraph H.*

Definition 1.4. *Let G and H be two multigraphs. An H-decomposition of G is a set D of H-subgraphs of G, such that the sum of $\omega(e)$ over all graphs in D which include an edge e, equals the multiplicity of e in G, for all edges e in G.*

Definition 1.5. *The multistar $S^{w_1,w_2,\cdots,w_n}$ is the multigraph whose underlying graph is an n-star and the multiplicities of its n edges are $w_1, w_2, \cdots, w_n$.*

There are considerable number of papers dealing with an H-decomposition of G and some them are provided in the reference [Shyu, 2013, Lin and Shyu, 1996, Lin, 2010, Lee and Lin, 2005, Lee et al., 2005, Bryant et al., 2001, Bialostocki and Roditty, 1982]. Priesler and Tarsi [Priesler and Tarsi, 2004] showed that, for any multistar H (except a few cases), H-decomposition is NP-complete. Priesler and Tarsi [Priesler and Tarsi, 2005] fully characterized $S^{1,2}$-decomposable multistars in the following theorem.

Theorem 1.1. *[Priesler and Tarsi, 2005] The multistar $S^{w_1,w_2,\cdots,w_n}$, $n \geq 2$ is $S^{1,2}$-decomposable if and only if*

1. $\Sigma_{i=1}^{n} w_i \equiv 0(mod\, 3)$
2. *The number of odd multiplicities among the w_i is at most $\frac{1}{3}(\Sigma_{i=1}^{n} w_i)$*
3. *The largest among the w_i is at most twice the sum of all the others.*

In this paper we fully characterize those multistars which are $S^{1,2,3}$-decomposable, $K_{1,m}$-decomposable, S^{2^m}-decomposable and S^{m^l}-decomposable where m^l denotes m repeated l times.

2 Main Results

2.1 $S^{1,2,3}$ decomposability of S^{w_1,w_2,w_3}

Theorem 2.1. *Let $w_1 \geq w_2 \geq w_3 \geq 2$ be positive integers and $n = \frac{w_1+w_2+w_3}{6}$. Then S^{w_1,w_2,w_3} is $S^{1,2,3}$-decomposable if*

1. $w_1 + w_2 + w_3 \equiv 0(mod\, 6)$
2. $2 \leq w_1 - n \leq 2n$, $2 \leq w_2 - n \leq 2n$
3. $5w_2 \leq w_1 + 7w_3 - 12$
4. $w_1 \leq w_2 + w_3 - 6$ *if m and $l - (\frac{m-1}{2})$ are odd where $m = w_1 - n, l = w_2 - n$.*

Proof. Consider the equations

$$3x_1 + 2x_2 + 1(n - (x_1 + x_2)) = w_1$$
$$3y_1 + 2y + 2 + 1(n - (y_1 + y_2)) = w_2$$
$$3(n - (x_1 + y_1)) + 2(n - (x_2 + y_2)) + 1(n - (2n - (x_1 + x_2 + y_1 + y_2))) = w_3$$

Let the above three equations be called as (A). Firstly we claim that under the given conditions (1),(2),(3) and (4) we can find non negative integers x_1, x_2, y_1, y_2 satisfying equations (A) such that $n - (x_l + x_2) \geq 0$, $n - (y_1 + y_2) \geq 0$, $n - (x_l + y_1) \geq 0$, $n - (x_2 + y_2) \geq 0$, $n - [2n - (x_l + x_2 + y_1 + y_2)] \geq 0$. Let these five inequalities be called as (B). The equations in (A) can be simplified as

$$2x_1 + x_2 = w_1 - n \tag{2.1.1}$$
$$2y_1 + y_2 = w_2 - n \tag{2.1.2}$$
$$2x_1 + 2y_1 + x_2 + y_2 = 4n - w_3 \tag{2.1.3}$$

Since $n = \frac{w_1+w_2+w_3}{6}$, $w_1 - n + w_2 - n = 4n - w_3$. Thus to prove our claim we have to solve the equations (2.1.1), (2.1.2) such that all the inequalities in (B) are satisfied. Observing (2.1.1) and (2.1.2), it is clear that they have a positive integral solution such that all the inequalities in (B) are satisfied if and only if $w_1 - n \leq 2n$ and $w_2 - n \leq 2n$. By condition (2), these inequalities holds. Let $m = w_1 - n,\ l = w_2 - n$.

Case 1: $w_2 - n \leq n$

Here we have to solve the equations $2x_1 + x_2 = m,\ 2y_1 + y_2 = l$.

Subcase 1.1: m **is even**

Take $x_1 = \frac{m}{2}, x_2 = 0, y_1 = 0, y_2 = l$. Thus $x_1+x_2 = \frac{m}{2} \leq n, y_1+y_2 = l \leq n$, $x_l + y_1 = \frac{m}{2} \leq n$, $x_2 + y_2 = l \leq n$. So the only inequality in (B), which has to be verified is $n - (2n - (x_l + x_2 + y_1 + y_2)) \geq 0$. Since $n - (x_l + x_2) \geq 0$ and $n-(y_1+y_2) \geq 0, x_l+x_2+y_1+y_2 \leq 2n$. Thus $n-(2n-(x_l+x_2+y_1+y_2)) \geq 0$ $\Leftrightarrow x_l + x_2 + y_1 + y_2 \geq n \Leftrightarrow \frac{m}{2} + l \geq n \Leftrightarrow \frac{w_1-n}{2} + w_2 - n \geq n \Leftrightarrow w_1 + 2w_2 \geq 5n$ $\Leftrightarrow w_1 + 2w_2 \geq 5(\frac{w_1+w_2+w_3}{6}) \Leftrightarrow 5w_3 \leq w_1 + 7w_2$, which is always true, since $w_1 \geq w_2 \geq w_3$. Thus in this subcase all the inequalities in (B) are satisfied.

Subcase 1.2: m **is odd**

Take $x_1 = \frac{m-1}{2}$, $x_2 = 1, y_1 = 1, y_2 = l - 2$. Here $x_1 + y_1 = \frac{m-1}{2} + 1 \leq n$ [since m is odd and $m \leq 2n$]. $x_2 + y_2 = 1 + l - 2 = l - 1 \leq n$ [since in this subcase $l = w_2 - n \leq n$], $y_1 + y_2 = l - 1 \leq n$ and $x_1 + x_2 = \frac{m-1}{2} + 1 \leq n$. As in the above subcase $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow x_1 + x_2 + y_1 + y_2 \geq n$ $\Leftrightarrow m - 1 + 2l \geq 2n \Leftrightarrow w_1 + 2w_2 - 1 \geq 5n \Leftrightarrow 5w_3 + 6 \leq w_1 + 7w_2$. Since $w_3 \geq 2, w_1 > 2 + n$, $w_2 \geq 2 + n$, we get $w_1 \geq 3$, $w_2 \geq 3$, $w_3 \geq 2$. Also $w_1 \geq w_2 \geq w_3$. Thus $7w_2 \geq 5w_3 + 2w_3 \geq 5w_3 + 4$. Thus $w_1 + 7w_2 \geq 5w_3 + 4 + w_1 \geq 5w_3 + 7 > 5w_3 + 6$. Hence $x_1 + x_2 + y_1 + y_2 \geq n$ and thus $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0$. Thus in this subcase also all the conditions in (B) are satisfied.

Case 2: $w_2 - n > n$

Here also we have to solve the equations $2x_1 + x_2 = m,\ 2y_1 + y_2 = l$.

Subcase 2.1: m **is even and** $l - \frac{m}{2}$ **is even**

Take $x_1 = \frac{m}{2}, x_2 = 0, y_1 = \frac{l-\frac{m}{2}}{2}, y_2 = \frac{m}{2}$. Here $n - (x_l + x_2) = n - \frac{m}{2} \geq 0$, since $\frac{m}{2} \leq n$. $n-(y_1+y_2) = n-(\frac{2l-m}{4}+\frac{m}{2}) = n-\frac{2l+m}{4}$. Thus $n-(y_1+y_2) \geq 0$ $\Leftrightarrow \frac{2l+m}{4} \leq n \Leftrightarrow 2(w_2 - n) + w_1 - n \leq 4n \Leftrightarrow 5w_2 \leq w_1 + 7w_3$, which is true by the given condition (3). Similarly $n - (x_1 + y_1) \geq 0$ and $n - (x_2 + y_2) \geq 0$. As in the above case, $n - (2n - (x_1 + x_2 + y_1 + y_2)) \geq 0 \Leftrightarrow x_1 + x_2 + y_1 + y_2 \geq n \Leftrightarrow$ $\frac{m}{2} + \frac{2l-m}{4} + \frac{m}{2} \geq n \Leftrightarrow 3m + 2l \geq 4n \Leftrightarrow 3w_1 + 2w_2 \geq 9n \Leftrightarrow 3w_3 \leq 3w_1 + w_2$, which is always true since $w_1 \geq w_2 \geq w_3$.

Subcase 2.2: m **is even and** $l - \frac{m}{2}$ **is odd**

Here take $x_1 = \frac{m}{2}$, $x_2 = 0$, $y_1 = \frac{l-\frac{m}{2}+1}{2}$, $y_2 = \frac{m}{2} - 1$. We can easily verify that $n - (x_l + x_2) \geq 0 \Leftrightarrow 5w_2 \leq w_1 + 7w_3 - 12$, which is true by condition (3).

Similarly $n-(y_1+y_2)\geq 0$. Also it easily follows that $n-(x_1+y_1)\geq 0$ and $n-(x_2+y_2)\geq 0$. $n-(2n-(x_1+x_2+y_1+y_2))\geq 0 \Leftrightarrow 3w_1+w_2\geq 3w_3+4$. But $w_1\geq w_2\geq w_3\geq 2$ and by condition (2), $w_1\geq n+2$ and $n=\frac{w_1+w_2+w_3}{6}$. Thus $w_1\geq 4$. Also $3w_1+w_2=w_1+2w_1+w_2\geq w_1+3w_3\geq 3w_3+4$ (since $w_1\geq 4$). Thus $n-(2n-(x_1+x_2+y_1+y_2))\geq 0$ and hence all the inequalities in (B) are satisfied.

Subcase 2.3: m is odd and $l-\frac{m-1}{2}$ is even

Take $x_1=\frac{m-1}{2}$, $x_2=1$, $y_1=\frac{l-\frac{m-1}{2}}{2}$, $y_2=\frac{m-1}{2}$. $n-(x_1+x_2)=n-(\frac{m-1}{2}+1)\geq 0\Leftrightarrow 1+\frac{m-1}{2}\leq n$. This is true since $m\leq 2n$ and m is odd. Similarly $n-(x_2+y_2)\geq 0$. Also $n-(x_1+y_1)\geq 0\Leftrightarrow 5w_2\leq w_1+7w_3+6$, which is true by condition (3). Similarly $n-(y_1+y_2)\geq 0$. Also $n-(2n-(x_1+x_2+y_1+y_2))\geq 0$ $\Leftrightarrow 3w_1+w_2+2\geq 3w_3$, which is always true since $w_1\geq w_2\geq w_3$. Thus all the inequalities in (B) are satisfied.

Subcase 2.4: m is odd and $l-\frac{m-1}{2}$ is odd

Take $x_1=\frac{m-1}{2}$, $x_2=1$, $y_1=\frac{l-\frac{m+1}{2}}{2}$, $y_2=\frac{m+1}{2}$. $n-(x_l+x_2)=n-\frac{m-1}{2}+1)=n-\frac{m+1}{2}$. Since m is odd and $m\leq 2n$, $\frac{m+1}{2}\leq n$. So $n-(x_l+x_2)\geq 0$. $n-(x_2+y_2)=n-(\frac{m+1}{2}+1)\geq 0\Leftrightarrow w_1\leq w_2+w_3-6$, which is true by condition(4). Also we can verify that $n-(x_l+y_1)\geq 0$ and $n-(y_1+y_2)\geq 0$ by condition(3). Similarly $n-(2n-(x_l+x_2+y_1+y_2))\geq 0\Leftrightarrow 3w_1+w_2+6\geq 3w_3$, which is always true. Hence all the conditions in (B) are satisfied in this subcase also. Hence our claim is proved in both cases. Thus using equations (A), we can properly partition w_1 into x_1 copies of 3, x_2 copies of 2 and $n-(x_1+x_2)$ copies of 1's. w_2 can be partitioned into $y_1$copies of 3, y_2 copies of 2 and $n-(y_1+y_2)$ copies of 1. w_3 can be partitioned into $n-(x_1+y_1)$ copies of 3, $n-(x_2+y_2)$ copies of 2 and $n-(2n-(x_1+x_2+y_1+y_2))$ copies of 1. Using these partitions of w_1,w_2,w_3, we can decompose S^{w_1,w_2,w_3} into copies of $S^{1,2,3}$.$\square$

2.2 $K_{1,m}$ decomposability of $S^{w_1,w_2,\cdots,w_n}$

Theorem 2.2. *The multistar $S^{w_1,w_2,\cdots,w_n}$, $w_1\geq w_2\geq\cdots\geq w_n$, is $K_{1,m}$-decomposable ($n\geq m$) if and only if*

1. $\Sigma_{i=1}^{n}w_i\equiv 0(mod\, m)$

2. *For each* $k=1,2,\cdots,m-1$, $\Sigma_{i=1}^{k}w_i\leq\frac{k}{m-k}(w_{k+1}+\cdots+w_n)$

Proof. Suppose the multistar the multistar $S^{w_1,w_2,\cdots,w_n}$, $w_1\geq w_2\geq\cdots\geq w_n$, is $K_{1,m}$-decomposable ($n\geq m$). Then clealy $\Sigma_{i=1}^{n}w_i\equiv 0(mod\, m)$.

To prove (2), assume the contrary. Suppose that $\sum_{i=1}^{k}w_i>\frac{k}{m-k}(w_{k+1}+\cdots+$

w_n), for some k with $1 \leq k \leq m-1$. This implies

$$(m-k)\sum_{i=1}^{k} w_i > k(w_{k+1}+\cdots+w_n) \Rightarrow m\sum_{i=1}^{k} w_i > k\sum_{i=1}^{n} w_i$$
$$\Rightarrow \sum_{i=1}^{k} w_i > \frac{k}{m}(\sum_{i=1}^{n} w_i).$$

This is not possible, since $\frac{1}{m}(\sum_{i=1}^{n} w_i)$ is the number of copies of $K_{1,m}$ to which $S^{w_1,w_2,\cdots,w_n}$ can be decomposed. Each copy of $K_{1,m}$ can contribute at most k to $\sum_{i=1}^{k} w_i$. Thus $\sum_{i=1}^{k} w_i \leq \frac{k}{m}(\sum_{i=1}^{n} w_i)$.

We prove sufficiency by induction on $w = \sum_{i=1}^{n} w_i$. For $w = m$, the multistar is $K_{1,m}$ itself. If $w \geq 2m$, one copy of $K_{1,m}$ is deleted from $s^{w_1,w_2,\cdots,w_m}$ by subtracting m number of 1's from the largest m multiplicities. The multistar obtained after this process still satisfies conditions 1 and 2. Hence by induction the proof follows.□

2.3 S^{2^m} decomposability of $S^{w_1,w_2,\cdots,w_n}$

Theorem 2.3. *The multistar $S^{w_1,w_2,\cdots,w_n}$, $w_1 \geq w_2 \geq \cdots \geq w_n$, is S^{2^m}-decomposable ($n \geq m$) if and only if*

1. $\Sigma_{i=1}^{n} w_i \equiv 0(mod\, 2m)$
2. *For* $1 \leq i \leq n$, $w_i \equiv 0(mod\, 2)$
3. *For each* $k = 1, 2, \cdots, m-1$, $\Sigma_{i=1}^{k} w_i \leq \frac{k}{m-k}(w_{k+1}+\cdots+w_n)$

Proof. Assume that the multistar $S^{w_1,w_2,\cdots,w_m}$ is S^{2^m}-decomposable. Then as in the above theorems conditions 1 and 3 hold. Since $S^{w_1,w_2,\cdots,w_m}$ is S^{2^m}-decomposable, clearly $w_i \equiv 0(mod\, 2)$.

We prove sufficiency by induction on $w = \sum_{i=1}^{n} w_i$. For $w = 2m$, the multistar is S^{2^m} itself. If $w \geq 4m$, delete one copy of S^{2^m} from $S^{w_1,w_2,\cdots,w_n}$ by subtracting m number of 2's from the largest m multiplicities. The multistar obtained after this deletion still satisfies all the three conditions. Hence by induction the proof follows.□

The above two theorems can be generalized to characterize S^{m^l}-decomposable multistars.

2.4 S^{m^l} decomposability of $S^{w_1,w_2,\cdots,w_n}$

Theorem 2.4. *The multistar $S^{w_1,w_2,\cdots,w_n}$, $w_1 \geq w_2 \geq \cdots \geq w_n$, is S^{m^l}-decomposable ($n \geq l$) if and only if*

1. $\Sigma_{i=1}^{n} w_i \equiv 0(mod\, lm)$
2. *For* $1 \leq i \leq n$, $w_i \equiv 0(mod\, m)$
3. *For each* $k = 1, 2, \cdots, l-1$, $\Sigma_{i=1}^{k} w_i \leq \frac{k}{l-k}(w_{k+1} + \cdots + w_n)$

Proof. Assume that the multistar $S^{w_1,w_2,\cdots,w_n}$ is S^{m^m}-decomposable. Then conditions 1,2 and 3 follows as in the above theorem.

The sufficiency can similarly be proved using induction by deleting one copy of S^{m^m} from $S^{w_1,w_2,\cdots,w_n}$ by subtracting l number of m's from the largest l multiplicities.□

3 Conclusions

In this paper we have characterized those multistars which are $S^{1,2,3}$-decomposable, $K_{1,m}$-decomposable, S^{2^m}-decomposable and S^{m^l}-decomposable.

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