MULTIVALUED FUNCTIONS, FUZZY SUBSETS AND JOIN SPACES

Piergiulio CORSINI,* Razieh MAHJOOB**

* Dept. of Biology and Agro-Industrial Economy, Via delle Scienze 208,
33100 UDINE (ITALY)
e-mail: corsini2002@yahoo.com
web site: http://ijpam.uniud.it/journal/curriculum_corsini.htm

** Dept. of Mathematics - Faculty of Basic Science, University of Semnan,
SEMNAN, (IRAN)
ra_mahjoob@yahoo.com

ABSTRACT

One has considered the Hypergroupoid $H_\Gamma = < H; o_\Gamma >$ associated with a multivalued function Γ from H to a set D, defined as follows:

$\forall\, x \in H,\ x\ o_\Gamma\ x = \{y \mid \Gamma(y) \cap \Gamma(x) \neq \emptyset\}$,

$\forall\, (y,z) \in H^2,\ y\ o_\Gamma\ z = y\ o_\Gamma\ y \cup z\ o_\Gamma\ z$,

and one has calculated the fuzzy grade $\partial(H_\Gamma)$ for several functions Γ defined on sets H, such that $|H| \in \{3, 4, 5, 6, 8, 9, 16\}$.

INTRODUCTION

The analysis of the connections between Hyperstructures and Fuzzy Sets dates since 1993 when Corsini defined and studied the join spaces H_μ obtained from the fuzzy set $< H, \mu >$, and a little later Zahedi and Ameri considered fuzzy hypergroups. These subjects were studied in the following years by several scientists in Romania, Iran, Greece, Italy, Canada.

In 1993 Corsini associated a hypergroupoid with every fuzzy subset, and he proved that this hypergroupoid is a join space [8].

In 2003 Corsini [14] associated a fuzzy set μ_H with every hypergroupoid $< H, o >$ and considered the sequence of the fuzzy subsets μ_H and of the join spaces H_μ constructed from a hypergroup. This sequence has been studied in depth for several classes of hypergroups by Corsini [14], Corsini–Cristea [16], [17], [18], Corsini–Leoreanu-Fotea [22], Corsini–Leoreanu–Iranmanesh [23], Cristea [25], [26], Stefanescu–Cristea [70], Leoreanu-Fotea V. – Leoreanu L. [53] .

In this paper one has considered the hypergroupoid $< H, o_\Gamma >$ associated with a multivalued function Γ from a set H to a set D, defined as follows

$\forall\, x \in H,\ x\ o_\Gamma\ x = \{y \mid \Gamma(y) \cap \Gamma(x) \neq \emptyset\}$,

$\forall (y,z) \in H^2$, $y\, o_\Gamma\, z = y\, o_\Gamma\, y \cup z\, o_\Gamma\, z$

and one has calculated the fuzzy grade $\partial(H_\Gamma)$, for several functions Γ defined on sets H such that $|H| \in \{3, 4, 5, 6, 8, 9, 16\}$.

We can remark that we have $\partial(H) = s+1$, for all the examinated cases with the exception of $(1_3^{\,6})$, $(2_3^{\,6})$, $(3_3^{\,6})$, $(1_2^{\,9})$, if $n = 2^s q$, where m.c.d. $(q,2) = 1$.

We remember here some definitions, notations and results which will be the basis of what follows.

With every fuzzy subset $(H; \mu_A)$ of a set H , it is possible to associate a hypergroupoid $\langle H; o_\mu \rangle$, where the hyperoperation $\langle o_\mu \rangle$ is defined by: $\forall (x,y) \in H^2$,

$$\text{(I)} \qquad x\, o_\mu\, y = \{ z \mid \min\{ \mu_A(x), \mu_A(y)\} \le \mu_A(z) \le \max\{ \mu_A(x), \mu_A(y)\} \}$$

One proved [8] that $\langle H; o_\mu \rangle$ is a join space.

With every hypergroupoid $\langle H; o \rangle$, it is possible to associate a fuzzy subset, as follows:

Set $\forall (x,y) \in H^2$, $\forall u \in H$, $\mu_{x,y}(u) = 0 \Leftrightarrow u \notin x\, o\, y$

if $u \in x\, o\, y$, $\mu_{x,y}(u) = 1/|x\, o\, y|$,

set $\forall u \in H$, $A(u) = \sum_{(x,y)\in H^2} \mu_{x,y}(u)$, $Q(u) = \{ (x,y) \mid u \in x\, o\, y \}$, $q(u) = |Q(u)|$,

$$\text{(II)} \qquad \mu_H(u) = A(u) / q(u) , \quad \text{see [14].}$$

So it is clear that, given a hypergroupoid $\langle H; o \rangle$, a sequence of fuzzy subsets and of join spaces is determined $\mu_H = \mu_1, \mu_2, \ldots \mu_{m+1}, \ldots$, $\langle H; o \rangle = {}_0H, {}_1H, \ldots {}_mH \ldots$, such that $\forall j \ge 1$, $\mu_j = \mu_{{}_{j-1}H}$, and ${}_jH$ is the join space associated, after (I), with μ_j .

We call "fuzzy grade of H", if it exists, the number $\partial(H)$ (or f.g.(H)) = $\min\{ s \mid {}_mH \approx {}_{m+1}H \}$ and "strong fuzzy grade of H" , if it exists, the number s.f.g.(H) = $\min\{ s \mid {}_mH = {}_{m+1}H \}$, see [17].

In this paper one has determined

6 hypergroupoids of 3 elements such that $\partial(H) = 0$,
4 hypergroupoids of 3 elements such that $\partial(H) = 1$,
5 hypergroupoids of 4 elements such that $\partial(H) = 0$,
8 hypergroupoids of 4 elements such that $\partial(H) = 1$,
12 hypergroupoids of 4 elements such that $\partial(H) = 2$,
5 hypergroupoids of 4 elements such that $\partial(H) = 3$,
2 hypergroupoids of 5 elements such that $\partial(H) = 1$,
2 hypergroupoids of 6 elements such that $\partial(H)= 1$,
8 hypergroupoids of 6 elements such that $\partial(H) = 2$,
3 hypergroupoids of 6 elements such that $\partial(H) = 3$,
1 hypergroupoid of 8 elements such that $\partial(H)= 4$,
1 hypergroupoid of 9 elements such that $\partial(H)= 2$,
1 hypergroupoid of 16 elements such that $\partial(H)= 5$.

\$ 1. Let Γ be a multivalued function from a set $H = \{u_1, u_2,..., u_n\}$ to a set D, i.e. $\Gamma : H \to P^*(D)$. Then we have the following

THEOREM 1 If there exists $d \in D$, such that $\forall i, \Gamma(u_i) \ni d$, then $\partial(H_\Gamma) = 0$.

Indeed, we have $\forall i, x_i \, o_\Gamma \, x_i = \{u_j \mid \Gamma(u_j) \cap \Gamma(u_j) \neq \emptyset\} = H$, therefore $\forall (i, j), u_i \, o_\Gamma \, u_j = H$. Whence ${}_oH = T$, from which $\forall s, {}_sH = {}_oH$, so $\partial(H_\Gamma) = 0$.

THEOREM 2 Let Γ be a multivalued function from a set H to a set D, that is $\Gamma : H \to P^*(D)$, and let $< o_\Gamma >$ be the hyperoperation defined in H :

$\forall x \in H, \qquad x o_\Gamma x = \{z \mid \Gamma(z) \cap \Gamma(x) \neq \emptyset\}$,

$\forall (y, z), \qquad y \, o_\Gamma \, z = y \, o_\Gamma \, y \cup z \, o_\Gamma \, z$.

Then the hypergroupoid $< H; o_\Gamma >$ is a commutative quasi-join space, that is

$\forall (a, b, c, d) \in H^4$,

(j) $\qquad a / b \cap c / d \neq \emptyset \Rightarrow a \, o_\Gamma \, d \cap b \, o_\Gamma \, c \neq \emptyset$.

Let's suppose $a / b \cap c / d \ni v$, that is $a \in b \, o_\Gamma \, v$, $c \in d \, o_\Gamma \, v$. Then, since

$b \, o \, v = b \, o_\Gamma \, b \cup v \, o_\Gamma \, v, \qquad d \, o_\Gamma \, v = d \, o_\Gamma \, d \cup v \, o_\Gamma \, v$, and

$\forall (x, y) \in H^2, \qquad y \in x \, o_\Gamma \, x \Rightarrow x \in y \, o_\Gamma \, y$,

at least one of the following cases is verified

(I) $a \in b \, o_\Gamma \, b, \quad c \in d \, o_\Gamma \, d$, (II) $a \in b \, o_\Gamma \, b, \quad c \in v \, o_\Gamma \, v$

(III) $a \in v \, o_\Gamma \, v, \quad c \in d \, o_\Gamma \, d$, (IV) $a \in v \, o_\Gamma \, v, \quad c \in v \, o_\Gamma \, v$

(I) implies $b \in a \, o_\Gamma \, a$, whence $b \in a \, o_\Gamma \, d$, and we have also $b \in b \, o_\Gamma \, b \subseteq b \, o_\Gamma \, c$

(II) We find $b \in a \, o_\Gamma \, d \cap b \, o_\Gamma \, c$ as in (I).

(III) We obtain $c \in d \, o_\Gamma \, d \subseteq a \, o_\Gamma \, d$, and also $c \in c \, o_\Gamma \, c \subseteq b \, o_\Gamma \, c$.

(IV) implies $v \in a \, o_\Gamma \, a \subseteq a \, o_\Gamma \, d$ and also $v \in c \, o_\Gamma \, c \subseteq b \, o_\Gamma \, c$.

Therefore the implication (j) is always satisfied whence $< H; o_\Gamma >$ is a quasi-join space.

\$ 2. Set $H = \{u_1, u_2, u_3\}$. Then there are functions $\Gamma : H \rightarrow P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 0, 1.

(1_0^3) Set $\Gamma(u_1) = \{d_1\}, \Gamma(u_2) = \Gamma(u_3) = \{d_2, d_3\}$. We have clearly

$_0H$	u_1	u_2	u_3
u_1	u_1	H	H
u_2		$u_2\, u_3$	$u_2\, u_3$
u_3			$u_2\, u_3$

So $\mu_1(u_1) = 0.467$, $\mu_1(u_2) = \mu_1(u_3) = 0.417$.

It follows $_1H = {}_0H$.

By consequence $\partial(1_0^3) = 0$.

(2_0^3) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	u_1	H	H
u_2		$u_2\, u_3$	$u_2\, u_3$
u_3			$u_2\, u_3$

One obtains $\mu_1(u_1) = 0.467$,

$\mu_1(u_2) = \mu_1(u_3) = 0.417$.

So $_1H = {}_0H$, then $\partial(2_0^3) = 0$.

(3_0^3) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_1\}$.

$_0H$	u_1	u_2	u_3
u_1	H	H	H
u_2		H	H
u_3			H

We have $_1H = {}_0H = T$,

$\partial(3_0^3) = 0$.

(4_0^3) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	$u_1\, u_2$	$u_1\, u_2$	H
u_2		$u_1\, u_2$	H
u_3			u_3

We obtain $\mu(1) = 0.417 = \mu(2)$,
$\mu(3) = 0.467$,
So $\partial(4_0^3) = 0$.

(5_0^3) Set $\Gamma(u_1) = \Gamma(u_2) = \{d_1\}$, $\Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	$u_1\ u_2$	$u_1\ u_2$	H
u_2		$u_1\ u_2$	H
u_3			u_3

As in (4_0^3), we obtain $\partial(5_0^3) = 0$.

(1_1^3) Let $|H| = 3 = |D|$. Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$.

So we have

$_0H$	u_1	u_2	u_3
u_1	u_1	$u_1\ u_2$	$u_1\ u_3$
u_2		u_2	$u_2\ u_3$
u_3			u_3

We have clearly

$\mu_1(u_1) = \mu_1(u_2) = \mu_1(u_3) = 0.6$.

Therefore we obtain $_1H = T$, whence $\partial(1_1^3) = 1$.

(2_1^3) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	H	H	H
u_2		$u_1\ u_2$	H
u_3			$u_1\ u_3$

We obtain : $\mu_1(u_1) = 0.37$,

$\mu_1(u_2) = \mu_1(u_3) = 0.354$.

By consequence,

$_1H$	u_1	u_2	u_3
u_1	u_1	H	H
u_2		$u_2\ u_3$	$u_2\ u_3$
u_3			$u_2\ u_3$

So we have : $\mu_2(u_1) = 0.467$,

$\mu_2(u_2) = \mu_2(u_3) = 0.417$.

From this, we obtain $_2H = {_1H}$, whence $\partial(2_1^3) = 1$.

(3_1^3) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_1, d_2\}$, $\Gamma(u_3) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3
u_1	H	H	H
u_2		$u_1\ u_2$	H
u_3			$u_1\ u_3$

See (2_1^3).

So we obtain again

$\partial(3_1^3) = 1$.

(4_1^3) Set $H = \{u_1, u_2, u_3\}$, $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_1\}$.

So we have

$_0H$	u_1	u_2	u_3
u_1	$u_1\ u_3$	H	H
u_2		$u_2\ u_3$	H
u_3			H

By consequence

$\mu_1(u_1) = 0.354 = \mu_1(u_2)$,

$\mu_1(u_3) = 0.370$.

Therefore we obtain

$_1H$	u_1	u_2	u_3
u_1	$u_1\ u_2$	$u_1\ u_2$	H
u_2		$u_1\ u_2$	H
u_3			u_3

Hence $\mu_2(u_1) = \mu_2(u_2) = 0.4167$, $\mu_2(u_3) = 0.467$. It follows $_2H = {}_1H$. Therefore $\partial(4_1^3) = 1$.

\$ 3. Set $H = \{u_1, u_2, u_3, u_4\}$. Then there are functions $\Gamma : H \to P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 0, 1, 2, 3.

(1_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \Gamma(u_4) = \{d_3, d_4\}$. Then we have

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

We obtain $\mu_1(u_1) = 0.357$,

$\mu_1(u_2) = \mu_1(u_3) = \mu_1(u_4) = 0.300$.

By consequence $_1H = {}_0H$

and therefore $\partial(1_0^4) = 0$.

(2_0^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. Then

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

We have as in (1)

${}_0H = {}_1H$ so $\partial(2_0^4) = 0$.

(3_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. Also in this case

${}_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

By consequence

${}_0H = {}_1H$ from which

$\partial(3_0^4) = 0$.

(4_0^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3, d_4\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4
u_1	H	H	H	H
u_2		H	H	H
u_3			H	H
u_4				H

Clearly, ${}_1H = {}_0H = T$.

So $\partial(4_0^4) = 0$.

(1_1^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	H
u_3			$u_2\ u_3$	$u_2\ u_3\ u_4$
u_4				u_4

Whence

$\mu_1(u_1) = \mu_1(u_3) = 0.333$

$\mu_1(u_2) = 0.344$, $\mu_1(u_4) = 0.405$.

from which we obtain ${}_1H$:

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_3$	$u_1\ u_3$	$u_1\ u_3\ u_2$	H
u_2		$u_1\ u_3$	$u_1\ u_3\ u_2$	H
u_3			u_2	$u_2\ u_4$
u_4				u_4

Hence

$\mu_2(u_1) = \mu_2(u_3) = 0.36$,

$\mu_2(u_2) = 0.394$,

$\mu_2(u_4) = 0.429$.

By consequence $_2H = {_1H}$, then $\partial(1_1^{\,4}) = 1$.

($2_1^{\,4}$) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. So we have

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_4$

Whence

$\mu_1(u_1) = 0.405$

$\mu_1(u_2) = 0.344$, $\mu(u_3) = \mu(u_4) = 0.3$.

We obtain $_1H$:

$_1H$	u_1	u_2	u_3	u_4
u_1	u_1	$u_1\ u_2$	H	H
u_2		u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

So we have : $\mu_2(u_1) = 0.429$,

$\mu_2(u_2) = 0.394$, $\mu_2(u_3) = \mu_2(u_4) = 0.361$

whence one finds that

$_2H = {_1H}$. It follows $\partial(2_1^{\,4}) = 1$.

($3_1^{\,4}$) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. So

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	$u_1\ u_2$	$u_1\ u_3$	$u_1\ u_4$
u_2		u_2	$u_2\ u_3$	$u_2\ u_4$
u_3			u_3	$u_3\ u_4$
u_4				u_4

Then $\forall$ i, $\mu_1(u_i) = 0.571$.

By consequence $_1H = T$ and

therefore $\partial(3_1^{\,4}) = 1$.

(4_1^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_1\}$, $\Gamma(u_3) = \{d_2\}$, $\Gamma(u_4) = \{d_3\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	H	H	H	H
u_2		$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_3			$u_1\ u_3$	$u_1\ u_3\ u_4$
u_4				$u_1\ u_4$

So $\mu_1(u_1) = 0.328$,

$\mu_1(u_2) = \mu_1(u_3) = \mu_1(u_4) = 0.299$.

Hence

$_1H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

So $\mu_2(u_1) = 0.357$

$\mu_2(u_2) = \mu_2(u_3) = \mu_2(u_4) = 0.3$

from which $_2H = {_1H}$. Therefore $\partial(4_1^4) = 1$.

(5_1^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

We have
$\mu_1(u_1) = \mu_1(u_2) = \mu_1(u_3) = \mu_1(u_4) = 0.333$.

So $_1H = T$, whence $\partial(5_1^4) = 1$.

(6_1^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

We have clearly

$\forall$ i, $\mu_1(u_2) = \mu_1(u_1)$.

Therefore $_1H = T$ and by consequence $\partial(6_1^4) = 1$.

(7_1^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_4\}$, $\Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

See (5_1^4) and (6_1^4).

We have $_1H = T$,

whence $\partial(7_1^{4)} = 1$.

(6_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

So $\partial(6_0^4) = 0$.

(5_0^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	H	H	H
u_2		$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_2\ u_3\ u_4$

So $\mu(1) = 0.357$,

$\mu(2) = \mu(3) = \mu(4)=0.3$.

It follows $\partial(5_0^4)=0$.

(1_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \Gamma(u_3) = \{d_2, d_3\}$, $\Gamma(u_4) = \{d_3, d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	H	H
u_2		H	H	H
u_3			H	H
u_4				$u_4\ u_2\ u_3$

One obtains $\mu_1(u_1) = 0.256 = \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.260$,

whence we obtain $_1H$:

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

Therefore $_2H = T$ (the total hypergroup). Then $\partial(1_2^4) = 2$.

(2_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. Then

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	H	H
u_2		$u_1\ u_2\ u_3$	H	H
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_3\ u_4$

whence

$\mu_1(u_1) = 0.292 = \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.3$.

So, we have

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

Then $\mu_1(u_1) = \mu_1(u_4) = \mu_1(u_2) = \mu_1(u_3)$,

whence $_2H = T$, and by consequence

$\partial(2_2^4) = 2$.

(3_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_2, d_4\}$, $\Gamma(u_4) = \{d_3\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	H	H	H	H
u_2		H	H	H
u_3			$u_1\ u_2\ u_3$	H
u_4				$u_1\ u_2\ u_4$

We have $\mu_1(u_1) = \mu_1(u_2) = 0.260$, $\mu_1(u_3) = \mu_1(u_4) = 0.256$, whence we obtain

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Then $_2H = T$, from which $\partial(3_2^4) = 2$.

(4_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_4\}$, $\Gamma(u_3) = \{ d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have:

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	$u_1\ u_2\ u_3$	H
u_2		$u_1\ u_2\ u_4$	H	$u_1\ u_2\ u_4$
u_3			$u_1\ u_3$	H
u_4				$u_2\ u_4$

Then $\mu_1(u_1) = 0.3 = \mu_1(u_2)$,

$\mu_1(u_3) = \mu_1(u_4) = 0.292$.

It follows

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Therefore $_2H = T$, whence $\partial(4_2^4) = 2$.

(5_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	H	H
u_2		H	H	H
u_3			H	H
u_4				$u_2\ u_3\ u_4$

So $\mu_1(u_1) = \mu_1(u_4) = 0.23$,

$\mu_1(u_2) = \mu_1(u_3) = 0.260$.

By consequence, we obtain

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

Therefore we have $_2H = T$, whence $\partial(5_2^4) = 2$.

(6_2^4) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_1, d_2\}$, $\Gamma(u_3) = \{d_2, d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	H	H
u_2		$u_1\ u_2\ u_3$	H	H
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_4				$u_3\ u_4$

Hence $\mu_1(u_1) = 0.292 = \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.3.$

We obtain

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

whence $\forall$ i, $\mu_1(u_i) = \mu_1(u_1)$.

Then $_2H = T$. Therefore $\partial(6_2^4) = 2$.

(7_2^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_4\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	H	H
u_2		H	H	H
u_3			H	H
u_4				$u_2\ u_3\ u_4$

We have $\mu_1(u_1) = \mu_1(u_4) = 0.256$,

$\mu_1(u_2) = \mu_1(u_3) = 0.260.$

So, we obtain $_1H$:

$_1H$	u_1	u_4	u_2	u_3
u_1	$u_1\ u_4$	$u_1\ u_4$	H	H
u_4		$u_1\ u_4$	H	H
u_2			$u_2\ u_3$	$u_2\ u_3$
u_3				$u_2\ u_3$

Then $\forall$ i, $\mu_2(u_i) = 0.389$, so $_2H = T$, and $\partial(7_2^4) = 2$.

(8_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_3			u_3	$u_3\ u_4$
u_4				u_4

We obtain $\mu_1(u_1) = 0.389 = \mu_1(u_2)$,

$\mu_1(u_3) = 0.476 = \mu_1(u_4)$.

By consequence,

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Therefore $_2H = T$ and $\partial(8_2^4) = 2$.

(9_2^4) Set $\Gamma(u_1) = \Gamma(u_2) = \{d_1, d_2\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_4$
u_3			u_3	$u_3\ u_4$
u_4				u_4

See (8_2^4).

Therefore $\partial(9_2^4) = 2$.

(10_2^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_3\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$. We obtain

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1	$u_1\ u_2$	$u_1\ u_3\ u_4$	$u_1\ u_3\ u_4$
u_2		u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

whence $\mu_1(u_1) = \mu_1(u_2) = 0.476$,

$\mu(u_3) = \mu(u_4) = 0.389$.

So, we obtain

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Hence $_2H = T$, from which $\partial(10_2^4) = 2$.

(11_2^4) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_1\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_4$	H	H	$u_1\ u_3\ u_4$
u_2		$u_2\ u_3$	$u_2\ u_3\ u_4$	H
u_3			$u_2\ u_3\ u_4$	H
u_4				$u_1\ u_3\ u_4$

Hence $\mu_1(u_1) = \mu_1(u_3) = 0.291$,

$\mu_1(u_3) = \mu_1(u_4) = 0.3$.

One obtains $_1H$ as follows:

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

From $_1H$, one finds $\mu_2(u_i) = \mu_2(u_j)$, $\forall(i, j)$.

Therefore $_2H = T$, so $\partial(11_2^4) = 2$.

(12_2^4) Let $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3\}$, $\Gamma(u_4) = \{d_1\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_4$	H	H	$u_1\ u_2\ u_4$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	H
u_3			$u_2\ u_3$	H
u_4				$u_1\ u_4$

Then $\mu_1(u_1) = 0.3$, $\mu_1(u_2) = 0.3$,

$\mu_1(u_3) = 0.2917$, $\mu_1(u_4) = 0.2917$.

By consequence,

$_1H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Therefore we have $_2H = T$ (the total hypergroup) whence $\partial(12_2^4) = 2$.

(1_3^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_2\}$, $\Gamma(u_4) = \{d_3\}$.

We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	H	$u_1\ u_2\ u_3$	H
u_2		H	H	H
u_3			$u_1\ u_2\ u_3$	H
u_4				$u_2\ u_4$

$\mu_1(u_1) = 0.272 = \mu_1(u_3)$,

$\mu_1(u_2) = 0.286$, $\mu_1(u_4) = 0.271$,

whence

$_1H$	u_2	u_1	u_3	u_4
u_2	u_2	$u_2\ u_1\ u_3$	$u_2\ u_1\ u_3$	H
u_1		$u_1\ u_3$	$u_1\ u_3$	$u_1\ u_3\ u_4$
u_3			$u_1\ u_3$	$u_1\ u_3\ u_4$
u_4				u_4

So we have $\mu_2(u_2) = 0.405 = \mu_2(u_4)$,

$\mu_2(u_1) = \mu_2(u_3) = 0.34$.

We obtain

$_2H$	u_1	u_3	u_2	u_4
u_1	$u_1\ u_3$	$u_1\ u_3$	H	H
u_3		$u_1\ u_3$	H	H
u_2			$u_2\ u_4$	$u_2\ u_4$
u_4				$u_2\ u_4$

We have clearly

$\mu_3(u_1) = \mu_3(u_3) = \mu_3(u_2) = \mu_3(u_4)$.

Therefore $_3H$ is the total hypergroup of order 4 and $\partial(1_3^4) = 3$.

(2_3^4) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_3, d_4\}$.

We obtain the following

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1 u_2	H	H	H
u_2		H	H	H
u_3			u_2 u_3 u_4	u_2 u_3 u_4
u_4				u_2 u_3 u_4

We have $\mu_1(u_1) = 0.27083$,

$\mu_1(u_2) = 0.286$, $\mu_1(u_3) = \mu_1(u_4) = 0.272$.

Therefore the second hypergroupoid is

$_1H$	u_2	u_3	u_4	u_1
u_2	u_2	u_2 u_3 u_4	u_2 u_3 u_4	H
u_3		u_3 u_4	u_3 u_4	u_3 u_4 u_1
u_4			u_3 u_4	u_3 u_4 u_1
u_1				u_1

Hence $\mu_2(u_2) = \mu_2(u_1) = 0.405$,

$\mu_2(u_3) = \mu_2(u_4) = 0.369$.

By consequence we have again

$_2H$	u_2	u_1	u_3	u_4
u_2	u_2 u_1	H	H	H
u_1		u_2 u_1	H	H
u_3			u_3 u_4	H
u_4				u_3 u_4

From $_2H$ we obtain $_3H = T$. Then $\partial(2_3^4) = 3$.

(3_3^4) If $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \Gamma(u_4) = \{d_4\}$.

$_0H$	u_1	u_2	u_3	u_4
u_1	u_1 u_2	H	H	H
u_2		H	H	H
u_3			u_2 u_3 u_4	u_2 u_3 u_4
u_4				u_2 u_3 u_4

One finds the same sequence as in (2_3^4).

Therefore $\partial(3_3^4) = 3$

(4_3^4) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$. We have

$_0H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3$	H	H
u_2		$u_1\ u_2\ u_3$	H	H
u_3			H	H
u_4				$u_3\ u_4$

So $\mu_1(u_1) = 0.272 = \mu_1(u_2)$,

$\mu_1(u_3) = 0.286$, $\mu_1(u_4) = 0.271$.

Hence

$_1H$	u_3	u_1	u_2	u_4
u_3	u_3	$u_3\ u_1\ u_2$	$u_3\ u_1\ u_2$	H
u_1		$u_1\ u_2$	$u_1\ u_2$	$u_1\ u_2\ u_4$
u_2			$u_1\ u_2$	$u_1\ u_2\ u_4$
u_4				u_4

Then $\mu_2(u_3) = \mu_2(u_4) = 0.405$,

$\mu_2(u_1) = \mu_2(u_2) = 0.369$,

from which we obtain

$_2H$	u_1	u_2	u_3	u_4
u_1	$u_1\ u_2$	$u_1\ u_2$	H	H
u_2		$u_1\ u_2$	H	H
u_3			$u_3\ u_4$	$u_3\ u_4$
u_4				$u_3\ u_4$

Hence $_3H = T$ and $\partial(4_3^4) = 3$.

$ 4. Set $H = \{u_1, u_2, u_3, u_4, u_5\}$. Then there are functions $\Gamma : H \to P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 1, 2.

(1_1^5) Let $|H| = 5 = |D|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4\}$, $\Gamma(u_5) = \{d_5\}$. We have

$_0H$	u_1	u_2	u_3	u_4	u_5
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_5$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_5$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5$
u_4				$u_3\ u_4$	$u_3\ u_4\ u_5$
u_5					u_5

So $\mu_1(u_1) = 0.29167= \mu_1(u_4)$,

$\mu_1(u_2) = \mu_1(u_3) = 0.2936$, $\mu_1(u_5) = 0.370$.

We obtain

${}_1H$	u_5	u_2	u_3	u_1	u_4
u_5	u_5	u_5 u_2 u_3	u_5 u_2 u_3	H	H
u_2		u_2 u_3	u_2 u_3	u_2 u_3 u_1 u_4	u_2 u_3 u_1 u_4
u_3			u_2 u_3	u_2 u_3 u_1 u_4	u_2 u_3 u_1 u_4
u_1				u_1 u_4	u_1 u_4
u_4					u_1 u_4

From this we have $\mu_2(u_5) = 0.348$, $\mu_2(u_2) = \mu_2(u_3) = 0.3067$, $\mu_2(u_1) = \mu_2(u_4) = 0.3$.

So $\mu_2(u_5) > \mu_2(u_2) = \mu(u_3) > \mu(u_1) = \mu(u_4)$.

It follows that ${}_2H = {}_1H$. Therefore $\partial(1_1{}^5) = 1$.

($2_2{}^5$) Set $|H| = 5 = |D|$, $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4\}$, $\Gamma(u_5) = \{d_5\}$. So we have

${}_0H$	u_1	u_2	u_3	u_4	u_5
u_1	u_1 u_2 u_3	u_1 u_2 u_3 u_4	H	u_1 u_2 u_3 u_4	u_1 u_2 u_3 u_5
u_2		u_1 u_2 u_3 u_4	H	u_1 u_2 u_3 u_4	H
u_3			H	H	H
u_4				u_2 u_3 u_4	u_2 u_3 u_4 u_5
u_5					u_3 u_5

Whence we obtain $\mu_1(u_1) = 0.228$, $\mu_1(u_2) = 0.234$, $\mu_1(u_3) = 0.2447$, $\mu_1(u_4) = \mu_1(u_1)$ $\mu_1(u_5) = 0.231$.

${}_1H$	u_3	u_2	u_5	u_1	u_4
u_3	u_3	u_3 u_2	u_3 u_2 u_5	H	H
u_2		u_2	u_2 u_5	u_2 u_5 u_1 u_4	u_2 u_5 u_1 u_4
u_5			u_5	u_5 u_1 u_4	u_5 u_1 u_4
u_1				u_1 u_4	u_1 u_4
u_4					u_1 u_4

We have $\mu_2(u_3) = 0.3852$, $\mu_2(u_2) = 0.3644$, $\mu_2(u_5) = 0.3412$, $\mu_2(u_1) = \mu_2(u_4) = 0.3208$.

So ${}_2H = {}_1H$ and by consequence $\partial(2_2{}^5) = 1$.

$ 5. Set $H = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then there are functions $\Gamma : H \to P^*(D)$ such that the fuzzy grade of the associated sequence is respectively 1, 2, 3.

(1_1^6) Set $|H| = 6 = |D|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \Gamma(u_6) = \{d_5, d_6\}$. We have

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	H	$u_1\ u_2\ u_4\ u_5\ u_6$	$u_1\ u_2\ u_4\ u_5\ u_6$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	H	H	H
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_4				$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$
u_5					$u_4\ u_5\ u_6$	$u_4\ u_5\ u_6$
u_6						$u_4\ u_5\ u_6$

So $\mu_1(u_1) = 0.231667$, $\mu_1(u_2) = 0.2284$, $\mu_1(u_3) = 0.22654$, $\mu_1(u_4) = 0.22656$, $\mu_1(u_5) = \mu_1(u_6) = 0.219$.

Hence we obtain

$_1H$	u_1	u_2	u_4	u_3	u_5	u_6
u_1	u_1	$u_1\ u_2$	$u_1\ u_2\ u_4$	$u_1\ u_2\ u_3\ u_4$	H	H
u_2		u_2	$u_2\ u_4$	$u_2\ u_4\ u_3$	$u_2\ u_4\ u_3\ u_5\ u_6$	$u_2\ u_4\ u_3\ u_5\ u_6$
u_4			u_4	$u_4\ u_3$	$u_4\ u_3\ u_5\ u_6$	$u_4\ u_3\ u_5\ u_6$
u_3				u_3	$u_3\ u_5\ u_6$	$u_3\ u_5\ u_6$
u_5					$u_5\ u_6$	$u_5\ u_6$
u_6						$u_5\ u_6$

Therefore $\mu_2(u_1) = 0.348$, $\mu_2(u_2) = 0.3315$, $\mu_2(u_4) = 0.317$, $\mu_2(u_3) = 0.303$, $\mu_2(u_5) = \mu_2(u_6) = 0.29$.

By consequence $_2H = {}_1H$, hence $\partial(1_1^6) = 1$.

(2_1^6) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_5, d_6\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	u_1	$u_1 u_2 u_3 u_4$	H	$u_1 u_2 u_3 u_4$	$u_1 u_3 u_5 u_6$	$u_1 u_3 u_5 u_6$
u_2		$u_2 u_3 u_4$	$u_2 u_3 u_4 u_5 u_6$	$u_2 u_3 u_4$	$u_2 u_3 u_4 u_5 u_6$	$u_2 u_3 u_4 u_5 u_6$
u_3			$u_2 u_3 u_4 u_5 u_6$	$u_2 u_3 u_4 u_5 u_6$	$u_2 u_3 u_4 u_5 u_6$	$u_2 u_3 u_4 u_5 u_6$
u_4				$u_2 u_3 u_4$	$u_2 u_3 u_4 u_5 u_6$	$u_2 u_3 u_4 u_5 u_6$
u_5					$u_3 u_5 u_6$	$u_3 u_5 u_6$
u_6						$u_3 u_5 u_6$

We obtain

$\mu_1(u_1) = 0.303$

$\mu_1(u_2) = 0.22469$

$\mu_1(u_3) = 0.24$

$\mu_1(u_4) = \mu_1(u_2) =$

$\mu_1(u_5) = \mu_1(u_6)$.

Setting $\{u_2, u_4, u_5, u_6\} = P$, we have

${}_1H$	u_1	u_3	u_2	u_4	u_5	u_6
u_1	u_1	$u_1 u_3$	H	H	H	H
u_3		u_3	H	H	H	H
u_2			P	P	P	P
u_4				P	P	P
u_5					P	P
u_6						P

One finds ${}_2H = {}_1H$. So $\partial(2_1{}^6) = 1$.

($1_2{}^6$) Set $|H| = 6 = |D|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$. So we have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1 u_2$	$u_1 u_2 u_3$	$u_1 u_2 u_3 u_4$	$u_1 u_2 u_3 u_4 u_5$	$u_1 u_2 u_4 u_5$	$u_1 u_2 u_6$
u_2		$u_1 u_2 u_3$	$u_1 u_2 u_3 u_4$	$u_1 u_2 u_3 u_4 u_5$	$u_1 u_2 u_3 u_4 u_5$	$u_1 u_2 u_3 u_6$
u_3			$u_2 u_3 u_4$	$u_2 u_3 u_4 u_5$	$u_2 u_3 u_4 u_5$	$u_2 u_3 u_4 u_6$
u_4				$u_3 u_4 u_5$	$u_3 u_4 u_5$	$u_3 u_4 u_5 u_6$
u_5					$u_4 u_5$	$u_4 u_5 u_6$
u_6						u_6

$\mu_1(u_1) = 0.268$,

$\mu_1(u_2) = 0.267$,

$\mu_1(u_3) = 0.260$,

$\mu_1(u_4) = \mu_1(u_2)$,

$\mu_1(u_5) = \mu_1(u_1)$,

$\mu_1(u_6) = 0.348$.

Therefore we obtain

$_1H$	u_6	u_1	u_5	u_2	u_4	u_3
u_6	u_6	$u_1\ u_5\ u_6$	$u_1\ u_5\ u_6$	$u_1\ u_5\ u_6\ u_2\ u_4$	$u_1\ u_5\ u_6\ u_2\ u_4$	H
u_1		$u_1\ u_5$	$u_1\ u_5$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4\ u_3$
u_5			$u_1\ u_5$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4$	$u_1\ u_5\ u_2\ u_4\ u_3$
u_2				$u_2\ u_4$	$u_2\ u_4$	$u_2\ u_4\ u_3$
u_4					$u_2\ u_4$	$u_2\ u_4\ u_3$
u_3						u_3

So $\mu_2(u_6) = \mu_2(u_3) = 0.315$

$\mu_2(u_1) = \mu_2(u_5) = \mu_2(u_2) = \mu_2(u_4) = 0.279$.

Setting $\Gamma = \{u_1, u_5, u_2, u_4\}$, $Q = \{u_6, u_3\}$ we have

$_2H$	u_1	u_5	u_2	u_4	u_6	u_3
u_1	P	P	P	P	H	H
u_5		P	P	P	H	H
u_2			P	P	H	H
u_4				P	H	H
u_6					Q	Q
u_3						Q

So we have

$\mu_1(u_1) = \mu_1(u_5) = \mu_1(u_2) = \mu_1(u_4) = 0.208$

$\mu_1(u_6) = \mu_1(u_3) = 0.233$

It follows $_3H = {}_2H$, by consequence $\partial(1_2^6) = 2$.

(2_2^6) Set $|D| = 6 = |H|$, $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \{d_5, d_6\}$, $\Gamma(u_6) = \{d_6\}$. We have

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_4\ u_5\ u_6$	$u_1\ u_2\ u_5\ u_6$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_5\ u_6$
u_3			$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_4				$u_3\ u_4\ u_5$	$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$
u_5					$u_4\ u_5\ u_6$	$u_4\ u_5\ u_6$
u_6						$u_5\ u_6$

So we obtain

$\mu_1(u_1) = 0.2467 = \mu_1(u_6)$,

$\mu_1(u_2) = 0.243 = \mu_1(u_5)$,

$\mu_1(u_3) = \mu_1(u_4) = 0.2407$

Whence

$_1H$	u_1	u_6	u_2	u_5	u_3	u_4
u_1	$u_1\ u_6$	$u_1\ u_6$	$u_1\ u_6$ $u_2\ u_5$	$u_1\ u_6$ $u_2\ u_5$	H	H
u_6		$u_1\ u_6$	$u_1\ u_6$ $u_2\ u_5$	$u_1\ u_6$ $u_2\ u_5$	H	H
u_2			$u_2\ u_5$	$u_2\ u_5$	$u_2\ u_5$ $u_3\ u_4$	$u_2\ u_5$ $u_3\ u_4$
u_5				$u_2\ u_5$	$u_2\ u_5$ $u_3\ u_4$	$u_2\ u_5$ $u_3\ u_4$
u_3					$u_3\ u_4$	$u_3\ u_4$
u_4						$u_3\ u_4$

Hence
$\mu_2(u_1) = \mu_2(u_6) = \mu_2(u_3) =$
$\mu_2(u_4) = 0.2667,$
$\mu_2(u_2) = \mu_2(u_5) = 0.2619.$

Therefore we set $P = \{u_1, u_6, u_3, u_4\}$. We obtain

$_2H$	u_1	u_6	u_3	u_4	u_2	u_5
u_1	P	P	P	P	H	H
u_6		P	P	P	H	H
u_3			P	P	H	H
u_4				P	H	H
u_2					$u_2\ u_5$	$u_2\ u_5$
u_5						$u_2\ u_5$

We have clearly $_3H = {}_2H$, whence $\partial(2_2^6) = 2$.

(3_2^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \{d_4, d_5\}$, $\Gamma(u_4) = \{d_5\}$, $\Gamma(u_5) = \{d_5, d_6\}$, $\Gamma(u_6) = \{d_6\}$.

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2$ u_3	$u_1\ u_2$ $u_3\ u_4\ u_5$	$u_1\ u_2$ $u_3\ u_4\ u_5$	H	$u_1\ u_2$ $u_5\ u_6$
u_2		$u_1\ u_2$ u_3	$u_1\ u_2$ $u_3\ u_4\ u_5$	$u_1\ u_2$ $u_3\ u_4\ u_5$	H	$u_1\ u_2$ $u_3\ u_5\ u_6$
u_3			$u_2\ u_3$ $u_4\ u_5$	$u_2\ u_3$ $u_4\ u_5$	$u_2\ u_3$ $u_4\ u_5\ u_6$	$u_2\ u_3$ $u_4\ u_5\ u_6$
u_4				$u_3\ u_4$ u_5	$u_3\ u_4$ $u_5\ u_6$	$u_3\ u_4$ $u_5\ u_6$
u_5					$u_3\ u_4$ $u_5\ u_6$	$u_3\ u_4$ $u_5\ u_6$
u_6						$u_5\ u_6$

So we have
$\mu_1(u_1) = 0.233,$
$\mu_1(u_2) = 0.230,$
$\mu_1(u_3) = 0.228,$
$\mu_1(u_4) = 0.218,$
$\mu_1(u_5) = \mu_1(u_3) = 0.228,$
$\mu_1(u_6) = 0.231.$

We obtain

$_1H$	u_1	u_6	u_2	u_3	u_5	u_4
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_2$	$u_1\ u_6\ u_2\ u_3\ u_5$	$u_1\ u_6\ u_2\ u_3\ u_5$	H
u_6		u_6	$u_6\ u_2$	$u_6\ u_2\ u_3\ u_5$	$u_6\ u_2\ u_3\ u_5$	$u_6\ u_2\ u_3\ u_5\ u_4$
u_2			u_2	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5\ u_4$
u_3				$u_3\ u_5$	$u_3\ u_5$	$u_3\ u_5\ u_4$
u_5					$u_3\ u_5$	$u_3\ u_5\ u_4$
u_4						u_4

We have

$\mu_2(u_1) = 0.345 > \mu_2(u_6) = 0.326 > \mu_2(u_4) = 0.324 > \mu_2(u_2) = 0.306 > \mu_2(u_3) = \mu_2(u_5) = 0.296.$

Therefore we have

$_2H$	u_1	u_6	u_4	u_2	u_3	u_5
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_4$	$u_1\ u_6\ u_4\ u_2$	H	H
u_6		u_6	$u_6\ u_4$	$u_6\ u_4\ u_2$	$u_6\ u_4\ u_2\ u_3\ u_5$	$u_6\ u_4\ u_2\ u_3\ u_5$
u_4			u_4	$u_4\ u_2$	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5$
u_2				u_2	$u_2\ u_3\ u_5$	$u_2\ u_3\ u_5$
u_3					$u_3\ u_5$	$u_3\ u_5$
u_5						$u_3\ u_5$

We obtain now

$\mu_3(u_1) = 0.348 > \mu_3(u_6) = 0.33158 > \mu_3(u_4) = 0.317 > \mu_3(u_2) = 0.303 > \mu_3(u_3) = \mu_3(u_5) = 0.29.$

Therefore $_3H = {}_2H$ and it follows that $\partial(3_2^6) = 2$.

(4_2^6) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_2\}$, $\Gamma(u_5) = \{d_3\}$, $\Gamma(u_6) = \{d_4\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\, u_2\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_6$
u_2		$u_1\, u_2\, u_3\, u_4\, u_5$	H	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_5$	H
u_3			$u_2\, u_3\, u_5\, u_6$	H	$u_2\, u_3\, u_5\, u_6$	$u_2\, u_3\, u_5\, u_6$
u_4				$u_1\, u_2\, u_4$	$u_1\, u_2\, u_3\, u_4\, u_5$	$u_1\, u_2\, u_3\, u_4\, u_6$
u_5					$u_2\, u_3\, u_5$	$u_2\, u_3\, u_5\, u_6$
u_6						$u_3\, u_6$

We find :

$\mu_1(u_1) = 0.210 = \mu_1(u_4)$

$\mu_1(u_2) = 0.221$

$\mu_1(u_3) = 0.216$

$\mu_1(u_5) = 0.208$

$\mu_1(u_6) = 0.219.$

Hence we obtain

${}_1H$	u_5	u_4	u_1	u_3	u_6	u_2
u_5	u_5	$u_5\, u_4\, u_1$	$u_5\, u_4\, u_1$	$u_5\, u_4\, u_1\, u_3$	$u_5\, u_4\, u_1\, u_3\, u_6$	H
u_4		$u_4\, u_1$	$u_4\, u_1$	$u_3\, u_1\, u_4$	$u_4\, u_1\, u_3\, u_6$	$u_4\, u_1\, u_3\, u_6\, u_2$
u_1			$u_4\, u_1$	$u_3\, u_1\, u_4$	$u_4\, u_1\, u_3\, u_6$	$u_4\, u_1\, u_3\, u_6\, u_2$
u_3				u_3	$u_3\, u_6$	$u_3\, u_6\, u_2$
u_6					u_6	$u_6\, u_2$
u_2						u_2

We have :

$\mu_2(u_5) = 0.324$

$\mu_2(u_2) = 0.345$

$\mu_2(u_6) = 0.326$

$\mu_2(u_3) = 0.3058$

$\mu_2(u_1) = \mu_2(u_4) = 0.296.$

Therefore $\mu_2(u_2) > \mu_2(u_6) > \mu_2(u_5) > \mu_2(u_3) > \mu_2(u_1) = \mu_2(u_4)$.

Therefore we have

${}_2H$	u_2	u_6	u_5	u_3	u_1	u_4
u_2	u_2	$u_2\, u_6$	$u_2\, u_6\, u_5$	$u_2\, u_6\, u_5\, u_3$	H	H
u_6		u_6	$u_6\, u_5$	$u_6\, u_5\, u_3$	$u_6\, u_5\, u_3\, u_1\, u_4$	$u_6\, u_5\, u_3\, u_1\, u_4$
u_5			u_5	$u_5\, u_3$	$u_5\, u_3\, u_1\, u_4$	$u_5\, u_3\, u_1\, u_4$
u_3				u_3	$u_3\, u_1\, u_4$	$u_3\, u_1\, u_4$
u_1					$u_1\, u_4$	$u_1\, u_4$
u_4						$u_1\, u_4$

We can see that ${}_3H = {}_2H$ and it follows that $\partial(4_2^6) = 2$.

(5_2^6) Set $\Gamma(u_1) = \{d_1\}$, $\Gamma(u_2) = \{d_2, d_3\}$, $\Gamma(u_3) = \{d_3, d_4\}$, $\Gamma(u_4) = \{d_4, d_5\}$, $\Gamma(u_5) = \{d_5, d_6\}$ $\Gamma(u_6) = \{d_6\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	u_1	u_1 u_2 u_3	u_1 u_2 u_3 u_4	u_1 u_3 u_4 u_5	u_1 u_4 u_5 u_6	u_1 u_5 u_6
u_2		u_2 u_3	u_2 u_3 u_4	u_2 u_3 u_4 u_5	u_2 u_3 u_4 u_5 u_6	u_2 u_3 u_5 u_6
u_3			u_2 u_3 u_4	u_2 u_3 u_4 u_5	u_2 u_3 u_4 u_5 u_6	u_2 u_3 u_4 u_5 u_6
u_4				u_3 u_4 u_5	u_3 u_4 u_5 u_6	u_3 u_4 u_5 u_6
u_5					u_4 u_5 u_6	u_4 u_5 u_6
u_6						u_5 u_6

We obtain $\mu_1(u_1) = 0.348$, $\mu_1(u_2) = 0.268 = \mu_1(u_6)$, $\mu_1(u_3) = 0.2667 = \mu_1(u_5)$, $\mu_1(u_4) = 0.260$.

So, we have

${}_1H$	u_1	u_2	u_6	u_3	u_5	u_4
u_1	u_1	u_1 u_2 u_6	u_1 u_2 u_6	u_1 u_2 u_6 u_3 u_5	u_1 u_2 u_6 u_3 u_5	H
u_2		u_2 u_6	u_2 u_6	u_2 u_6 u_3 u_5	u_2 u_6 u_3 u_5	u_2 u_6 u_5 u_4 u_3
u_6			u_2 u_6	u_2 u_6 u_3 u_5	u_2 u_6 u_3 u_5	u_2 u_6 u_5 u_4 u_3
u_3				u_3 u_5	u_3 u_5	u_5 u_4 u_3
u_5					u_3 u_5	u_5 u_4 u_3
u_4						u_4

Now we obtain $\mu_2(u_1) = 0.324$, $\mu_2(u_4) = 0.315$, $\mu_2(u_5) = \mu_2(u_3) = \mu_2(u_2) = \mu_2(u_6) = 0.279$.

Setting $P = \{ u_2, u_6, u_3, u_5 \}$, we find ${}_2H$

${}_2H$	u_1	u_4	u_2	u_6	u_3	u_5
u_1	u_1 u_4	u_1 u_4	H	H	H	H
u_4		u_1 u_4	H	H	H	H
u_2			P	P	P	P
u_6				P	P	P
u_3					P	P
u_5						P

We have clearly ${}_3H = {}_2H$ whence $\partial(5_2^6) = 2$.

(6_2^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_4, d_5\}$, $\Gamma(u_4) = \{d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$. So, denoting $\{u_i, u_{i+1},\ldots, u_{j-1}, u_j\}$ by u_i^j, we have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	u_1^3	u_1^5	H	u_1^5	$u_1\ u_2\ u_4\ u_6$
u_2		u_1^3	u_1^5	H	u_1^5	$u_1^4\ u_6$
u_3			u_2^5	u_2^6	u_2^5	u_2^6
u_4				u_3^6	u_3^6	u_3^6
u_5					u_3^5	u_3^6
u_6						$u_4\ u_6$

We have $\mu_1(u_1) = 0.233$,

$\mu_1(u_2) = 0.2302$,

$\mu_1(u_3) = \mu_1(u_4) = 0.228$,

$\mu_1(u_5) = 0.218$, $\mu_1(u_6) = 0.2308$.

Hence

${}_1H$	u_1	u_6	u_2	u_3	u_4	u_5
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_2$	$u_1\ u_6\ u_2\ u_3\ u_4$	$u_1\ u_6\ u_2\ u_3\ u_4$	H
u_6		u_6	$u_6\ u_2$	$u_6\ u_2\ u_3\ u_4$	$u_6\ u_2\ u_3\ u_4$	$u_6\ u_2\ u_3\ u_4\ u_5$
u_2			u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_5\ u_3\ u_4$
u_3				$u_3\ u_4$	$u_3\ u_4$	$u_3\ u_4\ u_5$
u_4					$u_3\ u_4$	$u_3\ u_4\ u_5$
u_5						u_5

We have $\mu_2(u_1) = 0.345454$,

$\mu_2(u_6) = 0.326316$,

$\mu_2(u_2) = 0.305797$,

$\mu_2(u_3) = \mu_2(u_4) = 0.29615$,

$\mu_2(u_5) = 0.32424$.

From this, we have ${}_2H$ as follows

${}_2H$	u_1	u_6	u_5	u_2	u_3	u_4
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_5$	$u_1\ u_6\ u_5\ u_2$	H	H
u_6		u_6	$u_6\ u_5$	$u_6\ u_5\ u_2$	$u_6\ u_5\ u_2\ u_3\ u_4$	$u_6\ u_5\ u_2\ u_3\ u_4$
u_5			u_5	$u_5\ u_2$	$u_5\ u_2\ u_3\ u_4$	$u_5\ u_2\ u_3\ u_4$
u_2				u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3					$u_3\ u_4$	$u_3\ u_4$
u_4						$u_3\ u_4$

One can see that ${}_3H = {}_2H$,

therefore $\partial(6_2^6) = 2$.

(7_2^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\,u_3$	$u_1\,u_2\,u_3\,u_4$	$u_1\,u_2\,u_3\,u_4\,u_5$	H	$u_1\,u_3\,u_4\,u_5$	$u_1\,u_3\,u_4\,u_6$
u_2		$u_2\,u_3\,u_4$	$u_1\,u_2\,u_3\,u_4\,u_5$	$u_2\,u_3\,u_4\,u_5\,u_6$	$u_2\,u_3\,u_4\,u_5$	$u_2\,u_3\,u_4\,u_6$
u_3			$u_1\,u_2\,u_3\,u_4\,u_5$	H	$u_1\,u_2\,u_3\,u_4\,u_5$	H
u_4				$u_2\,u_3\,u_4\,u_5\,u_6$	$u_2\,u_3\,u_4\,u_5\,u_6$	$u_2\,u_3\,u_4\,u_5\,u_6$
u_5					$u_3\,u_4\,u_5$	$u_3\,u_4\,u_5\,u_6$
u_6						$u_4\,u_6$

We have $\mu_1(u_1) = 0.22$, $\mu_1(u_2) = 0.20864$, $\mu_1(u_3) = 0.22762$, $\mu_1(u_4) = \mu_1(u_3)$, $\mu_1(u_5) = \mu_1(u_2) = 0.20864$, $\mu_1(u_6) = \mu_1(u_1) = 0.22$. Hence,

$\mu_1(u_3) = \mu_1(u_4) = 0.22762 > \mu_1(u_1) = \mu_1(u_6) = 0.22 > \mu_1(u_2) = \mu_1(u_5) = 0.20864$. We obtain

$_1H$	u_3	u_4	u_1	u_6	u_2	u_5
u_3	$u_3\,u_4$	$u_3\,u_4$	$u_3\,u_4\,u_1\,u_6$	$u_3\,u_4\,u_1\,u_6$	H	H
u_4		$u_3\,u_4$	$u_3\,u_4\,u_1\,u_6$	$u_3\,u_4\,u_1\,u_6$	H	H
u_1			$u_1\,u_6$	$u_1\,u_6$	$u_1\,u_6\,u_2\,u_5$	$u_1\,u_6\,u_2\,u_5$
u_6				$u_1\,u_6$	$u_1\,u_6\,u_2\,u_5$	$u_1\,u_6\,u_2\,u_5$
u_2					$u_2\,u_5$	$u_2\,u_5$
u_5						$u_2\,u_5$

We have $\mu_2(u_3) = \mu_2(u_4) = 0.2667$, $\mu_2(u_1) = 0.2619 = \mu_2(u_6)$, $\mu_2(u_2) = \mu_2(u_5) = \mu_2(u_3) = \mu_2(u_4) = 0.2667$.

Set $P = \{u_3, u_4, u_2, u_5\}$, $Q= \{u_1, u_6\}$. We obtain

$_2H$	u_3	u_4	u_2	u_5	u_1	u_6
u_3	P	P	P	P	H	H
u_4		P	P	P	H	H
u_2			P	P	H	H
u_5				P	H	H
u_1					Q	Q
u_6						Q

It follows that $\mu_3(u_3) = \mu_3(u_4) = \mu_3(u_2) = \mu_3(u_5) = 0.208$, $\mu_3(u_1) = \mu_3(u_6) = 0.233$.

We have clearly $_3H = {}_2H$, so $\partial(7_2^6) = 2$.

(8_2^6) Set $\Gamma(u_1) = \{d_1, d_2\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$ $\Gamma(u_4) = \{d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2$	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_4\ u_6$
u_2		$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_3\ u_4\ u_6$
u_3			$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_4				$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$	$u_3\ u_4\ u_5\ u_6$
u_5					$u_3\ u_4\ u_5$	$u_3\ u_4\ u_5\ u_6$
u_6						$u_4\ u_6$

We obtain $\mu_1(u_1) = 0.233$, $\mu_1(u_2) = 0.230247$, $\mu_1(u_3) = 0.228125$, $\mu_1(u_4) = \mu_1(u_3)$ $\mu_1(u_5) = 0.218518$, $\mu_1(u_6) = 0.230833$. From this, we have $_1H$.

$_1H$	u_1	u_6	u_2	u_3	u_4	u_5
u_1	u_1	$u_1\ u_6$	$u_1\ u_2\ u_6$	$u_1\ u_6\ u_2\ u_3\ u_4$	$u_1\ u_6\ u_2\ u_3\ u_4$	H
u_6		u_6	$u_2\ u_6$	$u_2\ u_6\ u_3\ u_4$	$u_2\ u_6\ u_3\ u_4$	$u_2\ u_6\ u_3\ u_4\ u_5$
u_2			u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4\ u_5$
u_3				$u_3\ u_4$	$u_3\ u_4$	$u_3\ u_4\ u_5$
u_4					$u_3\ u_4$	$u_3\ u_4\ u_5$
u_5						u_5

From $_1H$ we obtain : $\mu_2(u_1) = 0.34545$, $\mu_2(u_6) = 0.3263$, $\mu_2(u_5) = 0.324242$, $\mu_2(u_2) = 0.305797$, $\mu_2(u_3) = \mu_2(u_4) = 0.296$.

Therefore we find $_2H$ as follows

$_2H$	u_1	u_6	u_5	u_2	u_3	u_4
u_1	u_1	$u_1\ u_6$	$u_1\ u_6\ u_5$	$u_1\ u_6\ u_5\ u_2$	H	H
u_6		u_6	$u_6\ u_5$	$u_6\ u_5\ u_2$	$u_6\ u_5\ u_2\ u_3\ u_4$	$u_6\ u_5\ u_2\ u_3\ u_4$
u_5			u_5	$u_5\ u_2$	$u_5\ u_2\ u_3\ u_4$	$u_5\ u_2\ u_3\ u_4$
u_2				u_2	$u_2\ u_3\ u_4$	$u_2\ u_3\ u_4$
u_3					$u_3\ u_4$	$u_3\ u_4$
u_4						$u_3\ u_4$

From $_2H$ we obtain : $\mu_3(u_1) = 0.34848$, $\mu_3(u_6) = 0.331579$, $\mu_3(u_5) = 0.31739$ $\mu_3(u_2) = 0.302898$, $\mu_3(u_3) = \mu_3(u_4) = 0.29$

We have clearly $_3H = {_2H}$, by consequence $\partial(8_2{}^6) = 2$.

(1_3^6) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$ $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\ u_2\ u_3$	$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_3\ u_4\ u_6$
u_2		$u_1\ u_2\ u_3\ u_4$	$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_4\ u_5$	$u_1\ u_2\ u_3\ u_4\ u_6$
u_3			$u_1\ u_2\ u_3\ u_4\ u_5$	H	$u_1\ u_2\ u_3\ u_4\ u_5$	H
u_4				$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$	$u_2\ u_3\ u_4\ u_5\ u_6$
u_5					$u_3\ u_4\ u_5$	$u_3\ u_4\ u_5\ u_6$
u_6						$u_4\ u_6$

We obtain $\mu_1(u_1) = 0.2006173$, $\mu_1(u_2) = 0.2005208$, $\mu_1(u_3) = 0.20714$, $\mu_1(u_4) = 0.211905$, $\mu_1(u_5) = 0.198765$, $\mu_1(u_6) = 0.206667$.

By consequence we have ${}_1H$.

${}_1H$	u_4	u_3	u_6	u_1	u_2	u_5
u_4	u_4	$u_4\ u_3$	$u_4\ u_3\ u_6$	$u_4\ u_3\ u_6\ u_1$	$u_4\ u_3\ u_6\ u_1\ u_2$	H
u_3		u_3	$u_3\ u_6$	$u_3\ u_6\ u_1$	$u_3\ u_6\ u_1\ u_2$	$u_3\ u_6\ u_1\ u_2\ u_5$
u_6			u_6	$u_6\ u_1$	$u_6\ u_1\ u_2$	$u_6\ u_1\ u_2\ u_5$
u_1				u_1	$u_1\ u_2$	$u_1\ u_2\ u_5$
u_2					u_2	$u_2\ u_5$
u_5						u_5

Hence we have

$\mu_2(u_4) = 0.354545 = \mu_2(u_5)$,

$\mu_2(u_3) = 0.34035 = \mu_2(u_2)$

$\mu_2(u_6) = 0.33188 = \mu_2(u_1)$

from which we obtain ${}_2H$.

${}_2H$	u_4	u_5	u_3	u_2	u_6	u_1
u_4	$u_4\ u_5$	$u_4\ u_5$	$u_4\ u_5\ u_3\ u_2$	$u_4\ u_5\ u_3\ u_2$	H	H
u_5		$u_4\ u_5$	$u_4\ u_5\ u_3\ u_2$	$u_4\ u_5\ u_3\ u_2$	H	H
u_3			$u_3\ u_2$	$u_3\ u_2$	$u_3\ u_2\ u_6\ u_1$	$u_3\ u_2\ u_6\ u_1$
u_2				$u_3\ u_2$	$u_3\ u_2\ u_6\ u_1$	$u_3\ u_2\ u_6\ u_1$
u_6					$u_6\ u_1$	$u_6\ u_1$
u_1						$u_6\ u_1$

From ${}_2H$ it follows

$\mu_3(u_4) = \mu_3(u_5) = \mu_3(u_6) = \mu_3(u_1) = 0.26667$

$\mu_3(u_2) = \mu_3(u_3) = 0.26190$.

Set $P = \{u_4, u_5, u_6, u_1\}$, $Q = \{u_3, u_2\}$. Then we obtain ${}_3H$ as follows

${}_3H$	u_4	u_5	u_6	u_1	u_3	u_2
u_4	P	P	P	P	H	H
u_5		P	P	P	H	H
u_6			P	P	H	H
u_1				P	H	H
u_3					Q	Q
u_2						Q

From ${}_3H$, it follows that ${}_4H = {}_3H$ and we have finally $\partial(1_3^{\,6}) = 3$.

($2_3^{\,6}$) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$, $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$.

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\,u_2$ u_3	$u_1\,u_2$ $u_3\,u_4$	$u_1\,u_2$ $u_3\,u_4\,u_5$	H	$u_1\,u_2$ $u_3\,u_4\,u_5$	$u_1\,u_2$ $u_3\,u_4\,u_6$
u_2		$u_1\,u_2$ $u_3\,u_4$	$u_1\,u_2$ $u_3\,u_4\,u_5$	H	$u_1\,u_2$ u_3,u_4,u_5	u_1,u_2 $u_3\,u_4\,u_6$
u_3			$u_1\,u_2$ $u_3\,u_4\,u_5$	H	$u_1\,u_2$ $u_3\,u_4\,u_5$	H
u_4				$u_2\,u_3$ $u_4\,u_5\,u_6$	$u_2\,u_3$ $u_4\,u_5\,u_6$	$u_2\,u_3$ $u_4,u_5\,u_6$
u_5					$u_3\,u_4$ u_5	$u_3\,u_4$ $u_5\,u_6$
u_6						$u_4\,u_6$

whence $\partial(1_3^{\,6}) = \partial(2_3^{\,6}) = 3$.

($3_3^{\,6}$) Set $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$ $\Gamma(u_5) = \{d_5\}$, $\Gamma(u_6) = \{d_6\}$. We have

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$u_1\,u_2$ u_3	$u_1\,u_2$ $u_3\,u_4$	$u_1\,u_2$ $u_3\,u_4\,u_5$	H	$u_1\,u_2$ $u_3\,u_4\,u_5$	$u_1\,u_2$ $u_3\,u_4\,u_6$
u_2		$u_1\,u_2$ $u_3\,u_4$	$u_1\,u_2$ $u_3\,u_4\,u_5$	H	$u_1\,u_2\,u_3$ $u_4\,u_5$	$u_1\,u_2$ $u_3\,u_4\,u_6$
u_3			$u_1\,u_2\,u_3$ $u_4\,u_5$	H	$u_1\,u_2\,u_3$ $u_4\,u_5$	H
u_4				$u_2\,u_3$ $u_4\,u_5\,u_6$	$u_2\,u_3$ $u_4\,u_5\,u_6$	$u_2\,u_3$ $u_4\,u_5\,u_6$
u_5					$u_3\,u_4$ u_5	$u_3\,u_4$ $u_5\,u_6$
u_6						$u_4\,u_6$

See ($1_3^{\,6}$).

We have $\partial(3_3^{\,6}) = \partial(1_3^{\,6}) = 3$.

\$ 6. (1_4^8) Set $H = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ and $\Gamma(u_1) = \{d_1, d_2, d_3\}$, $\Gamma(u_2) = \{d_2, d_3, d_4\}$, $\Gamma(u_3) = \{d_3, d_4, d_5\}$, $\Gamma(u_4) = \{d_4, d_5, d_6\}$, $\Gamma(u_5) = \{d_5, d_6, d_7\}$, $\Gamma(u_6) = \{d_7, d_8\}$, $\Gamma(u_7) = \{d_7\}$, $\Gamma(u_8) = \{d_8\}$.

So, denoting $\{u_i, u_{i+1},\ldots, u_{j-1}, u_j\}$ by u_i^j, we have

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	u_1^3	u_1^4	u_1^5	u_1^5	u_1^7	$u_1^3\ u_5^8$	$u_1^3\ u_5^7$	$u_1^3\ u_6\ u_8$
u_2		u_1^4	u_1^5	u_1^5	u_1^7	H	u_1^7	$u_1^4\ u_6\ u_8$
u_3			u_1^5	u_1^5	u_1^7	H	u_1^7	$u_1^6\ u_8$
u_4				u_2^5	u_2^7	u_2^8	u_2^7	$u_2^6\ u_8$
u_5					u_3^7	u_3^8	u_3^7	u_3^8
u_6						u_5^8	u_5^8	u_5^8
u_7							u_5^7	u_5^8
u_8								$u_6\ u_8$

We have

$\mu_1(u_1) = 0.1756$,

$\mu_1(u_2) = 0.17470$,

$\mu_1(u_3) = 0.1754978$,

$\mu_1(u_4) = 0.1729$,

$\mu_1(u_5) = 0.1803$,

$\mu_1(u_6) = 0.1813$,

$\mu_1(u_7) = 0.175641 = \mu_1(u_1)$,

$\mu_1(u_8) = 0.19073$.

So $\mu_1(u_8) > \mu_1(u_6) > \mu_1(u_5) > \mu(u_7) = \mu(u_1) > \mu_1(u_3) > \mu_1(u_2) > \mu(u_4)$.

One obtains $_1H$ as follows

$_1H$	u_4	u_2	u_3	u_1	u_7	u_5	u_6	u_8
u_4	u_4	$u_4\ u_2$	$u_4\ u_2\ u_3$	$u_4\ u_2\ u_3\ u_1\ u_7$	$u_4\ u_2\ u_3\ u_1\ u_7$	$u_4\ u_2\ u_3\ u_1\ u_7\ u_5$	$u_4\ u_2\ u_3\ u_1\ u_7\ u_5\ u_6$	H
u_2		u_2	$u_2\ u_3$	$u_2\ u_3\ u_1\ u_7$	$u_2\ u_3\ u_1\ u_7$	$u_2\ u_3\ u_1\ u_7\ u_5$	$u_2\ u_3\ u_1\ u_7\ u_5\ u_6$	$u_2\ u_3\ u_1\ u_7\ u_5\ u_6\ u_8$
u_3			u_3	$u_3\ u_1\ u_7$	$u_3\ u_1\ u_7$	$u_3\ u_1\ u_5\ u_7$	$u_3\ u_1\ u_7\ u_5\ u_6$	$u_3\ u_1\ u_7\ u_5\ u_6\ u_8$
u_1				$u_1\ u_7$	$u_1\ u_7$	$u_1\ u_7\ u_5$	$u_1\ u_7\ u_5\ u_6$	$u_1\ u_7\ u_5\ u_6\ u_8$
u_7					$u_1\ u_7$	$u_1\ u_7\ u_5$	$u_1\ u_7\ u_5\ u_6$	$u_1\ u_7\ u_5\ u_6\ u_8$
u_5						u_5	$u_5\ u_6$	$u_5\ u_6\ u_8$
u_6							u_6	$u_6\ u_8$
u_8								u_8

from which $\mu_2(u_4) = \mu_2(u_8) = 0.2890$, $\mu_2(u_2) = \mu_2(u_6) = 0.2724$,

$\mu_2(u_1) = \mu_2(u_7) = 0.247567$, $\mu_2(u_3) = \mu_2(u_5) = 0.2549$. So we obtain $_2H$.

$_2H$	u_4	u_8	u_2	u_6	u_3	u_5	u_1	u_7
u_4	$u_4\,u_8$	$u_4\,u_8$	$u_4\,u_8$ $u_2\,u_6$	$u_4\,u_8$ $u_2\,u_6$	$u_4\,u_8\,u_2$ $u_6\,u_3\,u_5$	$u_4\,u_8\,u_2$ $u_6\,u_3\,u_5$	H	H
u_8		$u_4\,u_8$	$u_4\,u_8$ $u_2\,u_6$	$u_4\,u_8$ $u_2\,u_6$	$u_4\,u_8\,u_2$ $u_6\,u_3\,u_5$	$u_4\,u_8\,u_2$ $u_6\,u_3\,u_5$	H	H
u_2			$u_2\,u_6$	$u_2\,u_6$	$u_2\,u_6$ $u_3\,u_5$	$u_2\,u_6$ $u_3\,u_5$	$u_2\,u_6\,u_3$ $u_5\,u_1\,u_7$	$u_2\,u_6\,u_3$ $u_5\,u_1\,u_7$
u_6				$u_2\,u_6$	$u_2\,u_6$ $u_3\,u_5$	$u_2\,u_6$ $u_3\,u_5$	$u_2\,u_6\,u_3$ $u_5\,u_1\,u_7$	$u_2\,u_6\,u_3$ $u_5\,u_1\,u_7$
u_3					$u_3\,u_5$	$u_3\,u_5$	$u_3\,u_5$ $u_1\,u_7$	$u_3\,u_5$ $u_1\,u_7$
u_5						$u_3\,u_5$	$u_3\,u_5$ $u_1\,u_7$	$u_3\,u_5$ $u_1\,u_7$
u_1							$u_1\,u_7$	$u_1\,u_7$
u_7								$u_1\,u_7$

Hence we have :

$\mu_3(u_4) = \mu_3(u_8) = \mu_3(u_1) = \mu_3(u_7) = 0.22619$, $\mu_3(u_2) = \mu_3(u_6) = \mu_3(u_3) = \mu_3(u_5) = 0.2197$.

Setting $P= \{u_4, u_8, u_1, u_7\}$, $Q = \{u_3, u_5, u_2, u_6\}$, we find $_3H$

$_3H$	u_4	u_8	u_1	u_7	u_2	u_6	u_3	u_5
u_4	P	P	P	P	H	H	T	H
u_8		P	P	P	H	H	H	H
u_1			P	P	H	H	H	H
u_7				P	H	H	H	H
u_2					Q	Q	Q	Q
u_6						Q	Q	Q
u_3							Q	Q
u_5								Q

From this, we obtain :
$\forall i,\ \mu_4(u_i) = 0.166$.

It follows $_4H = T$, whence $\partial(1_4{}^8) = 4$.

\$7. ($1_2{}^9$) Let $H=\{ u_i \mid 1 \le i \le 9\}$ and for $i \le 7$, set

$\Gamma(u_i) = \{ d_i, d_{i+1}, d_{i+2}\}$, $\Gamma(u_8) = \{d_8\}$, $\Gamma(u_9) = \{d_9\}$.

We obtain

$_0H$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
u_1	$u_1 u_2$ u_3	$u_1 u_2$ $u_3 u_4$	$u_1 u_2$ $u_3 u_4$ u_5	$u_1 u_2$ $u_3 u_4$ $u_5 u_6$	u_1 u_2 u_3 $u_4 u_5 u_6 u_7$	u_1 u_2 u_3 u_4 u_5 u_6 $u_7 u_8$	$u_1 u_2 u_3$ $u_5 u_6 u_7 u_8$ u_9	u_1 u_2 u_3 $u_6 u_7 u_8$	$u_1 u_2 u_3$ $u_7 u_9$
u_2		$u_1 u_2$ $u_3 u_4$	$u_1 u_2$ $u_3 u_4$ u_5	$u_1 u_2$ $u_3 u_4$ $u_5 u_6$	u_1 u_2 u_3 $u_4 u_5 u_6$ u_7	u_1 u_2 u_3 u_4 u_5 u_6 $u_7 u_8$	H	u_1 u_2 u_3 u_4 $u_6 u_7$ u_8	$u_1 u_2 u_3$ $u_4 u_7 u_9$
u_3			$u_1 u_2$ $u_3 u_4$ u_5	$u_1 u_2$ $u_3 u_4$ $u_5 u_6$	$u_1 u_2 u_3$ $u_4 u_5 u_6$ u_7	u_1 u_2 u_3 u_4 u_5 u_6 $u_7 u_8$	H	u_1 $u_2 u_3$ u_4 u_5 u_6 $u_7 u_8$	$u_1 u_2 u_3$ u_4 $u_5 u_7 u_9$
u_4				u_2 u_3 $u_4 u_5$ u_6	$u_2 u_3$ u_4 $u_5 u_6$ u_7	u_2 u_3 u_4 u_5 $u_6 u_7$ u_8	$u_2 u_3 u_4 u_5$ $u_6 u_7 u_8$ u_9	$u_2 u_3$ u_4 u_5 $u_6 u_7$ u_8	$u_2 u_3$ u_4 u_5 $u_6 u_7 u_9$
u_5					u_3 $u_4 u_5$ u_6 u_7	u_3 $u_4 u_5$ u_6 u_7 u_8	$u_3 u_4 u_5 u_6$ $u_7 u_8$ u_9	u_3 $u_4 u_5$ u_6 $u_7 u_8$	u_3 $u_4 u_5$ u_6 $u_7 u_9$
u_6						u_4 u_5 u_6 $u_7 u_8$	u_4 u_5 u_6 $u_7 u_8$ u_9	u_4 u_5 u_6 $u_7 u_8$	u_4 u_5 u_6 $u_7 u_8$ u_9
u_7							u_5 $u_6 u_7$ u_8 u_9	u_5 $u_6 u_7$ u_8 u_9	u_5 $u_6 u_7 u_8$ u_9
u_8								$u_6 u_7 u_8$	$u_6 u_7 u_8$ u_9
u_9									$u_7 u_9$

From $_0H$ we have

$\mu_1(u_9) = 0.1729 > \mu_1(u_7) = 0.1648 > \mu_1(u_1) = 0.1616 > \mu_1(u_3) = 0.160 > \mu_1(u_6) = 0.1599$

$> \mu_1(u_8) = 0.1597 > \mu_1(u_2) = 0.159169 > \mu_1(u_5) = 0.157387 > \mu_1(u_4) = 0.159218.$

From these data, we obtain $_1H$ as follows

$_1H$	u_9	u_7	u_1	u_3	u_6	u_8	u_4	u_2	u_5
u_9	u_9	u_9 u_7	u_9 u_7 u_1	u_9 u_7 u_1 u_3	u_9 u_7 u_1 $u_3 u_6$	u_9 u_7 u_1 $u_3 u_6$ u_8	u_9 u_7 u_1 $u_3 u_6$ $u_8 u_4$	u_9 u_7 $u_1 u_3$ u_6 u_8 u_2 u_4	H
u_7		u_7	$u_7 u_1$	u_7 u_1 u_3	$u_7 u_1$ $u_3 u_6$	$u_7 u_1$ $u_3 u_6 u_8$	$u_7 u_1 u_3 u_6$ $u_8 u_4$	$u_7 u_1 u_3 u_6$ u_8 $u_2 u_4$	$u_7 u_1 u_3 u_6$ u_8 u_2 u_5 u_4
u_1			u_1	u_1 u_3	$u_1 u_3 u_6$	u_1 $u_3 u_6$ u_8	$u_1 u_3 u_6$ u_8 u_4	$u_1 u_3 u_6$ u_8 u_2 u_4	$u_1 u_3 u_6 u_8$ u_2 $u_5 u_4$
u_3				u_3	$u_3 u_6$	$u_3 u_6 u_8$	$u_3 u_6$ $u_8 u_4$	$u_3 u_6$ u_8 $u_2 u_4$	$u_3 u_6 u_8$ u_2 $u_5 u_4$
u_6					u_6	$u_6 u_8$	u_6 $u_8 u_4$	u_6 $u_8 u_2 u_4$	$u_6 u_8 u_2$ u_5 u_4
u_8						u_8	u_8 u_4	u_8 u_2 u_4	$u_8 u_2$ $u_5 u_4$
u_4							u_4	u_2 u_4	$u_2 u_5 u_4$
u_2								u_2	u_5 u_2
u_5									u_5

From ${}_1H$ we obtain as follows, μ_2 and then ${}_2H$.

$\mu_2(u_9) = \mu_2(u_5) = 0.2740 > \mu_2(u_7) = \mu_2(u_2) = 0.261085 > \mu_2(u_1) = \mu_2(u_4) =$

$= 0.250716 > \mu_2(u_3) = \mu_2(u_8) = 0.24495 > \mu_2(u_6) = 0.243116.$

${}_2H$	u_9	u_5	u_7	u_2	u_1	u_4	u_3	u_8	u_6
u_9	u_5 u_9	u_5 u_9	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 u_7 u_2 u_3 u_8 u_1 u_4	u_5 u_9 u_7 u_3 u_8 u_2 u_1 u_4	H
u_5		u_5 u_9	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 u_7 u_2 u_1 u_4	u_5 u_9 $u_7 u_2$ u_1 u_4 u_3 u_8	u_5 u_9 u_7 u_2 u_1 u_4 $u_3 u_8$	H
u_7			u_7 u_2	u_7 u_2	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_6 u_1 u_4 u_3 u_8
u_2				u_7 u_2	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_1 u_4 u_3 u_8	u_7 u_2 u_6 u_1 u_4 u_3 u_8
u_1					u_1 u_4	u_1 u_4	u_1 u_4 u_3 u_8	u_1 u_4 u_3 u_8	u_1 u_4 u_6 u_3 u_8
u_4						u_1 u_4	u_1 u_4 u_3 u_8	u_1 u_4 u_3 u_8	u_1 u_4 u_6 u_3 u_8
u_3							u_3 u_8	u_3 u_8	u_3 u_8 u_6
u_8								u_3 u_8	u_3 u_8 u_6
u_6									u_6

From ${}_2H$ we obtain $\mu_3(u_9) = \mu_3(u_5) = 0.211805$, $\mu_3(u_7) = \mu_3(u_2) = 0.205433$,

$\mu_3(u_1) = \mu_3(u_4) = 0.20504$, $\mu_3(u_3) = \mu_3(u_8) = 0.21155$, $\mu_3(u_6) = 0.24407$.

Then we have ${}_3H$ as follows

${}_3H$	u_6	u_9	u_5	u_3	u_8	u_2	u_7	u_1	u_4
u_6	u_6	u_6 u_9 u_5	u_6 u_9 u_5	u_6 u_9 u_5 u_3 u_8	u_6 u_9 u_5 u_3 u_8	u_6 u_9 u_5 u_3 u_8 $u_2 u_7$	u_6 u_9 u_5 u_3 u_8 u_2 u_7	H	H
u_9		u_9 u_5	u_9 u_5	u_9 $u_5 u_3$ u_8	u_9 u_5 u_3 u_8	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 $u_3 u_8$ u_2 u_7 u_1 u_4	u_9 u_5 u_3 u_8 u_2 u_7 u_1 u_4
u_5			u_9 u_5	u_9 u_5 u_3 u_8	u_9 u_5 u_3 u_8	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 u_3 u_8 u_2 u_7	u_9 u_5 u_3 u_8 u_2 u_7 u_1 u_4	u_9 u_5 $u_3 u_8$ u_2 u_7 u_1 u_4
u_3				u_3 u_8	u_3 u_8	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7 u_1 u_4	u_3 u_8 u_2 u_7 u_1 u_4
u_8					u_3 u_8	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7	u_3 u_8 u_2 u_7 u_1 u_4	u_3 u_8 u_2 u_7 u_1 u_4
u_2						u_2 u_7	u_2 u_7	u_2 u_7 u_1 u_4	u_2 u_7 u_1 u_4
u_7							u_2 u_7	u_2 u_7 u_1 u_4	u_2 u_7 u_1 u_4
u_1								u_1 u_4	u_1 u_4
u_4									u_1 u_4

It is possible to verify that the function $\varphi : {}_2H \rightarrow {}_3H$ defined as follows

$\varphi(u_3) = u_9$, $\varphi(u_8) = u_5$, $\varphi(u_1) = u_3$,

$\varphi(u_4) = u_8$, $\varphi(u_9) = u_1$, $\varphi(u_5) = u_4$, $\varphi(u_7) = u_7$,

$\varphi(u_2) = u_2$, $\varphi(u_6) = u_6$,

is a hypergroup isomorphism.

It follows that the fuzzy grade of $(1_2{}^9)$ is 2.

$ 8. ($1_5{}^{16}$) Set $H = \{u_i \mid 1 \leq i \leq 16\}$, $D = \{d_i \mid 1 \leq i \leq 16\}$, $\Gamma(u_1) = \{d_1, d_2, d_3\}$,

$\Gamma(u_2) = \{d_2, d_3, d_4\}$, and

$\forall$ i : $i \leq 13$, $\Gamma(u_i) = \{d_i, d_{i+1}, d_{i+2}\}$,

$\Gamma(u_{14}) = \{d_{15}, d_{16}\}$, $\Gamma(u_{15}) = \{d_{15}\}$, $\Gamma(u_{16}) = \{d_{16}\}$.

Since $\forall$ i, we have $u_i \text{ o } u_i = \{u_j \mid \Gamma(u_j) \cap \Gamma(u_i) \neq \emptyset\}$, it follows that we have

$u_1 \text{ o } u_1 = \{ u_1, u_2, u_3\}$, $u_2 \text{ o } u_2 = \{ u_1, u_2, u_3, u_4\}$,

$u_3 \text{ o } u_3 = \{ u_1, u_2, u_3, u_4, u_5\}$,

$\forall$ i : $4 \leq i \leq 13$, $u_i \text{ o } u_i = \{ u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$,

$u_{14} \text{ o } u_{14} = \{ u_{13}, u_{14}, u_{15}, u_{16}\}$,

$u_{15} \text{ o } u_{15} = \{ u_{13}, u_{14}, u_{15}\}$,

$u_{16} \text{ o } u_{16} = \{ u_{14}, u_{16}\}$.

For ${}_0H$ we have the following table :

${}_0H$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
u_1	u_1^{3}	u_1^{4}	u_1^{5}	u_1^{6}	u_1^{7}	u_1^{8}	$u_1^{3} u_5^{9}$	$u_1^{3} u_6^{10}$	$u_1^{3} u_7^{11}$	$u_1^{3} u_8^{12}$	$u_1^{3} u_9^{13}$	$u_1^{3} u_{10}^{14}$	$u_1^{3} u_{11}^{15}$	$u_1^{3} u_{13}^{16}$	$u_1^{3} u_{13}^{15}$	$u_1^{3} u_{14} u_{16}$
u_2		u_1^{4}	u_1^{5}	u_1^{6}	u_1^{7}	u_1^{8}	u_1^{9}	$u_1^{4} u_6^{10}$	$u_1^{4} u_7^{11}$	$u_1^{4} u_8^{12}$	$u_1^{4} u_9^{13}$	$u_1^{4} u_{10}^{14}$	$u_1^{4} u_{11}^{15}$	$u_1^{4} u_{13}^{16}$	$u_1^{4} u_{13}^{15}$	$u_1^{4} u_{14} u_{16}$
u_3			u_1^{5}	u_1^{6}	u_1^{7}	u_1^{8}	u_1^{9}	u_1^{10}	$u_1^{5} u_7^{11}$	$u_1^{5} u_8^{12}$	$u_1^{5} u_9^{13}$	$u_1^{5} u_{10}^{14}$	$u_1^{5} u_{11}^{15}$	$u_1^{5} u_{13}^{16}$	$u_1^{5} u_{13}^{15}$	$u_1^{5} u_{14} u_{16}$
u_4				u_2^{6}	u_2^{7}	u_2^{8}	u_2^{9}	u_2^{10}	u_2^{11}	$u_2^{6} u_8^{12}$	$u_2^{6} u_9^{13}$	$u_2^{6} u_{10}^{14}$	$u_2^{6} u_{11}^{15}$	$u_2^{6} u_{13}^{16}$	$u_2^{6} u_{13}^{15}$	$u_2^{6} u_{14} u_{16}$
u_5					u_3^{7}	u_3^{8}	u_3^{9}	u_3^{10}	u_3^{11}	u_3^{12}	$u_3^{7} u_9^{13}$	$u_3^{7} u_{10}^{14}$	$u_3^{7} u_{11}^{15}$	$u_3^{7} u_{13}^{16}$	$u_3^{7} u_{13}^{15}$	$u_3^{7} u_{14} u_{16}$
u_6						u_4^{8}	u_4^{9}	u_4^{10}	u_4^{11}	u_4^{12}	u_4^{13}	$u_4^{8} u_{10}^{14}$	$u_4^{8} u_{11}^{15}$	$u_4^{8} u_{13}^{16}$	$u_4^{8} u_{13}^{15}$	$u_4^{8} u_{14} u_{16}$
u_7							u_5^{9}	u_5^{10}	u_5^{11}	u_5^{12}	u_5^{13}	u_5^{14}	$u_5^{9} u_{11}^{15}$	$u_5^{9} u_{13}^{16}$	$u_5^{9} u_{13}^{15}$	$u_5^{9} u_{14} u_{16}$
u_8								u_6^{10}	u_6^{11}	u_6^{12}	u_6^{13}	u_6^{14}	u_6^{15}	$u_6^{10} u_{13}^{16}$	$u_6^{10} u_{13}^{15}$	$u_6^{10} u_{14} u_{16}$
u_9									u_7^{11}	u_7^{12}	u_7^{13}	u_7^{14}	u_7^{15}	$u_7^{11} u_{13}^{16}$	u_7^{15}	$u_7^{11} u_{14} u_{16}$
u_{10}										u_8^{12}	u_8^{13}	u_8^{14}	u_8^{15}	u_8^{16}	u_8^{15}	$u_8^{12} u_{14} u_{16}$
u_{11}											u_9^{13}	u_9^{14}	u_9^{15}	u_9^{16}	u_9^{15}	$u_9^{14} u_{16}$
u_{12}												u_{10}^{14}	u_{10}^{15}	u_{10}^{16}	u_{10}^{15}	$u_{10}^{14} u_{16}$
u_{13}													u_{11}^{15}	u_{11}^{16}	u_{11}^{15}	u_{11}^{16}
u_{14}														u_{13}^{16}	u_{13}^{16}	u_{13}^{16}
u_{15}															u_{13}^{15}	u_{13}^{16}
u_{16}																$u_{14} u_{16}$

From ${}_0H$ we obtain $\mu_1(u_{16}) = 0.15673$, $\mu_1(u_{15}) = 0.13992$, $\mu_1(u_{14}) = 0.141293$, $\mu_1(u_1) = 0.13867$, $\mu_1(u_2) = 0.134942$, $\mu_1(u_{13}) = 0.134215$, $\mu_1(u_3) = 0.132574$, $\mu_1(u_4) = 0.129700$, $\mu_1(u_5) = 0.128076$, $\mu_1(u_6) = 0.127554$, $\mu_1(u_7) = 0.126581$, $\mu_1(u_{12}) = 0.1283654$, $\mu_1(u_{11}) = 0.126441$, $\mu_1(u_{10}) = 0.126878$, $\mu_1(u_8) = 0.12671$, $\mu_1(u_9) = 0.126608$.

For ${}_1H$, set $v_1 = u_{16}$, $v_2 = u_{14}$, $v_3 = u_{15}$, $v_4 = u_1$, $v_5 = u_2$, $v_6 = u_{13}$, $v_7 = u_3$, $v_8 = u_4$, $v_9 = u_{12}$, $v_{10} = u_5$, $v_{11} = u_6$, $v_{12} = u_{10}$, $v_{13} = u_8$, $v_{14} = u_9$, $v_{15} = u_7$, $v_{16} = u_{11}$.

$\forall(i, j)$, such that $i \leq j$ set $v_i^j = \{v_i, v_{i+1},...v_j\}$. So we have $v_i \; o_1 \; v_j = v_i^j$. For ${}_2H$ we have $v_1 \; o_2 \; v_1 = v_1 \; o_2 \; v_{16} = v_{16} \; o_2 \; v_{16} = \{v_1, v_{16}\}$, $v_2 \; o_2 \; v_2 = v_2 \; o_2 \; v_{15} = v_{15} \; o_2 \; v_{15} = \{v_2, v_{15}\}$. Generally, $v_i \; o_2 \; v_i = v_i \; o_2 \; v_{16-(i-1)} = v_{16-(i-1)} \; o_2 \; v_{16-(i-1)} = \{v_i, v_{16-(i-1)}\}$. For $i < j$, $v_i \; o_2 \; v_j = \bigcup_{i \leq s \leq j} v_s \; o_2 \; v_s$.

Set $P_1 = \{v_1, v_{16}, v_8, v_9\}$, $P_2 = \{v_2, v_{15}, v_7, v_{10}\}$, $P_3 = \{v_3, v_{14}, v_6, v_{11}\}$, $P_4 = \{v_4, v_{13}, v_5, v_{12}\}$. Then for ${}_3H$, $\forall k$: $1 \leq k \leq 14$, we have $\forall(v_i, v_j) \in P_k \times P_k$, $v_i \; o_3 \; v_j = P_k$.

If $s < t$, $\forall(v_i, v_j) \in P_s \times P_t$, we have $v_i \; o_3 \; v_j = \bigcup_{s \leq u \leq t} P_u$. For ${}_4H$, setting

$Q_1 = P_1 \bigcup P_4$, $Q_2 = P_2 \bigcup P_3$, we have $\forall(v_i, v_j) \in Q_i \times Q_j$, $v_i \; o_4 \; v_j = Q_i \bigcup Q_j$.

By consequence, if $i \neq j$, $v_i \; o_4 \; v_j = H$ and $v_i \; o_4 \; v_i = Q_i$. Since $|Q_1| = |Q_2|$, we have $\forall \; v_i \in Q_1$, $\forall \; v_j \in Q_2$, $\mu_4(v_i) = \mu_4(v_j)$. It follows that ${}_5H = T$ (total hypergroup) and by consequence $\partial(1_5^{16}) = 5$.

REFERENCES

[1] Ameri R. and Zahedi M.M., Hypergroup and join space induced by a fuzzy subset, PU.M.A. vol. 8, (1997)

[2] Ameri R. and Shafiiyan, Fuzzy Prime and Primary Hyperideals of Hyperrings, Advances in Fuzzy Math., n. 1-2, Research India Publications (2007)

[3] Ameri R., Hedayati H, Molaee A., On fuzzy hyperideals of Γ-hyperrings, Iranian J. of Fuzzy Systems, vol. 6, n. 2, (2009)

[4] Bakhshi M., Mashinki M., Borzooei R.A., Representation Theorem for some Algebraic Hyperstructures, International Review of Fuzzy Mathematics (IRFM), vol. 1, n.1, (2006)

[5] Borzooei R.A., Jun Young Bae, Intuitionistic Fuzzy Hyper BCK- Ideals of BCK- Algebras, Iranian J. of Fuzzy Systems, vol. 1, n.1, (2004)

[6] Borzooei R.A, Zahedi M.M., Fuzzy structures on hyper K-algebras, International J. of Uncertainty Fuzzyness and Knowledge-Based Systems 112, (2), (2000)

[7] Corsini P., Prolegomena of hypergroup theory, Aviani Editore (1993)

[8] Corsini P., Join Spaces, Power Sets, Fuzzy Sets, Proc. Fifth International Congress on A.H.A. 1993, Iasi. Romania, Hadronic Press, (1994)

[9] Corsini P., Rough Sets, Fuzzy Sets and Join Spaces, Honorary Volume dedicated to Prof. Emeritus Ioannis Mittas, Aristotle Univ. of Thessaloniki, 1999-2000, Editors M. Konstantinidou, K. Serafimidis, G. Tsagas

[10] Corsini P., On Chinese Hyperstructures, Proc. of the Seventh Congress A.H.A., Taormina, 1999, Journal of Discrete Mathematical Sciences & Criptography, vol. 6 (2003)

[11] Corsini P., Fuzzy sets, join spaces and factor spaces, PU.M.A. vol. 11, n. 3, (2000)

[12] Corsini P., Properties of hyperoperations associated with fuzzy sets and with factor spaces, International Journal of Sciences and Technology, Kashan University, vol. 1, n. 1, (2000)

[13] Corsini P., Binary Relations, Interval Structures and Join Spaces, Korean J.Math Comput. Appl. Math., 9(1) (2002)

[14] Corsini P., A new connection between Hypergroups and Fuzzy Sets, Southeast Asian Bulletin of Math., 27. (2003)

[15] Corsini P., Hyperstructures associated with ordered sets, Bull. Greek. Math. Soc. vol. 48, (2003)

[16] Corsini P., Cristea I., Fuzzy grade of i.p.s. hypergroups of order less or equal to 6, PU.M.A. vol. 14, n. 4, (2003)

[17] Corsini P., Cristea I., Fuzzy grade of i.p.s. hypergroups of order 7, Iranian J.of Fuzzy Systems, vol. 1, n. 2 (2004)

[18] Corsini P., Cristea I., Fuzzy sets and non complete 1-hypergroups, An. St. Univ. Ovidius Constanta, 13(1), (2005)

[19] Corsini P. and Leoreanu V., Applications of Hyperstructure Theory, Kluwer Academic Publishers, Advances in Mathematics, n. 5, (2003)

[20] Corsini P. and Leoreanu V., Join Spaces associated with Fuzzy Sets, J. of Combinatorics, Information and System Sciences, vol. 20, n.1. (1995)

[21] Corsini P., Leoreanu V., Fuzzy Sets and Join Spaces associated with Rough Sets, Circolo Matematico di Palermo, S. II, T. LI, (2002)

[22] Corsini P. and Leoreanu-Fotea V., On the grade of a sequence of fuzzy sets and join spaces determined by a hypergraph, accepted by Southeast Asian Bulletin of Mathematics (2007)

[23] Corsini P., Leoreanu-Fotea V., Iranmanesh A., On the sequence of hypergroups and membership functions determined by a hypergraph, J. of Multiple Valued Logic and Soft Computing, vol. 14, issue 6, (2008)

[24] Corsini P., Tofan I., On Fuzzy Hypergroups, PU.M.A. vol. 8, n. 1, (1997)

[25] Cristea I., Complete Hypergroups, 1-Hypergroups and Fuzzy Sets, An. St. Univ. Ovidius Constanta, Vol. 10(2), (2002)

[26] Cristea I., A property of the connection between Fuzzy Sets and Hypergroupoids, Italian Journal of Pure and Applied Mathematics, Vol. 21, (2007)

[27] Cristea I., On the fuzzy subhypergroups of some particular complete hypergroup (II), accepted by Proc. of the 10th International Congress on A.H.A., Brno, Czech Republic (2008)

[28] Cristea I., Hyperstructures and fuzzy sets endowed with two membership functions, Fuzzy Sets and Systems, 160, (2009)

[29] Davvaz B., Interval-valued fuzzy subhypergroups, Korean J.Comput. Appl. Math., 6, n. 1, (1999)

[30] Davvaz B., Fuzzy hyperideals in semihypergroups, Italian J. of Pure and Appl. Math., 8, (2000)

[31] Davvaz B., Fuzzy Hv-submodules, Fuzzy Sets and Systems, 117, (2001)

[32] Davvaz B., Interval-valued ideals of a hyperring, Italian J. of Pure and Applied Math., 10, (2001)

[33] Davvaz B., Corsini P., Generalized fuzzy sub-hyperquasigroups of hyperquasigroups, Soft Computing, 10(11), (2006)

[34] Davvaz B., Corsini P., Fuzzy n-ary hypergroups, J. of Intelligent and Fuzzy Systems, 18 (4), (2007)

[35] Davvaz B, Leoreanu-Fotea V., On a product of Hv–submodules, International J.of Fuzzy Systems, v.19 (2), (2008)

[36] Davvaz B., Corsini P, Leoreanu-Fotea V., Atanassov's intuitionistic (S,T)-fuzzy n-ary subhypergroups and their properties, Information Sciences, 179 (2009)

[37] Dramalidis A, Vougiouklis T., Two Fuzzy geometric-like hyperoperations defined on the same set, Proc. of the 9th International Congress on A.H.A., (2005), University of Mazandaran, Babolsar, Iran, Journal of Basic Sciences, 2008, vol.4, n.1

[38] Feng Yuming, Algebraic hyperstructures obtained from algebraic structures with fuzzy binary relations. , Italian J. of Pure and Applied Math. 25, (2009)

[39] Feng Yuming, L-fuzzy * and / hyperoperations, Fuzzy Seys, Rough Sets and Multivalued Operations and Applications, International Sciences Press n.1, (2009)

[40] Feng Yuming, The L-fuzzy hyperstructures $(X, \wedge, \vee)$ and $(X, \vee, \wedge)$, Italian J. of Pure and Applied Math. 26, (2009)

[41] Feng Yuming, P-Fuzzy Hypergroupoids associated with the product of Fuzzy Hypergraphs, accepted by Italian J. of Pure and Applied Math., (2009)

[42] Feng Yuming, Interval-valued Fuzzy Hypergraphs and Interval-valued Fuzzy Hyperoperations, accepted by Italian J. of Pure and Applied Math., (2009)

[43] Horry M., Zahedi M.M., Hypergroups and fuzzy general automata, Iranian J. of Fuzzy Systems, vol. 6, n. 2, (2009)

[44] Horry M., Zahedi M.M., Join Spaces and Max-Min general fuzzy automata, Italian J. of Pure and Applied Math. 26, (2009)

[45] Kehagias Ath., An example of L-Fuzzy Join Space, Rendiconti del Circolo Mat. di Palermo, vol. 51, (2002)

[46] Kehagias Ath., Lattice-Fuzzy Meet and Join Hyperoperations, Proc. 8th International Congress on A.H.A., Samothraki 2002, Edited by T. Vougiouklis, Spanidis Press

[47] Kehagias Ath., L-fuzzy Join and Meet Hyperoperations and the Associated L-fuzzy Hyperalgebras, Rendiconti del Circolo Mat. di Palermo, vol. 52, (2003)

[48] Kehagias Ath., Serafimidis K, The L-Fuzzy Nakano Hypergroup, Information Sciences, vol. 169 (2005)

[49] Kyung Ho Kim, B. Davvaz, Eun Hwan Ro, On fuzzy hyper R-subgroups of hypernear-rings, Italian J. of Pure and Applied Math.20, (2006)

[50] Leoreanu V., Direct limit and inverse limit of join spaces associated with rough sets, Honorary Volume dedicated to Prof. Emeritus Ioannis Mittas, Aristotle Univ. of Thessaloniki, 1999-2000

[51] Leoreanu V., Direct limit and inverse limit of join spaces associated with fuzzy sets, PU.M.A. vol. 11, n. 3, (2000)

[52] Leoreanu V., About hyperstructures associated with fuzzy sets of type 2, Italian Journal of Pure and Applied Mathematics, n. 17, (2005)

[53] Leoreanu-Fotea V., Leoreanu L., About a sequence of hyperstructures associated with a rough set, accepted by Southeast Asian Bulletin of Mathematics (2007)

[54] Leoreanu-Fotea V., Fuzzy rough n-ary subhypergroups, Iranian Journal of Fuzzy Systems, vol. 5, n. 2, (2008)

[55] Leoreanu-Fotea V., Fuzzy hypermodules, Computers and Mathematics with Applications, vol. 57, issue 3, (2009)

[56] Leoreanu-Fotea V.,Davvaz B., Roughness in n-ary hypergroups, Information Sciences, 178. (2008)

[57] Leoreanu-Fotea V., Rosenberg Ivo, Homomorphisms of hypergroupoids associated with L-fuzzy sets, accepted by Journal of Multiple Valued Logic Soft Comput. (2008)

[58] Leoreanu-Fotea V., Davvaz B., Fuzzy Hyperrings, Fuzzy Sets and Systems, vol.160, issue 16 (2008)

[59] Maturo A. A geometrical approach to the coherent conditional probability and its fuzzy extensions, Scientific Annals of University of A. S. V. M., "Ion Ionescu de la Brad",Iasi XLIX, (2006)

[60] Maturo A., Alternative Fuzzy Operations and Applications to Social Sciences, International Journal of Intelligent Systems, to appear.

[61] Maturo A., Squillante M., and Ventre A.G.S., (2006a), Consistency for assessments of uncertainty evaluations in non-additive settings in: Amenta, D'Ambra, Squillante and Ventre, Metodi, modelli e tecnologie dell'informazione a supporto delle decisioni, Franco Angeli, Milano.

[62] Maturo A., Squillante M., and Ventre A. G. S., (2006b), Consistency for nonadditive measures: analytical and algebraic methods, in B. Reusch (ed.), Computational Intelligence, Theory and Applications, in Advances in Soft Computing, Springer, Berlin, Heidelberg, New York.

[63] Maturo A., Squillante M., and Ventre A. G. S., (2009), Coherence for fuzzy Measures and Applications to Decision Making, Proceeding of the Intern. Conference on Preferences and Decisions, Trento, 6th-8-th april 2009.

[64] Maturo A., Squillante M., and Ventre A. G. S., (2009), Decision Making, Fuzzy Measures, and Hyperstructures, IV meeting on Dynamics of Social and Economic systems, April 14-18, 2009, Argentina, submitted to Advances and Applications in Statistical Sciences.

[65] Maturo A., On some structures of fuzzy numbers, accepted for publication in Proceedings of 10th International AHA Congress Brno, 3 - 9. 9. 2008.

[66] Maturo A., Tofan I., Iperstrutture, strutture fuzzy ed applicazioni, Pubblicazioni Progetto Internazionale Socrates, Italia-Romania, San Salvo. Dierre Edizioni

[67] Prenowitz W and Jantosciak J., Geometries and Join Spaces, J. reine und angewandte Math. 257, (1972)

[68] Serafimidis K., Konstantinidou M., Kehagias Ath., L - Fuzzy Nakano Hyperlattices, Proc. 8th International Congress on A.H.A., Samothraki (2002), Spanidis Press

[69] Serafimidis K., Kehagias Ath., Konstantinidou M., The L-Fuzzy Corsini Join Hyperoperation, Italian Journal of Pure and Applied Mathematics, n. 12, (2003)

[70] Stefanescu M., Cristea I., On the Fuzzy Grade of Hypergroups, Fuzzy Sets and Systems, (2008)

[71] Tofan I., Volf C., On some connections between hyperstructures and fuzzy sets, Italian Journal of Pure and Applied Mathematics, n. 7, (2000)

[72] Zadeh L.A., Fuzzy Sets, Information and Control, 8, (1965)

[73] Zahedi M.M., Bolurian M., Hasankhani A., On polygroups and fuzzy subpolygroups, J. of Fuzzy Mathematics, vol. 3, n.1, (1995)

[74] Zahedi, MM, Hasankhani, A, F-polygroups (II), Information Sciences, v 89, n 3-4, Mar, 1996,

[75] M. M. Zahedi, L. Torkzadeh: Intuitionistic Fuzzy Dual Positive Implicative Hyper K- Ideals. WEC (5) 2005

[76] Zhan Janming, Davvaz B., Shum K.P., A new view of hypermodules, Acta Mathematica Sinica, English Series 23(8), (2007)

[77] Zhan Janming, Davvaz B., Shum K.P., On fuzzy isomorphism theorems of hypermodules, Soft Computing, 11, (2007)