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On integer cordial labeling of some families of graphs

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Abstract

An integer cordial labeling of a graph $G(p,q)$ is an injective map $f: V \to [-\frac{p}{2}...\frac{p}{2}]^*$ or $[-\lfloor\frac{p}{2}\rfloor...\lfloor\frac{p}{2}\rfloor]$ as p is even or odd, which induces an edge labeling $f^*: E \to \{0,1\}$ as $f^*(uv) = \begin{cases} 1, f(u)+f(v) \geq 0 \\ 0, \text{ otherwise} \end{cases}$ such that the number of edges labelled with 1 and the number of edges labelled with 0 differ at most by 1. If a graph has integer cordial labeling, then it is called integer cordial graph. In this paper, we have proved that the Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jewel graph, Jahangir graph, Crown graph admits integer cordial labeling.

Keywords: Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jewel graph, Jahangir graph, Crown graph, Integer cordial labeling.

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1 Introduction

Let G be a simple, finite, undirected graph. For terms not defined here, we refer to Harary [3]. A graph labeling is an assignment of integers to the vertices or edges, or both, under some conditions. It has wide applications in mathematics as well as in other fields such as circuit design, communication network addressing, date base management and so on. In this paper, we have proved that the Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jahangir graph, Jewel graph, Crown graph are integer cordial.

2 Preliminaries

Definition 2.1 [8] A mapping $f : V(G) \to \{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f. If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$.

Definition 2.2 [1] A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ = number of vertices having label i under f and $e_f(i)$ = number of edges having label i under f^*. A graph G is cordial if it admits cordial labeling. I. Cahit [1] introduced the concept of cordial labeling as a weaker version of graceful and harmonious graphs.

Definition 2.3 [7] An integer cordial labeling of a graph $G(p,q)$ is an injective map $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ or $[-\lfloor\frac{p}{2}\rfloor...\lfloor\frac{p}{2}\rfloor]$ as p is even or odd, which induces an edge labeling $f^* : E \to \{0,1\}$ defined by $f^*(uv) = \begin{cases} 1, f(u) + f(v) \geq 0 \\ 0, \text{ otherwise} \end{cases}$ such that the number of edges labelled with 1 and the number of edges labelled with 0 differ at most by 1. If a graph has integer cordial labeling, then it is called integer cordial graph.

Definition 2.4 [4] A banana tree $B_{n,k}$ is a graph obtained by connecting one leaf of each of n copies of a k - star graph with a single root vertex. It has $nk + 1$ vertices and nk edges.

Definition 2.5 [5] $K_{1,n} * K_{1,m}$ is the graph obtained from $K_{1,n}$ by attaching root of a star $K_{1,m}$ at each pendant vertex of $K_{1,n}$.

Definition 2.6 [6] Olive tree T_k is a rooted tree consisting of k branches where the i th branch is a path of length i and it consists of $\frac{k(k+1)}{2} + 1$ vertices.

Definition 2.7 [5] The Jewel graph J_n is the graph with vertex set $V(J_n) = \{u, v, x, y, w_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, uy, xy, xv, yv, uw_i, vw_i : 1 \leq i \leq n\}$.

Definition 2.8 [2] Jahangir graph $J_{n,m}$ for $m \geq 3$, is a graph on $nm + 1$ vertices, consisting of a cycle C_{nm} with one additional vertex which is adjacent to m ver-

tices of C_{nm} at a distance n to each other on C_{nm}.
Definition 2.9 [5] The crown $C_n \bigodot K_1$ is the graph obtained from a cycle by attaching a pendant edge to each vertex of the cycle.

In [7], Nicholas et al. introduced the concept of integer cordial labeling of graphs and proved that some standard graphs such as Path P_n, Star graph $K_{1,n}$, Cycle C_n, Helm graph H_n, Closed helm graph CH_n are integer cordial. K_n is not integer cordial, $K_{n,n}$ is integer cordial iff n is even and $K_{n,n} \backslash M$ is integer cordial for any n, where M is perfect matching of $K_{n,n}$. In [8], Sarah et al. proved that the Sierpinski Sieve graph, the graph obtained by joining two friendship graphs by a path of arbitrary length, $(n, k)-$ kite graph and Prism graph are integer cordial.

3 Main Results

Theorem 3.1. *Banana tree $B_{n,k}$ is integer cordial.*

Proof. **Case1:** When n is even (the total number of vertices is odd).
Let u denote the root vertex. Let $v_1, v_2, \ldots, v_{\frac{nk}{2}}$ denote the vertices on $\frac{n}{2}$ leaves and $v_{\frac{nk}{2}+1}, v_{\frac{nk}{2}+2}, \ldots, v_{nk}$ denote the vertices on the remaining $\frac{n}{2}$ leaves of $B_{n,k}$.
We define $f : V \to [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows:

$$\begin{aligned}
f(u) &= 0 \\
f(v_i) &= -i; 1 \leq i \leq \frac{nk}{2} \\
f(v_i) &= i - \frac{nk}{2}; \frac{nk}{2} + 1 \leq i \leq nk
\end{aligned}$$

Case 2: When n is odd and number of vertices is odd.
Let u denote the root vertex. Let $v_1, v_2, \ldots, v_{\frac{(n-1)k}{2}}$ denote the vertices on $\lfloor \frac{n}{2} \rfloor$ leaves and $v_{\frac{(n-1)k}{2}+1}, v_{\frac{(n-1)k}{2}+2}, \ldots, v_{(n-1)k}$ denote the vertices of remaining $\lfloor \frac{n}{2} \rfloor$ leaves. Let $u_1, u_2, \ldots, u_k$ denote the k vertices of the another leaf such that u_1 is adjacent to u and u_k is adjacent to u_i where $1 \leq i \leq k$.

We define $f : V \to [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows:

$$f(u) = 0$$
$$f(v_i) = -i; 1 \leq i \leq \frac{(n-1)k}{2}$$
$$f(v_i) = i - \frac{(n-1)k}{2}; \frac{(n-1)k}{2} + 1 \leq i \leq k(n-1)$$
$$f(u_i) = -(\frac{nk}{2} + 1 - i); 1 \leq i \leq \frac{k}{2}$$
$$f(u_i) = \frac{k(n+1)}{2} + 1 - i; \frac{k}{2} + 1 \leq i \leq k$$

Case 3: When n is odd and the number of vertices is even.
Let u denote the root vertex. Let $v_1, v_2, \ldots, v_{\frac{(n-1)k}{2}}$ denote the vertices on $\lfloor \frac{n}{2} \rfloor$ leaves and $v_{\frac{(n-1)k}{2}+1}$,$v_{\frac{(n-1)k}{2}+2}$,...,$v_{k(n-1)}$ denote the vertices of another $\lfloor \frac{n}{2} \rfloor$ leaves where v_1 not adjacent to u. Let $u_1, u_2, ..., u_k$ denote the k vertices of the remaining leaf such that u_1 is adjacent to u and u_k is adjacent to u_i, $1 \leq i \leq k$.
We define $f : V \to [-\frac{p}{2} ... \frac{p}{2}]^*$ as follows:

$$f(u) = 1$$
$$f(v_i) = -i; 1 \leq i \leq \frac{(n-1)k}{2}$$
$$f(v_i) = i + 1 - \frac{(n-1)k}{2}; \frac{(n-1)k}{2} + 1 \leq i \leq k(n-1)$$
$$f(u_i) = -\Big(\lceil \frac{nk}{2} \rceil + 1 - i\Big); 1 \leq i \leq \lceil \frac{k}{2} \rceil$$
$$f(u_i) = \frac{k(n+1)}{2} + 2 - i; \lceil \frac{k}{2} \rceil + 1 \leq i \leq k$$

Hence in all the possible cases, we have $|e_f(1) - e_f(0)| \leq 1$. Therefore, Banana tree $B_{n,k}$ admits integer cordial labeling(See Figure 1).

□

Theorem 3.2. *The graph $K_{1,n} * K_{1,m}$ is integer cordial.*

Proof. **Case 1:** When n is even and m can be either odd or even.
Let $u_1, u_2, ...u_{\frac{n}{2}(1+m)}$ be the vertices of $\frac{n}{2}$ leaves and let $w_1, w_2, ...w_{\frac{n}{2}(1+m)}$ be the vertices of the other $\frac{n}{2}$ leaves and let u_0 be the center vertex.
We define $f : V \to [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows.

$$f(u_0) = 0$$
$$f(u_i) = i, 1 \leq i \leq \frac{n}{2}(1+m)$$
$$f(w_i) = -i, 1 \leq i \leq \frac{n}{2}(1+m)$$

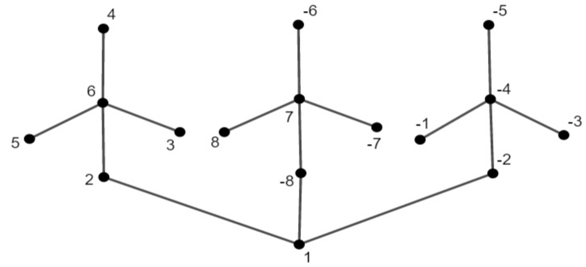

Figure 1: Integer cordial labeling of $B_{3,5}$

Case 2: When n is odd.
Let the center vertex be u_0. Let $u_1, u_2, ...u_{\lfloor \frac{n}{2} \rfloor(1+m)}$ be the vertices of $\lfloor \frac{n}{2} \rfloor$ leaves and let $w_1, w_2, ...w_{\lfloor \frac{n}{2} \rfloor(1+m)}$ be the vertices of the other $\lfloor \frac{n}{2} \rfloor$ leaves and let $v_1, v_2, ...v_{(m+1)}$ be the remaining vertices on the left out leaf, where v_1 is adjacent to u.
Case 2.1: When m is odd.
We define $f : V \to [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows:

$$f(u_0) = 0$$
$$f(u_i) = i, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor (1+m)$$
$$f(w_i) = -i, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor (1+m)$$
$$f(v_1) = -\Big(\lfloor \frac{n}{2} \rfloor (2+m)\Big)$$
$$f(v_i) = \lfloor \frac{n}{2} \rfloor (1+m+i), 2 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$$
$$f(v_{\lfloor \frac{m+1}{2} \rfloor + i}) = -\lfloor \frac{n}{2} \rfloor (1+m+i), \lceil \frac{m+1}{2} \rceil \leq i \leq m+1$$

Case 2.2: When m is even.
We define $f : V \to [-\frac{p}{2} ... \frac{p}{2}]^*$ as follows:

$$f(u_0) = 1$$
$$f(u_i) = i+1, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor (1+m)$$
$$f(w_i) = -(i+1), 1 \leq i \leq \lfloor \frac{n}{2} \rfloor (1+m)$$
$$f(v_1) = -1$$
$$f(v_i) = \lfloor \frac{n}{2} \rfloor (1+m+i), 2 \leq i \leq \frac{m+1}{2}$$
$$f(v_{\lfloor \frac{m+1}{2} \rfloor + i}) = -\Big(\lfloor \frac{n}{2} \rfloor (1+m+i)\Big), \frac{m+1}{2} \leq i \leq m+1$$

Here, for all possible cases, we have $|e_f(1) - e_f(0)| \leq 0$. Therefore $K_{1,n} * K_{1,m}$ is integer cordial(See Figure 2). □

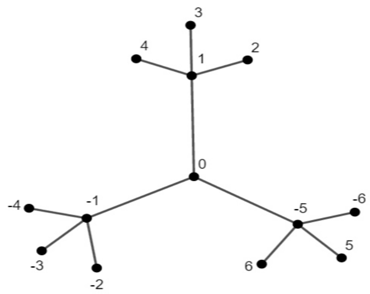

Figure 2: Integer cordial labeling of $K_{1,3} * K_{1,3}$

Theorem 3.3. *Olive tree T_k admits integer cordial labeling.*

Proof. Let U_i denote the $(k+1-i)^{th}$ branch and U denote the root vertex.
Case 1: When $\frac{k(k+1)}{2} + 1$ is an odd number.
An integer cordial labeling of T_k is obtained by assigning the positive integer from 1 to $\frac{k(k+1)}{4}$ to the vertices of the branches namely $U_2, U_3, U_6, U_7, U_{10}, U_{11}, U_{14}$,..., in any order and the negative integers from -1 to $\frac{-k(k+1)}{4}$ to the vertices of the branches namely $U_1, U_4, U_5, U_8, U_9, U_{12}, U_{13}, ...$, in any order and let $U = 0$.

Case 2: When $\frac{k(k+1)}{2} + 1$ is an even number.
An integer cordial labeling of T_k is obtained by assigning the positive integers from 2 to $\frac{(k(k+1)+2)}{4}$ to the vertices of the branches namely $U_2, U_3, U_6, U_7, U_{10}, U_{11}, ...$, in any order and the negative integers from -1 to $\frac{-(k(k+1)+2)}{4}$ to the vertices of the branches namely $U_1, U_4, U_5, U_8, U_9, U_{12}, U_{13}, ...$, in any order and let $U = 1$.
Hence, we have $|e_f(1) - e_f(0)| \leq 1$. Therefore, Olive tree admits integer cordial labeling(See Figure 3).

□

Theorem 3.4. *The Jewel graph J_n admits integer cordial labeling.*

Proof. Let $V(G_n) = \{u, v, x, y, w_i : 1 \leq i \leq n\}$ and $E(G) = \{ux, uy, xy, xv, yv, uw_i, vw_i : 1 \leq i \leq n\}$.
Case 1: When n is even.

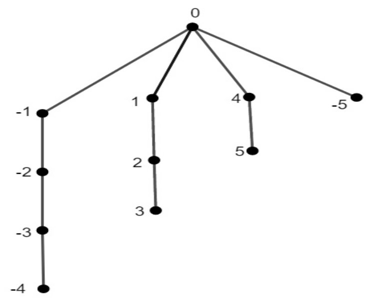

Figure 3: Integer cordial labeling of T_4

We define $f : V \rightarrow [-\frac{p}{2}...\frac{p}{2}]^*$ as follows:

$$\begin{aligned}
f(u) &= 1 \\
f(v) &= -1 \\
f(x) &= 2 \\
f(y) &= -2 \\
f(w_i) &= i+2, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
f(w_i) &= -(i - \lfloor \frac{n}{2} \rfloor + 2), \lceil \frac{n}{2} \rceil \leq i \leq n-1
\end{aligned}$$

Case 2: When n is odd.
We define $f : V \rightarrow [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows:

$$\begin{aligned}
f(u) &= 1 \\
f(v) &= -1 \\
f(x) &= 2 \\
f(y) &= -2 \\
f(w_i) &= i+2, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
f(w_i) &= -(i - \lfloor \frac{n}{2} \rfloor + 2), \lceil \frac{n}{2} \rceil \leq i \leq n-1 \\
f(w_n) &= 0
\end{aligned}$$

Here, for both the cases, we have $n+3$ edges with label 1 and $n+2$ edges with label 0. Hence in all possible cases, we have $|e_f(1) - e_f(0)| = 1$. Therefore, J_n is integer cordial(See Figure 4).

□

Theorem 3.5. *Jahangir graph $J_{n,m}$ is integer cordial except when $n = 1$.*

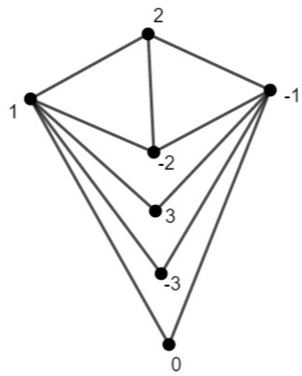

Figure 4: Integer cordial labeling of J_3

Proof. When $n = 1$, we have $J_{1,m}$ to be a complete graph with $m+1$ vertices and hence not integer cordial.
Let u denote the central vertex adjacent to m vertices of C_{nm} and let $v_1, v_2, ..., v_{nm}$ denote the vertices in the cycle C_{nm}.

Case 1: When the number of vertices $(nm + 1)$ is odd.
We define $f : V \to [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows:

$$f(u) = 0$$
$$f(v_i) = i; 1 \leq i \leq \frac{nm}{2}$$
$$f(v_{(\frac{nm}{2}+i)}) = -i; 1 \leq i \leq \frac{nm}{2}$$

Case 2: When the number of vertices $(nm + 1)$ is even.
We define $f : V \to [-\frac{p}{2} ... \frac{p}{2}]^*$ as follows.

$$f(u) = 1$$
$$f(v_i) = i + 1; 1 \leq i \leq \lfloor \frac{nm}{2} \rfloor$$
$$f(v_{\lfloor \frac{nm}{2} \rfloor + i}) = -i; 1 \leq i \leq \lceil \frac{nm}{2} \rceil$$

Hence in all possible cases, we have $|e_f(1) - e_f(0)| \leq 1$. Therefore $J_{n,m}$ admits integer cordial labeling except when $n = 1$(See Figure 5).

□

Theorem 3.6. *The Crown $C_n \bigodot K_1$ admits integer cordial labeling.*

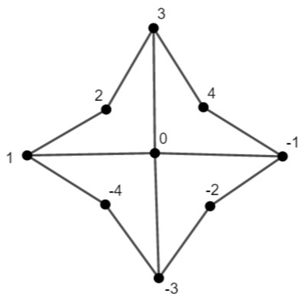

Figure 5: Integer cordial labeling of $J_{2,4}$

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of the inner cycle and let $u_1, u_2, ..., u_n$ be the pendent vertices where u_i is adjacent to v_i.
We define $f : V \rightarrow [-\frac{p}{2}...\frac{p}{2}]^*$ as follows:

$$f(v_i) = -i$$
$$f(u_i) = i$$

Here, we have n edges with label 1 and n edges with label 0. Hence, $|e_f(1) - e_f(0)| = 0$. Therefore, the Crown $C_n \bigodot K_1$ is integer cordial(See Figure 6).

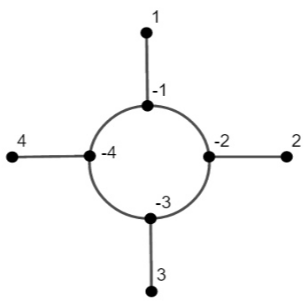

Figure 6: Integer cordial labeling of $C_4 \bigodot K_1$

□

4 Conclusion

In this paper, we proved that the Banana tree, $K_{1,n} * K_{1,m}$, Olive tree, Jewel graph, Jahangir graph, Crown graph are integer cordial. Obtaining the integer cordial labeling of other class of graphs is still open. Further investigation can be done for all the networks.

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